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Rational Pairing Rank of a Map

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Abstract

We define a rational homotopy invariant, the rational pairing rank $v_0(f)$ of a map $f: X \to Y$, which is a natural generalization of the rational pairing rank $v_0(X)$ of a space X [16]. It is upper-bounded by the rational LS-category $cat_0(f)$ and lower-bounded by an invariant $g_0(f)$ related to the rank of Gottlieb group. Also it has a good estimate for a fibration $X \overset{j}{\to} E \overset{p}{\to} Y$ such as $v_0(E) \leq v_0(j) + v_0(p) \leq v_0(X) + v_0(Y)$.

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1 Introduction

In this paper, all spaces are connected and simply connected based CW complexes of finite rational LS-category [4] and maps are based. In [16], the author has introduced a homotopy invariant, which is called the rational pairing rank of a space, being inspired by the notion of pairing of a map in [10]. We begin with the definition of the invariant.

Definition 1.1. ([16]) The pairing rank $v_0(X)$ of a space X in the rational homotopy group is the maximal integer n such that there is a map μ_X in the homotopy commutative diagram:

$$\begin{array}{ccc} S^{l_1} \times \cdots \times S^{l_n} & \xrightarrow{\mu_X} & X_{\mathbb{Q}} \\ \cup\uparrow & & \| \\ S^{l_1} \vee \cdots \vee S^{l_n} & \xrightarrow[\langle a_{i_1}, \cdots, a_{i_n} \rangle]{} & X_{\mathbb{Q}} \end{array}$$

for some linearly independent elements $a_{i_1}, .., a_{i_n}$ of $\pi_{odd}(X)_{\mathbb{Q}} = \oplus_{i>0} \pi_{2i+1}(X) \otimes \mathbb{Q}$ with $|a_{i_k}| = l_k$.

For example, for a map $f: G \to X$ from a compact Lie group G to a space X, $\dim \mathrm{Im} \pi_*(f)_{\mathbb{Q}} \leq v_0(X)$. In particular, if $\pi_*(f)_{\mathbb{Q}}$ is injective, then rank $G \leq v_0(X)$.

Note that the restriction on the odd degree elements in $\pi_*(X)_{\mathbb{Q}}$ in Definition 1.1 is suitable because $\pi_{4k-1}(S^{2k})_{\mathbb{Q}} \cong \mathbb{Q}$ for any $k > 0$ [5]. The definition is naturally generalized as follows.

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Definition 1.2. The pairing rank $v_0(f)$ of a map $f : X \to Y$ in the rational homotopy group is the maximal integer n such that there is a map μ_f in the homotopy commutative diagram $(*)$:

$$\begin{array}{ccc} S^{l_1}\times\cdots\times S^{l_n} & \xrightarrow{\mu_f} & X_{\mathbb{Q}} \\ \cup\uparrow & & \downarrow f_{\mathbb{Q}} \\ S^{l_1}\vee\cdots\vee S^{l_n} & \xrightarrow[\langle a_{i_1},\cdots,a_{i_n}\rangle]{} & Y_{\mathbb{Q}} \end{array}$$

for some linearly independent elements $a_{i_1},..,a_{i_n}$ of $\pi_{odd}(Y)_{\mathbb{Q}}$ with $|a_{i_k}| = l_k$.

Then it induces $v_0(id_X) = v_0(X)$ and $v_0(f) = v_0(g)$ if $f_{\mathbb{Q}} \simeq g_{\mathbb{Q}}$. In this paper, we consider this rational homotopy invariant $v_0(f)$ of a map f.

Lemma 1.3. *Let $f : X \to Y$ be a map. Then*

(1) $v_0(f) \leq \dim \mathrm{Im}(\pi_*(f)_{\mathbb{Q}})$.

(2) $v_0(f) \leq \min\{v_0(X), v_0(Y)\}$.

(3) $v_0(f) = 0$ *if f is rationally constant, i.e.; $f \simeq_{\mathbb{Q}} *$.*

(4) $v_0(f) = v_0(X)$ *if $\pi_*(f)_{\mathbb{Q}}$ is injective.*

(5) $v_0(f) = v_0(Y)$ *if f has a rational homotopy section, i.e.; there is a map $s : Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ with $f_{\mathbb{Q}} \circ s \simeq id_{Y_{\mathbb{Q}}}$.*

(6) $v_0(g \circ f) \leq \min\{v_0(f), v_0(g)\}$ *for a map $g : Y \to Z$.*

(7) $v_0(f_1 \vee f_2) = \max\{v_0(f_1), v_0(f_2)\}$ *for maps $f_i : X_i \to Y_i$ ($i = 1,2$).*

(8) $v_0(f_1 \times f_2) = v_0(f_1) + v_0(f_2)$ *for maps $f_i : X_i \to Y_i$ ($i = 1,2$).*

