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Pairing Rank in Rational Homotopy Group

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Abstract. Let X be a simply connected CW complex of finite rational LS-category. The dimension of rational Gottlieb group $G_*(X)\otimes\mathbb{Q}$ is upper-bounded by the rational LS-category $cat_0(X)$ [2]. Then we introduce a new rational homotopical invariant between them, denoted as the pairing rank $v_0(X)$ in the rational homotopy group $\pi_*(X)\otimes\mathbb{Q}$. If $\pi_*(f)\otimes\mathbb{Q}$ is injective for a map $f : X \to Y$, then we have $v_0(X) \leq v_0(Y)$. Also it has a good estimate for a fibration $X{\to}E{\to}Y$ as $v_0(E) \leq v_0(X)+v_0(Y)$.

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1 Introduction

In this paper, all spaces are connected and simply connected based CW complexes of finite rational LS-category [2] and maps are based unless otherwise noted. Let $G_n(X)$ be the n-th Gottlieb group (evaluation subgroup) of X, which consists of elements a of $\pi_n(X)$ with the homotopy commutative diagram:

$$\begin{array}{ccc} S^n\times X & \xrightarrow{\mu_a} & X \\ \cup\uparrow & & \| \\ S^n\vee X & \xrightarrow[\langle a,id_X\rangle]{} & X \end{array}$$

where $\langle a,id_X\rangle(x) = \nabla_X\circ(a\vee id_X)$. Here $\nabla_X : X\vee X\to X$ is the folding map of X. Let $G_*(X)$ be the total Gottlieb group $\oplus_{n>0}G_n(X)$. For any (homogeneous) elements $a_{i_1},\cdots,a_{i_n}$ of $G_*(X)$ with $\deg a_{i_k} = l_k$, there is a map $\mu_a : S^{l_1}\times\cdots\times S^{l_n}\to X$ such that $(*)$:

$$\begin{array}{ccc} S^{l_1}\times\cdots\times S^{l_n}\times X & \xrightarrow{\mu_a} & X \\ \cup\uparrow & & \| \\ S^{l_1}\vee\cdots\vee S^{l_n}\vee X & \xrightarrow[\langle a_{i_1},\cdots,a_{i_n},id_X\rangle]{} & X \end{array}$$

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since there is a composition of affiliated maps $\{1\times\cdots\times 1\times\mu\}$:

$$S^{l_1}\times\cdots\times S^{l_{n-1}}\times(S^{l_n}\times X)\to S^{l_1}\times\cdots\times S^{l_{n-2}}\times(S^{l_{n-1}}\times X)\to\cdots\to X.$$

As a general formula of $(*)$,

Definition 1.1. [9] We say that maps $f_i : X_i \to Y$ $(i = 1,..,n)$ have an n-pairing if there is a homotopy commutative diagram:

$$\begin{array}{ccc} X_1\times\cdots\times X_n & \xrightarrow{\mu} & Y \\ \cup\uparrow & & \| \\ X_1\vee\cdots\vee X_n & \xrightarrow[\langle f_1,\cdots,f_n\rangle]{} & Y \end{array}$$

with a map μ, which is called an affiliated map. Then we write $f_1 \perp f_2 \perp \cdots \perp f_n$.

Definition 1.2. Let the pairing rank $v_0(X)$ of X in the rational homotopy group be

$$\max\Big\{n \mid a_{i_1}\perp\cdots\perp a_{i_n}\ \ for\ \ \{a_{i_1},..,a_{i_n}\}\subset A \text{ with some basis } A \text{ of } \pi_{odd}(X)_{\mathbb{Q}}\Big\},$$

where A is a homogeneous basis of the graded vector space $\pi_{odd}(X)_{\mathbb{Q}} = \oplus_{k>0}\pi_{2k+1}(X)\otimes\mathbb{Q}$; i.e., $A = \cup_i A_i$ with A_i a basis of $\pi_i(X)_{\mathbb{Q}}$ (i is odd).