Recall the definition of the rational LS(Lusternik-Schnirelmann)-category $cat_0(f)$ of a map $f : X \to Y$ [2]. It is the minimal integer n such that there exists a map $\eta(n)$ which makes the diagram $(**)$:

$$\begin{array}{ccc} X_{\mathbb{Q}} & \xrightarrow{\eta(n)} & E_n(Y)_{\mathbb{Q}} \\ \| & & \downarrow p_{n\mathbb{Q}} \\ X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}} \end{array}$$

homotopy commutative. Here $X_{\mathbb{Q}}$ and $f_{\mathbb{Q}}$ are the rationalizations of X and f, respectively [7] and $p_n : E_n(Y) \to Y$ is the n-th Ganea map of Y [2]. Then $cat_0(id_X) = cat_0(X)$, where id_X is the identity map of X and $cat_0(X)$ is the rational LS-category of a space X. It does not hold that $cat_0(f_1 \times f_2) = cat_0(f_1) + cat_0(f_2)$ as (8) in general [12]. By using Sullivan models [13] in §2, we have

Theorem 1.4. *For a map $f : X \to Y$, $v_0(f) \leq cat_0(f)$.*

Recall the n-th *Gottlieb group* $G_n(X)$ [6] of a CW complex X for $n > 0$, which is the subgroup of the $\pi_n(X)$ consisting of homotopy classes of maps $a : S^n \to X$ such that the wedge $(a|id_X) : S^n \vee X \to X$ extends to a map $F_a : S^n \times X \to X$ in the homotopy commutative diagram:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{F_a} & X \\ \text{incl.}\uparrow & & \uparrow\nabla \\ X \vee S^n & \xrightarrow{id_X \vee a} & X \vee X. \end{array}$$

Note that a map $f : X \to Y$ does not induce $\pi_n(f) : G_n(X) \to G_n(Y)$ in general. Let $G_*(X) = \oplus_{n>0} G_n(X)$ and $G_n(X)_{\mathbb{Q}} = G_n(X) \otimes \mathbb{Q}$. Note $G_*(X)_{\mathbb{Q}} = G_{odd}(X)_{\mathbb{Q}}$ [4]. Recall that it holds that $\dim G_*(X)_{\mathbb{Q}} \leq v_0(X) \leq cat_0(X)$ [16].

Definition 1.5. The Gottlieb rank $g_0(f)$ of a map $f : X \to Y$ is given by

$$g_0(f) := \dim \mathrm{Im}(\pi_*(f)_{\mathbb{Q}} : G_*(X)_{\mathbb{Q}} \to \pi_*(Y)_{\mathbb{Q}}).$$

Then $g_0(f) = g_0(f')$ if $f_{\mathbb{Q}} \simeq f'_{\mathbb{Q}}$ and $g_0(f) \leq \dim G_*(X)_{\mathbb{Q}}$. In particular, $g_0(id_X) = \dim G_*(X)_{\mathbb{Q}}$ and $g_0(f) = 0$ when f is a rationally constant map. We often denote $\dim G_*(X)_{\mathbb{Q}}$ as $g_0(X)$. For maps $f_i : X_i \to Y_i$ for $i = 1,2$, $g_0(f_1 \times f_2) = g_0(f_1) + g_0(f_2)$. There do not hold (6) and (7) in Lemma 1.3 for Gottlieb rank of a map (see Example 3.5).

Theorem 1.6. *For a map $f : X \to Y$, $g_0(f) \leq v_0(f)$.*

Proof. Let $g_0(f) = n$. Then there is a homotopy commutative diagram:

$$\begin{array}{ccc} X_{\mathbb{Q}} \times S^{l_1} \times \cdots \times S^{l_n} & \longrightarrow & X_{\mathbb{Q}} \\ \cup \uparrow & & \downarrow f_{\mathbb{Q}} \\ X_{\mathbb{Q}} \vee S^{l_1} \vee \cdots \vee S^{l_n} & \xrightarrow[\langle f_{\mathbb{Q}}, a_{i_1}, \cdots, a_{i_n} \rangle]{} & Y_{\mathbb{Q}} \end{array}$$

for some linearly independent elements $a_{i_1}, .., a_{i_n}$ of $\pi_{odd}(Y)_{\mathbb{Q}}$ with $|a_{i_k}| = l_k$, as §1 $(*)$ in [16]. It means $n \leq v_0(f)$ since the diagram induces the above $(*)$ by restrictions. □

From Theorem 1.4 and Theorem 1.6, we have $g_0(f) \leq v_0(f) \leq cat_0(f)$. In particular, when a map f is the projection $p_Y : X \times Y \to Y$ or the inclusion $i_X : X \to X \times Y$, they are equal. The author does not know when does it hold that $g_0(f) = v_0(f) = cat_0(f)$ in general. Finally we consider a relation between $v_0(j) + v_0(p)$ and $v_0(E)$ for a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} Y$. Recall the inequation $v_0(E) \leq v_0(X) + v_0(Y)$ [16]. In this paper, we see

Theorem 1.7. *For a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} Y$, $v_0(E) \leq v_0(j) + v_0(p)$.*

In §2, we give the proofs of the above theorems by using Sullivan models. In §3, we illustrate some examples. In §4, we comment a relation with Halperin conjecture on fibration [5, page 516].

2 Sullivan model

Recall the *Sullivan minimal model* $M(X)$ [13] of a simply connected space X of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(\Lambda V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i>1} V^i$ of $\dim V^i < \infty$ and a decomposable differential d. Denote the degree of a homogeneous element x of a graded algebra as $|x|$. A fibration $p : E \to Y$ has a minimal model which is a DGA-map $M(p) : M(Y) \to M(E)$. It is induced by a relative model

$$M(Y) = (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D),$$

where $(\Lambda V,\overline{D})=(\Lambda V,d_X)$ is the minimal model of the homotopy fibre X of p and there is a quasi-isomorphism $\rho_E : M(E)\xrightarrow{\sim}(\Lambda W\otimes\Lambda V,D)$. Notice that $M(X)$ determines the rational homotopy type of X, especially $H^*(X;\mathbb{Q})\cong H^*(M(X))$ as graded algebras and $\pi_i(X)\otimes\mathbb{Q}\cong Hom(V^i,\mathbb{Q})$. We refer to [5] for a general introduction and the standard notations. The above Definition 1.2 is replaced with

Lemma 2.1. *For a map $f : X\to Y$, $v_0(f)\geq n$ if and only if there is a DGA-map:*

$$\mu_f : (\Lambda W\otimes\Lambda V,D)\to(\Lambda(w_1,\cdots,w_n),0) \qquad (1)$$

such that $\mu_f(w_i)=w_i$ for some linearly independent elements $w_1,\cdots,w_n$ of W^{odd}.

Proof of Lemma 1.3. We can check that (5) follows from Lemma 2.1 since, after a suitable change of basis, $DV\subset\Lambda W\otimes\Lambda^+V$ [14]. The others immediately hold from Definition 1.2. □

In the following, we often use the same symbols $\mu_X : M(X)=(\Lambda V,d)\to(\Lambda(v_1,\cdots,v_n),0)$ with some linearly independent elements $v_1,\cdots,v_n$ of V in [16, Lemma 2.1] for an n-pairing $\mu_X : S^{k_1}\times\cdots\times S^{k_n}\to X$ of $k_i=|v_i|$ and μ_f in Lemma 2.1(1) for μ_f in Definition 1.2.

Proof of Theorem 1.4. In the rational models, the diagram $(**)$ of §1 is given as the DGA-commutative diagram:

$$\begin{array}{ccc}
(\Lambda W\otimes\Lambda U,D_Y) & \underset{\sim}{\xrightarrow{\rho_n}} & (\Lambda W/\Lambda^{>n}W,\overline{d_Y}) \\
\| & & \uparrow{\scriptstyle proj_n} \\
(\Lambda W\otimes\Lambda U,D_Y) & \xleftarrow{i_n} & (\Lambda W,d_Y) \\
{\scriptstyle \eta(n)}\downarrow & & \| \\
(\Lambda W\otimes\Lambda V,D) & \longleftarrow & (\Lambda W,d_Y)
\end{array} \qquad (2)$$

where i_n is the relative model of proj_n (see [3, Theorem 10.6], [4], [2]). Suppose $v_0(f)=n$. From Lemma 2.1, there is a map μ_f in (1). Then there is no map $\eta(n-1)$ in the DGA-commutative diagram induced from (2):

$$\begin{array}{ccccc}
 & & (\Lambda W\otimes\Lambda U',D'_Y) & \underset{\sim}{\xrightarrow{\rho_{n-1}}} & (\Lambda W/\Lambda^{\geq n}W,\overline{d_Y}) \\
 & & \| & & \uparrow{\scriptstyle proj_{n-1}} \\
 & & (\Lambda W\otimes\Lambda U',D'_Y) & \xleftarrow{i_{n-1}} & (\Lambda W,d_Y) \\
 & & {\scriptstyle \eta(n-1)}\downarrow & & \| \\
(\Lambda(w_1,\cdots,w_n),0) & \xleftarrow{\mu_f} & (\Lambda W\otimes\Lambda V,D) & \longleftarrow & (\Lambda W,d_Y)
\end{array}$$

Indeed, if it exists, the zero element is sent to the non-zero element $w_1\cdots w_n$ in the composition

$$H^*(\Lambda W/\Lambda^{\geq n}W)\overset{(\rho^*_{n-1})^{-1}}{\cong}H^*(\Lambda W\otimes\Lambda U')\overset{\mu_f^*\circ\eta(n-1)^*}{\longrightarrow}\Lambda(w_1,\cdots,w_n)$$

since $\rho_{n-1}(w_1\cdots w_n+a)=0$ and $\mu_f\circ\eta(n-1)(w_1\cdots w_n+a)=w_1\cdots w_n$ for a suitable element $a\in\Lambda W\otimes\Lambda^+U'$ such that $w_1\cdots w_n+a$ is a D'_Y-(exact) cocycle. Thus we have $cat_0(f)>n-1$. □