Let $X_{\mathbb{Q}}$ and $f_{\mathbb{Q}}$ be the rationalizations of a space X and a map $f : X \to Y$, respectively [5]. It is known that $G_*(X_{\mathbb{Q}}) = G_*(X)_{\mathbb{Q}}$ when X is finite [6] (in general, $G_*(X_{\mathbb{Q}}) \supset G_*(X)_{\mathbb{Q}}$) and $G_{even}(X)_{\mathbb{Q}} = 0$ [2, 6.12]. Y. Félix and S. Halperin [2, p.35] conjecture that $G_n(X)_{\mathbb{Q}} = 0$ for all $n \geq 2q$ if X is a complex of dimension q. When $G_*(X_{\mathbb{Q}}) = \mathbb{Q}\langle a_{i_1},\cdots,a_{i_n}\rangle$ with $\deg a_{i_k} = l_k$, there is the restriction map $\mu'_a : S^{l_1}\times\cdots\times S^{l_n} \to X_{\mathbb{Q}}$ of μ_a in $(*)$ such that

$$\begin{array}{ccc} S^{l_1}\times\cdots\times S^{l_n} & \xrightarrow{\mu'_a} & X_{\mathbb{Q}} \\ \cup\uparrow & & \| \\ S^{l_1}\vee\cdots\vee S^{l_n} & \xrightarrow[\langle a_{i_1},\cdots,a_{i_n}\rangle]{} & X_{\mathbb{Q}} \end{array}$$

homotopically commutes. Thus we have $\dim G_*(X_{\mathbb{Q}}) \leq v_0(X) \leq \dim \pi_*(X)_{\mathbb{Q}}$.

Let $cat(X)$ be the Lusternik-Schnirelmann (LS) category of X, which is the least integer n such that X is the union of $n+1$ open subsets contractible in X [1]. Let $cat_0X := cat(X_{\mathbb{Q}})$ be the rational LS-category of X. Then $cat_0X \leq catX$. It is known that $\dim G_*(X_{\mathbb{Q}}) \leq cat_0X$ [2, 6.12]. Recall Y. Félix and S. Halperins'

Mapping theorem [2, Theorem I]. If $\pi_*(f)\otimes\mathbb{Q}$ is injective for a map $f : X \to Y$, then $cat_0X \leq cat_0Y$.

We have a similar result about our pairing rank:

Proposition 1.3. *If $\pi_{odd}(f)\otimes\mathbb{Q}$ is injective for a map $f : X \to Y$, then $v_0(X) \leq v_0(Y)$.*

Proof. It is given by the homotopy commutative diagram:

$$\begin{array}{ccccc} S^{l_1}\times\cdots\times S^{l_n} & \longrightarrow & X_{\mathbb{Q}} & & \\ \cup\uparrow & & \| & & \\ S^{l_1}\vee\cdots\vee S^{l_n} & \xrightarrow[\langle a_{i_1},\cdots,a_{i_n}\rangle]{} & X_{\mathbb{Q}} & \xrightarrow[f_{\mathbb{Q}}]{} & Y_{\mathbb{Q}} \end{array}$$

when $v_0(X) = n$. It means $f_{\mathbb{Q}} \circ a_{i_1} \perp \cdots \perp f_{\mathbb{Q}} \circ a_{i_n}$ for the sub-basis $\{f_{\mathbb{Q}} \circ a_{i_1}, \cdots, f_{\mathbb{Q}} \circ a_{i_n}\}$ of $\pi_{odd}(Y)_{\mathbb{Q}}$. □

In particular, when $X_{\mathbb{Q}} \simeq Y_{\mathbb{Q}}$, we have $v_0(X) = v_0(Y)$.

Theorem 1.4. $v_0(X) \le cat_0(X)$.