Proof of Theorem 1.7. It is the similar argument as the proof of Thorem 1.6(1) in [16]. Let $v_0(E)=n$ by $\mu_E:S^{l_1}\times\cdots\times S^{l_n}\to E$. Then there is an integer $m(\leq n)$ such that there are the homotopy commutative diagrams:

$$(i)\qquad \begin{array}{ccc} S^{k_1}\times\cdots\times S^{k_m} & \xrightarrow{\mu_p} & E_{\mathbb{Q}} \\ \cup\big\uparrow & & \big\downarrow p_{\mathbb{Q}} \\ S^{k_1}\vee\cdots\vee S^{k_m} & \xrightarrow[\langle a_{i_1},\cdots,a_{i_m}\rangle]{} & Y_{\mathbb{Q}} \end{array}$$

and

$$(ii)\qquad \begin{array}{ccc} S^{k_{m+1}}\times\cdots\times S^{k_n} & \xrightarrow{\mu_j} & X_{\mathbb{Q}} \\ \cup\big\uparrow & & \big\downarrow j_{\mathbb{Q}} \\ S^{k_{m+1}}\vee\cdots\vee S^{k_n} & \xrightarrow[\langle a_{i_{m+1}},\cdots,a_{i_n}\rangle]{} & E_{\mathbb{Q}} \end{array}$$

with $\{k_1,..,k_n\}=\{l_1,..,l_n\}$. Here μ_p is a homotopy restriction of μ_E and μ_j is a homotopy lift of a restriction of μ_E.

Indeed, let

$$(\Lambda W,d_Y)\to(\Lambda W\otimes\Lambda V,D)\xrightarrow{q}(\Lambda V,d_X)$$

be the model (Koszul-Sullivan extension) of ξ and $M(E)=(\Lambda U,d_E)$. Then there is an inclusion $U\subset W\oplus V$ inducing $U\cong H^*(W\oplus V,Q(D))$ ($Q(D)$ is the linear part of D) so that a diagram

$$\begin{array}{ccccc} W & \longrightarrow & U & \longrightarrow & V \\ \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\ \pi_*(Y)^\vee_{\mathbb{Q}} & \xrightarrow{p^\vee_\sharp} & \pi_*(E)^\vee_{\mathbb{Q}} & \xrightarrow{j^\vee_\sharp} & \pi_*(X)^\vee_{\mathbb{Q}} \end{array}$$

is commutative (up to sign) [5, Proposition 15.13]. Suppose that there is a DGA-map $\mu_E:(\Lambda U,d_E)\to(\Lambda(u_1,\cdots,u_n),0)$ for $u_i\in U$ given as Lemma 2.1 of [16]. Without loss of generality, we can assume that there is an integer $m(\leq n)$ with $\{u_1,..,u_n\}=\{w_1,..,w_m,v_1,..,v_{n-m}\}\subset W^{odd}\oplus V^{odd}$. Then (i) is obvious and (ii) is guaranteed by the DGA homotopy commutative diagram

$$\begin{array}{ccc} (\Lambda(v_1,\cdots,v_{n-m}),0) & \xleftarrow{\overline{\mu_E^*}} & (\Lambda V,d_X) \\ \overline{q}\big\uparrow & & q\big\uparrow \\ (\Lambda(w_1,..,w_m,v_1,..,v_{n-m}),0) & \xleftarrow{\mu_E^*} & (\Lambda W\otimes\Lambda V,D), \end{array}$$

where the induced map $\overline{\mu_E^*}$ gives the model of μ_j. □

Corollary 2.2. *For a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} Y$ with $v_0(E) = n$, there is an integer $m \le n$ such that a diagram with k_i odd*

$$\begin{array}{ccccc} S^{k_{m+1}}\times\cdots\times S^{k_n} & \longrightarrow & S^{k_1}\times\cdots\times S^{k_n} & \longrightarrow & S^{k_1}\times\cdots\times S^{k_m} \\ {\scriptstyle \mu'_j}\downarrow & & {\scriptstyle \mu_E}\downarrow & & {\scriptstyle \mu'_p}\downarrow \\ X_{\mathbb{Q}} & \xrightarrow{j} & E_{\mathbb{Q}} & \xrightarrow{p} & Y_{\mathbb{Q}} \end{array}$$

is homotopy commutative. Here μ'_j and μ'_p are certain restrictions of μ_j and μ_p as in Definition 1.2 $(*)$, respectively.

Remark 2.3. In the above corollary, the integer m is not unique since μ_E is not unique. For example, for a fibration $S^3\times S^7 \to E \to S^5$ given by

$$(\Lambda(w),0) \to (\Lambda(w,x,y),D) \to (\Lambda(x,y),0)$$

with $|x| = 3$, $|y| = 7$, $|w| = 5$, $Dy = wx$, $Dx = Dw = 0$, we have $v_0(E) = v_0(j) = 2$ and $v_0(p) = 1$. Then there are two diagrams as

$$\begin{array}{ccccc} S^7 & \longrightarrow & S^7\times S^5 & \longrightarrow & S^5 \\ \downarrow & & {\scriptstyle \mu_E}\downarrow & & \| \\ X_{\mathbb{Q}} & \xrightarrow{j} & E_{\mathbb{Q}} & \xrightarrow{p} & Y_{\mathbb{Q}} \end{array} \quad \text{and} \quad \begin{array}{ccccc} S^3\times S^7 & \longrightarrow & S^3\times S^7 & \longrightarrow & \bullet \\ \| & & {\scriptstyle \mu_E}\downarrow & & \downarrow \\ X_{\mathbb{Q}} & \xrightarrow{j} & E_{\mathbb{Q}} & \xrightarrow{p} & Y_{\mathbb{Q}}, \end{array}$$

where $m = 1$ and $m = 0$, respectively.