Proof. When $v_0(X) = n$, the induced map of an affiliated map $\pi_*(\mu)_{\mathbb{Q}} : \pi_*(S^{i_1} \times \cdots \times S^{i_n})_{\mathbb{Q}} = \pi_{i_1}(S^{i_1})_{\mathbb{Q}} \oplus \cdots \oplus \pi_{i_n}(S^{i_n})_{\mathbb{Q}} \to \pi_*(X)_{\mathbb{Q}}$ is injective. Recall $cat_0(S^{i_1} \times \cdots \times S^{i_n}) = n$. From the above Mapping theorem, we have $n \le cat_0(X)$. □

Accordingly, we have the main inequalities:

$$\dim G_*(X)_{\mathbb{Q}} \le v_0(X) \le cat_0(X). \qquad (**)$$

If X is the product of spheres, $\dim G_*(X)_{\mathbb{Q}} = v_0(X) = cat_0(X)$. In Theorem 2.5, we give a relaxed condition in the terms of Sullivan models [11],[3].

Recall a (rationalized) result of Varadarajan and Hardie:

Theorem 1.5. *[3, Proposition 30.6] For a fibration $X \to E \to Y$, $cat_0 E$ is upper bounded by $cat_0 X$ and $cat_0 Y$ as*

$$cat_0 E + 1 \le (cat_0 X + 1)(cat_0 Y + 1)$$

and this inequality is best possible.

For example, the projectivization of a complex n-bundle over S^{2n} is given by a non-trivial fibration $\mathbb{C}P^{n-1} \to \mathbb{C}P^{2n-1} \to S^{2n}$, where $\mathbb{C}P^n$ is the n-dimensional complex projective space. It induces the equation $cat_0 E + 1 = 2n = n \cdot 2 = (cat_0 X + 1)(cat_0 Y + 1)$. Recall that K. Hess showed that $cat_0(X \times Y) = cat_0 X + cat_0 Y$ in 1991 [1]. There is a problem: when X is elliptic (see §2 for the definition), $\dim H^*(X;\mathbb{Q}) \le 2^{cat_0(X)}$? [12].

Claim 1.6. *(1) Gottlieb group is not functorial, that is, a map $f : X \to Y$ does not induce $G_*(X) \to G_*(Y)$ in general. Thus, even if $\pi_*(f)\otimes\mathbb{Q}$ is injective, it does not hold that* $\dim G_*(X)_{\mathbb{Q}} \le \dim G_*(Y)_{\mathbb{Q}}$. *For example, let $M(Y) = (\Lambda(v_1, v_2, v_3), d)$ with $|v_i|$ odd, $dv_1 = dv_2 = 0$, $dv_3 = v_1 v_2$ and $M(X) = (\Lambda(v_2, v_3), 0)$. When $M(f)$ is the projection removing v_1, $\pi_*(f) \otimes \mathbb{Q}$ is injective. But* $\dim G_*(X)_{\mathbb{Q}} = 2 > 1 = \dim G_*(Y)_{\mathbb{Q}}$.

(2) Although $\dim G_*(X \times Y)_{\mathbb{Q}} = \dim G_*(X)_{\mathbb{Q}} + \dim G_*(Y)_{\mathbb{Q}}$, *there is no good estimate of* $\dim G_*(E)_{\mathbb{Q}}$ *in terms of* $\dim G_*(X)_{\mathbb{Q}}$ *and* $\dim G_*(Y)_{\mathbb{Q}}$ *for a fibration $X \to E \to Y$. Indeed, they can be arbitrary ([14, Example 1]).*

Our pairing rank has a good evaluation inequality induced by an inclusion $\pi_{odd}(E)_{\mathbb{Q}} \subset \pi_{odd}(X)_{\mathbb{Q}} \oplus \pi_{odd}(Y)_{\mathbb{Q}}$ as

Theorem 1.7. *For a fibration $\xi : X \xrightarrow{i} E \xrightarrow{p} Y$,*
(1) $v_0(E) \le v_0(X) + v_0(Y)$.
(2) $v_0(X) \le v_0(E)$ if it is weakly rational trivial; i.e., $\pi_*(E)_{\mathbb{Q}} = \pi_*(X)_{\mathbb{Q}} \oplus \pi_*(Y)_{\mathbb{Q}}$.
(3) In particular, $v_0(X \times Y) = v_0(X) + v_0(Y)$.