Corollary 2.4. *If a fibration $\xi : X \xrightarrow{j} E \to Y$ is weakly rational trivial; i.e., $\pi_*(E)_{\mathbb{Q}} = \pi_*(X)_{\mathbb{Q}} \oplus \pi_*(Y)_{\mathbb{Q}}$, we have $v_0(j) = v_0(X) \le v_0(E)$.*

3 Examples

Let $\mathbb{C}P^n$ be the n-dimensional complex projective space. A space X is *formal* if there is a quasi-isomorphism $M(X) \to (H^*(X;\mathbb{Q}),0)$. For example, S^n, $\mathbb{C}P^n$, Lie groups and their products are formal. It is known that $cup_0(X) = cat_0(X)$ when X is formal [2]. Recall the cup-length of a map

$$cup_0(f) := \max\{n \mid f^*(b_1\cdots b_n) \neq 0 \text{ for some } b_i \in H^+(Y;\mathbb{Q})\ \}$$

for a map $f : X \to Y$. It is known that $cup_0(f) \le cat_0(f)$ [2, p.43] and $cup_0(f) = cat_0(f)$ when f is a map between formal spaces X and Y.

Example 3.1. In general, it does not hold that $v_0(f) \le cup_0(f)$ though $v_0(f) \le cat_0(f)$. Let Y be a simply connected 11-dimensional manifold such that $M(Y) = (\Lambda(w_1,w_2,w_3),d)$ with $|w_1| = |w_2| = 3$, $|w_3| = 5$, $d(w_1) = d(w_2) = 0$, $d(w_3) = w_1w_2$. It is the pullback of the sphere bundle of the tangent bundle of S^6 by the canonical degree 1 map $S^3\times S^3 \to S^6$. It is not formal since $H^*(Y;\mathbb{Q})$ contains indecomsable elements $[w_1w_3]$ and $[w_2w_3]$. Then $cat_0(Y) = 3$ but $cup_0(Y) = 2$ since $[w_1][w_2w_3]$ is the fundamental class of Y. Consider a map $f : X = S^3\times S^5 \to Y$ with $f^* : M(Y) = (\Lambda(w_1,w_2,w_3),d) \to (\Lambda(w_1,w_3),0) = M(X)$ given by $f^*(w_1) = w_1$, $f^*(w_2) = 0$ and $f^*(w_3) = w_3$. Then $cup_0(f) = 1$. On the other hand, $v_0(f) = cat_0(f) = 2$ from Lemma 2.1(1) and (2).

Example 3.2. The following fibrations $\xi : X \xrightarrow{j} E \xrightarrow{p} Y$ satisfy the condition that $v_0(E) = v_0(j) + v_0(p)$. We can verify them by using Lemma 2.1. Of course, it holds if ξ is a trivial fibration.

(1) Let a fibration $S^3 \xrightarrow{j} S^2 \times S^{2n-1} \xrightarrow{p} \mathbb{C}P^n$ be given by the model

$$(\Lambda(x,y), d_Y) \to (\Lambda(x,y,v), D) \to (\Lambda(v), 0)$$

with $|x| = 2$, $|y| = 2n+1$, $|v| = 3$, $Dx = d_Y x = 0$, $Dv = x^2$ and $Dy = d_Y y = x^{n+1}$. Then the following diagram is DGA-commutative:

$$\begin{array}{ccccc} (\Lambda(x,y), d_Y) & \longrightarrow & (\Lambda(x,y,v), D) & \longrightarrow & (\Lambda(v), 0) \\ {\scriptstyle \mu_p}\downarrow & & {\scriptstyle \mu_E}\downarrow & & {\scriptstyle \mu_j}\downarrow \\ (\Lambda(y), 0) & \longrightarrow & (\Lambda(y,v), 0) & \longrightarrow & (\Lambda(v), 0). \end{array}$$

Thus $v_0(E) = 2 = 1 + 1 = v_0(p) + v_0(j)$.

(2) For the Hopf fibration $S^3 \xrightarrow{j} S^7 \xrightarrow{p} S^4$, the model is given by

$$(\Lambda(x,y), d_Y) \to (\Lambda(x,y,v), D) \to (\Lambda(v), 0)$$

with $|x| = 4$, $|y| = 7$, $|v| = 3$, $Dy = d_Y y = x^2$ and $Dv = x$. Notice $v_0(j) = 0$ since $M(S^7) = (\Lambda(y), 0) \simeq (\Lambda(x,y,v), D)$. Also the projectivization $P(E^n)$ of a non-trivial complex n-vector bundle E^n over S^{2n} is given as the total space of a fibration: $\mathbb{C}P^{n-1} \xrightarrow{j} \mathbb{C}P^{2n-1} \xrightarrow{p} S^{2n}$ [1]. The model is given by

$$(\Lambda(x,y), d_Y) \to (\Lambda(x,y,u,v), D) \to (\Lambda(u,v), d_X)$$

with $|x| = 2n$, $|y| = 4n-1$, $|u| = 2$, $|v| = 2n-1$, $d_Y y = x^2$, $Dv = u^n + x$ and $d_X v = u^n$. Then $v_0(j) = 0$ since $M(\mathbb{C}P^{2n-1}) = (\Lambda(u,y), d_E) \simeq (\Lambda(x,y,u,v), D)$ with $d_E u = 0$ and $d_E y = u^{2n}$.

(3) For an even interger m, let a fibration $S^{3m-1} \xrightarrow{j} E \xrightarrow{p} S^3_1 \times \cdots \times S^3_m$ be given as $M(E) = (\Lambda(w_1, .., w_m, v), D)$ with $|w_i| = 3$, $|v| = 3m-1$, $Dw_i = 0$ and $Dv = w_1 \cdots w_m$. Then the following diagram is DGA-commutative:

$$\begin{array}{ccccc} (\Lambda(w_1,..,w_m), 0) & \longrightarrow & (\Lambda(w_1,..,w_m,v), D) & \longrightarrow & (\Lambda(v), 0) \\ {\scriptstyle \mu_p}\downarrow & & {\scriptstyle \mu_E}\downarrow & & {\scriptstyle \mu_j}\downarrow \\ (\Lambda(w_2,..,w_m), 0) & \longrightarrow & (\Lambda(w_2,..,w_m,v), 0) & \longrightarrow & (\Lambda(v), 0). \end{array}$$

Thus $v_0(E) = m = (m-1) + 1 = v_0(p) + v_0(j)$.

(4) Let a fibration $S^6 \times S^9 \xrightarrow{j} E \xrightarrow{p} S^3 \times S^4$ be given by the model

$$(\Lambda(x,y,z), d_Y) \to (\Lambda(x,y,z,a,b,c), D) \to (\Lambda(a,b,c), d_X)$$

where $|x| = 4$, $|y| = 3$, $|z| = 7$, $|a| = 6$, $|b| = 9$, $|c| = 11$, $Dx = Dy = 0$, $Dz = x^2$, $Da = xy$, $Db = xa + yz$, $Dc = a^2 + 2yb$, $d_X a = d_X b = 0$ and $d_X c = a^2$. Then the following diagram is DGA-commutative:

$$\begin{array}{ccccc} (\Lambda(x,y,z), d_Y) & \longrightarrow & (\Lambda(x,y,z,a,b,c), D) & \longrightarrow & (\Lambda(a,b,c), d_X) \\ {\scriptstyle \mu_p}\downarrow & & {\scriptstyle \mu_E}\downarrow & & {\scriptstyle \mu_j}\downarrow \\ (\Lambda(z), 0) & \longrightarrow & (\Lambda(z,b,c), 0) & \longrightarrow & (\Lambda(b,c), 0). \end{array}$$

Thus $v_0(E) = 3 = 1+2 = v_0(p)+v_0(j)$.

The computation above is summarized as follows.

ξ	$v_0(E)$	$v_0(X)$	$v_0(Y)$	$v_0(j)$	$v_0(p)$
(1)	2	1	1	1	1
(2)	1	1	1	0	1
(3)	m	1	m	1	$m-1$
(4)	3	2	2	2	1

Remark 3.3. The total space E of Example 3.2 (4) is also one of a fibration $SU(6)/SU(3)\times SU(3) \to E \to S^3$, where the fiber is the (non-formal) homogeneous space of special unitary groups $SU(3)\times SU(3) \subset SU(6)$ (with blockwise inclusion). It is given as

$$(\Lambda(y),0) \to (\Lambda(x,y,z,a,b,c),D) \to (\Lambda(x,a,z,b,c),d_X)$$

with $d_Xx = d_Xa = 0$, $d_Xb = xy$ and $d_Xc = a^2$. Then we have $v_0(E) = 3 < 4 = 3+1 = v_0(j)+v_0(p) = v_0(X)+v_0(Y)$ since there is a DGA-map $\mu_j : (\Lambda(x,a,z,b,c),d_X) \to (\Lambda(z,b,c),0)$.

Problem 3.4. When $v_0(E) = v_0(j)+v_0(p)$?

Let A be a DGA $A = (A^*,d_A)$ with $A^* = \oplus_{i\geq 0}A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the argumentation $\epsilon : A \to \mathbb{Q}$. Define Der_iA the vector space of derivations of A decreasing the degree by $i > 0$, where $\theta(xy) = \theta(x)y + (-1)^{i|x|}x\theta(y)$ for $\theta \in Der_iA$. We denote $\oplus_{i>0}Der_iA$ by $DerA$. The boundary operator $\delta : Der_*A \to Der_{*-1}A$ is defined by $\delta(\sigma) = d_A\circ\sigma - (-1)^{|\sigma|}\sigma\circ d_A$. For the minimal model $M(Z) = (\Lambda V,d)$ of a finite complex Z and the argumentation $\epsilon : \Lambda V \to \mathbb{Q}$, according to [4],

$$G_n(Z)_{\mathbb{Q}} \cong \mathrm{Im}(H_n(\epsilon_*) : H_n(Der(\Lambda V,d)) \to \mathrm{Hom}(V^n,\mathbb{Q})).$$

Example 3.5. (1) For maps $f : X \to Y$ and $g : Y \to Z$, it does not hold that $g_0(g\circ f) \leq \min\{g_0(f), g_0(g)\}$ in general. Let $X = Z = S^3\times S^3\times S^3$ be given by the Sullivan model $M(X) = M(Z) = (\Lambda(w_1,w_2,w_3),0)$ with $|w_i| = 3$. Let Y be given by the Sullivan model $M(Y) = (\Lambda(w_1,w_2,w_3,u,v),d_Y)$ with $|u| = 3$, $|v| = 11$, $d_Yw_i = d_Yu = 0$ and $d_Yv = w_1w_2w_3u$. Then there are DGA-maps $M(f) : M(Y) \to M(X)$ and $M(g) : M(Z) \to M(Y)$ preserving w_i and $M(f)(u) = M(f)(v) = 0$. They induces $g_0(f) = 3$, $g_0(g) = 0$ and $g_0(g\circ f) = g_0(id_X) = \dim G_*(X)_{\mathbb{Q}} = 3$.

(2) For maps $f_i : X_i \to Y_i$ for $i = 1,2$, it does not hold that $g_0(f_1\vee f_2) = \max\{g_0(f_1), g_0(f_2)\}$ in general. Let f_i be the identity maps $id_{S^3} : S^3 \to S^3$ of $X_i = S^3 = Y_i$. Then $g_0(f_1) = g_0(f_2) = 1$ but $g_0(f_1\vee f_2) = 0$ since $G_*(S^3\vee S^3)_{\mathbb{Q}} = 0$ [11].

Example 3.6. It does not hold that $cat_0(E) \leq cat_0(j)+cat_0(p)$ in general. For example, for the fibration $\mathbb{C}P^{n-1} \to \mathbb{C}P^{2n-1} \to S^{2n}$ in Example 3.2(2), we have $cat_0(\mathbb{C}P^{2n-1}) = 2n-1$, $cat_0(j) = n-1$ and $cat_0(p) = 1$.

Example 3.7. Let a fibration $S^3_1\times\cdots\times S^3_n\times S^5 \xrightarrow{j} E \xrightarrow{p} S^3_1\times\cdots\times S^3_n$ be given by

$$(\Lambda(w_1,..,w_n),0) \to (\Lambda(w_1,..,w_n,v_1,..v_n,v),D) \to (\Lambda(v_1,..v_n,v),0)$$

with $|w_i| = |v_i| = 3$, $|v| = 5$, $Dv_i = 0$ and $Dv = w_1v_1 + \cdots + w_nv_n$. Then $v_0(j) = n+1$, $v_0(p) = n$ and $v_0(E) = n+1$. Also $g_0(j) = n+1$, $g_0(p) = 0$ and $\dim G_*(E)_{\mathbb{Q}} = 1$. Thus both $v_0(j) + v_0(p) - v_0(E)$ and $g_0(j) + g_0(p) - g_0(E)$ can be arbitrarily large. Note that $g_0(E) = g_0(j) + g_0(p)$ for the fibrations in Example 3.2 (1),(2),(3) but not (4) as

ξ	$g_0(E)$	$g_0(X)$	$g_0(Y)$	$g_0(j)$	$g_0(p)$
(1)	2	1	1	1	1
(2)	1	1	1	0	1
(3)	1	1	m	1	0
(4)	1	2	2	2	0

We see that $g_0(j) = 3$, $g_0(p) = 0$ and $g_0(E) = 1$ for the fibration of Ramark 3.3.

Problem 3.8. For all fibrations $X \xrightarrow{j} E \xrightarrow{p} Y$ of finite complexes, does it hold that $g_0(E) \leq g_0(j) + g_0(p)$?

Refer [15] for an estimate of $\dim G_*(E)_{\mathbb{Q}}$.

Example 3.9. (1) The integer $cat_0(f) - v_0(f)$ can be arbitarily large. For example, for the natural inclusion map $f : \mathbb{C}P^n \to \mathbb{C}P^{n+1}$, we have $v_0(f) = 0$ and $cat_0(f) = n$.

(2) The integer $v_0(f) - g_0(f)$ can be arbitarily large. For example, for the map $E \xrightarrow{p} S^3_1 \times \cdots \times S^3_m$ in Example 3.2 (3), we have $g_0(p) = 0$ and $v_0(p) = m - 1$.

4 Halperin conjecture

A space X is said to be elliptic when $\dim H^*(X;\mathbb{Q}) < \infty$ and $\dim \pi_*(X)_{\mathbb{Q}} < \infty$. An elliptic space X is said to be an F_0-space when $H^*(X;\mathbb{Q})$ is evenly graded, which is equivalent to be isomorphic to $\mathbb{Q}[x_1, .., x_n]/(f_1, .., f_n)$ for some $x_1, .., x_n$ and homogeneous polynomials $f_1, .., f_n \in \mathbb{Q}[x_1, .., x_n]$. Then $M(X) = (\Lambda(x_1, \cdots, x_n) \otimes \Lambda(y_1, \cdots, y_n), d)$ with $|x_i|$ even, $|y_i|$ odd, $dx_i = 0$ and $dy_i = f_i$. Halperin has conjectured that any fibration $\xi : X \xrightarrow{j} E \to B$ with X an F_0-space c-splits; i.e., $H^*(E;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \otimes H^*(B;\mathbb{Q})$ additively. It is equivalent to that ξ is totally non-cohomologous to zero (abbreviated TNCZ); i.e., $j^* : H^*(E;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is surjective. The Halperin conjecture is equivalent to requiring that any fibration $X \to E \to S^{odd}$ is rationally trivial [9, Theorem 2.2]. Here S^{odd} means S^{2n+1} for any $n > 0$.

Proposition 4.1. *For a fibration $X \xrightarrow{j} E \to S^{2n+1}$ with X an F_0-space given by*

$$(\Lambda w, 0) \to (\Lambda w \otimes \Lambda V, D) \to (\Lambda V, d) = (\Lambda(x_1, \cdots, x_n) \otimes \Lambda(y_1, \cdots, y_n), d),$$

it holds that $v_0(E) = n+1 = v_0(j) + 1 = v_0(X) + 1$ if and only if

$$Dy_i \in \Lambda(w) \otimes \Lambda^+(x_1, \cdots, x_n) \otimes \Lambda(y_1, \cdots, y_n) \quad (3)$$

for $i = 1, \cdots, n$.

Proof. It follows since $(\Lambda w \otimes \Lambda(y_1, \cdots, y_n), \overline{D})$ is DGA-isomorphic to $(\Lambda w \otimes \Lambda(y_1, \cdots, y_n), 0)$ only under the condition (3). □

Theorem 4.2. *For a fibration* $\xi : X \stackrel{j}{\to} E \to S^{2n+1}$ *over an odd-sphere,* $v_0(j) = v_0(X) \leq v_0(E) \leq v_0(X)+1$. *In particular, when* X *is an* F_0*-space,* $v_0(E) = v_0(X)+1$ *if Halperin conjecture is true.*

Proof. The former follows from Corollary 2.4 since ξ is weakly rational trivial [14]. The latter follows since ξ is rationally trivial [9, Theorem 2.2]. □

Remark 4.3. A comment that "We know $v_0(E) = n+1$" in [16, Remark 2.6] may be incorrect from Proposition 4.1. On the other hand, even if $v_0(E) = v_0(X)+1$ for any fibration $X \to E \to S^{odd}$, it does not indicate Halperin conjecture to be true, again from Proposition 4.1. Notice that, for any fibration $X \to E \to S^{odd}$, $g_0(E) = g_0(X)+1$ if and only if Halperin conjecture is true [15, Corollary A]. But $cat_0(E) = cat_0(X)+1$ for any F_0-space X [9, Theorem 4.7].

Finally, we propose a weak form of Halperin conjecture.

Problem 4.4. When X is an F_0-space, does it hold that $v_0(E) = v_0(X)+1$ for any fibration $X \to E \to S^{odd}$?

References

[1] R.Bott and L.Tu, *Differential forms in algebraic topology*, G.T.M. **82** Springer 1982

[2] O.Cornea, G.Lupton, J.Oprea and D.Tanré, *Lusternik-Schnirelmann category*, Math. Survey and Mono. **103**, A.M.S. 2003

[3] Y.Félix, *La dichotomie elliptique-hyperbolique en homotopie rationnelle*, Astérisque **176** S.M.F. 1989

[4] Y.Félix and S.Halperin, *Rational LS category and its applications,* Trans.A.M.S. **273** (1982) 1-38

[5] Y.Félix, S.Halperin and J.-C.Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics **205**, Springer-Verlag, 2001

[6] D.H.Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math., **91** (1969) 729-756

[7] P.Hilton, G.Mislin and J.Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies, **15** 1975

[8] B.Jessup and G.Lupton, *Free torus actions and two-stage spaces*, Math. Proc. Cambridge Philos. Soc. **137**(1) (2004) 191-207

[9] G.Lupton, *Variations on a conjecture of Halperin*, Homotopy and Geometry Banach Center Publ. **45** (1998) 115-135

[10] N.Oda, *Pairings and copairings in the category of topological spaces*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **28** (1992) 83-97

[11] S.B.Smith, *Rational evaluation subgroups*, Mat.Z.**221** (1996) 387-400

[12] D.Stanley, *On the Lusternik-Schnirelmann category of maps*, Cand. J. Math. **54** (2002) 608-633

[13] D.Sullivan, *Infinitesimal computations in topology*, I.H.E.S. **47** (1978) 269-331

[14] J.-C.Thomas, *Rational homotopy of Serre fibrations*, Ann. Inst. Fourier, Grenoble **31** (1981) 71-90

[15] T.Yamaguchi, *An estimate in Gottlieb ranks of fibration*, Bull. Belg. Math. Soc. **15** (2008) 663-675

[16] T.Yamaguchi, *Pairing rank in rational homotopy group*, African Diaspora J. of Math. **17** (2014) 85-92