In general, even if $v_0(E) = v_0(X) + v_0(Y)$, the fibration $\xi : X \to E \to Y$ may not be trivial (See Example 3.5(2)(3) in §3). In the future works, it is expected to find some relations between other numerical invariants as in [1], [4].

2 Sullivan model

Recall the *Sullivan minimal model* $M(X)$ of a simply connected space X of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(\Lambda V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i>1} V^i$ of $\dim V^i < \infty$ and a decomposable differential d. Denote the degree of a homogeneous element x of a graded algebra as $|x|$, the $\mathbb{Q}$-vector space of basis $\{v_i\}_i$ as $\mathbb{Q}\langle v_i\rangle_i$. A fibration $p : E \to Y$ has a minimal model which is a DGA-map $M(p) : M(Y) \to M(E)$. It is induced by a relative model (KS-extension)

$$M(Y) = (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D),$$

where $(\Lambda V, \overline{D}) = (\Lambda V, d_X)$ is the minimal model of the homotopy fibre X of p and there is a quasi-isomorphism $\rho : M(E) \xrightarrow{\sim} (\Lambda W \otimes \Lambda V, D)$. Notice that $M(X)$ determines the rational homotopy type of X, especially $H^*(X;\mathbb{Q}) \cong H^*(M(X))$ as graded algebras and $\pi_i(X)\otimes\mathbb{Q} \cong Hom(V^i, \mathbb{Q})$. We refer to [3] for a general introduction and the standard notations. The next lemma immediately follows:

Lemma 2.1. *The inequality $v_0(X) \geq n$ is given by an affiliated map*

$$\mu : S^{a_1} \times \cdots \times S^{a_n} \to X_{\mathbb{Q}}$$

where $|a_i|$ are odd if and onl if there is a subspace $\mathbb{Q}\langle v_1, \cdots v_n\rangle$ of V with $|v_i| = a_i$ for $M(X) = (\Lambda V, d)$ such that there is a DGA-map

$$M(\mu) : (\Lambda V, d) \to (\Lambda(v_1, \cdots, v_n), 0),$$

where $M(\mu)(v_i) = v_i$.

Proof of Theorem 1.7. (1) Suppose $\mu : S^{n_1}\times\cdots\times S^{n_b} \to E_{\mathbb{Q}}$ is an affiliated map. Then we can assume that it is $\mu : S^{n_1}\times\cdots\times S^{n_a}\times S^{n_{a+1}}\times\cdots\times S^{n_b} \to E_{\mathbb{Q}}$ such that $\alpha_i : S^{n_i} \to E_{\mathbb{Q}}$ is an element of $\pi_{n_i}(X)_{\mathbb{Q}}$ for $1 \leq i \leq a$ and $\beta_i : S^{n_i} \to E_{\mathbb{Q}}$ is an element of $\pi_{n_i}(Y)_{\mathbb{Q}}$ for $a+1 \leq i \leq b$. (The existence of such elements are guaranteed by the construction of Sullivan relative model as we see below.) Then there is a homotopy commutative diagram

$$\begin{tikzcd}
S^{n_1}\times\cdots\times S^{n_a} \arrow[r] \arrow[d, "\mu_\alpha"] & S^{n_1}\times\cdots\times S^{n_a}\times S^{n_{a+1}}\times\cdots\times S^{n_b} \arrow[r] \arrow[d, "\mu"] & S^{n_{a+1}}\times\cdots\times S^{n_b} \arrow[d, "\mu_\beta"] \\
X_{\mathbb{Q}} \arrow[r, "i"'] & E_{\mathbb{Q}} \arrow[r, "p"'] & Y_{\mathbb{Q}}
\end{tikzcd}$$

and it induces $\mu' = \mu_\alpha\times\mu_\beta : (S^{n_1}\times\cdots\times S^{n_a})\times(S^{n_{a+1}}\times\cdots\times S^{n_b}) \to X_{\mathbb{Q}}\times Y_{\mathbb{Q}}$. From Lemma 2.1, it is equivalent to the homotopy commutative diagram of DGAs:

$$\begin{tikzcd}
(\Lambda(v_1,..,v_a),0) & (\Lambda(v_1,..,v_a,w_{a+1},..,w_b),0) \arrow[l] & (\Lambda(w_{a+1},..,w_b),0) \arrow[l] \\
(\Lambda V, d_X) \arrow[u, "M(\mu_\alpha)"] \arrow[d, equal] & (\Lambda U, d_E) \arrow[l, "M(i)"'] \arrow[u, "M(\mu)"'] \arrow[d, "\rho"', "\sim"] & (\Lambda W, d_Y) \arrow[l, "M(p)"'] \arrow[u, "M(\mu_\alpha)"'] \arrow[d, equal] \\
(\Lambda V, d_X) & (\Lambda V\otimes\Lambda W, D) \arrow[l] & (\Lambda W, d_Y) \arrow[l]
\end{tikzcd}$$

where $(\Lambda V, d_X)$, $(\Lambda U, d_E)$ and $(\Lambda W, d_Y)$ are the Sullivan minimal models of X, E and Y with $U \subset V\oplus W$. It induces the DGA-map $M(\mu_\alpha)\otimes M(\mu_\beta) : (\Lambda V, d_X)\otimes(\Lambda W, d_Y) \to (\Lambda(v_1,..,v_a,w_{a+1},..,w_b),0)$.

(2) It follows from Proposition 1.3. In this case, $(\Lambda U, d_E)$ is identified to $(\Lambda V\otimes\Lambda W, D)$, it follows from the DGA-map $M(\mu_\alpha)\circ M(i)$ in the above diagram and (3) is obvious. □

Let A be a DGA $A = (A^*, d_A)$ with $A^* = \oplus_{i\geq 0} A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the augmentation $\epsilon : A \to \mathbb{Q}$. Define $Der_i A$ the vector space of derivations of A decreasing the degree by $i > 0$, where $\theta(xy) = \theta(x)y + (-1)^{i|x|} x\theta(y)$ for $\theta \in Der_i A$. We denote $\oplus_{i>0} Der_i A$ by $Der A$. The boundary operator $\delta : Der_* A \to Der_{*-1} A$ is defined by $\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$.

Proposition 2.2. *[2] For the minimal model $M(X) = (\Lambda V, d)$ of a simply connected finite complex X and the argumentation $\epsilon : \Lambda V \to \mathbb{Q}$,*

$$G_n(X_{\mathbb{Q}}) \cong \mathrm{Im}(H_n(\epsilon_*) : H_n(Der(\Lambda V, d)) \to Hom_n(V, \mathbb{Q}) = Hom(V^n, \mathbb{Q}))$$

for all $n > 0$.

A space X or a model $M(X) = (\Lambda V, d)$ is said to be *elliptic* if $\dim H^*(X; \mathbb{Q}) = H^*(\Lambda V, d) < \infty$ and $\dim \pi_*(X)_{\mathbb{Q}} = \dim V < \infty$. When X is elliptic, $cat_0(X) = e_0(\Lambda V, d) := \max\{n \mid [\alpha] \neq 0 \in H^+(\Lambda V, d)$ for $\alpha \in \Lambda^{\geq n} V\}$ [1]. A model $(\Lambda V, d)$ is called *pure* when $dV^{even} = 0$ and $dV^{odd} \subset \Lambda V^{even}$.

Lemma 2.3. *For a pure minimal model $M = (\Lambda(x_1, .., x_m, y_1, .., y_n), d)$ with $|x_i|$ even and $|y_i|$ odd, we have $v_0(M) = n$.*

Proof. The model of an affiliated map is given by the DGA-projection $M(\mu) : M \to (\Lambda(y_1, .., y_n), 0)$ from Lemma 2.1. □

L. Lechuga and A. Murillo give

Theorem 2.4. *[7, Theorem 1] For an elliptic model with $M(X) = (\Lambda V, d)$ with $dV \subset \Lambda^{\geq k} V$, $cat_0(X) = (k-2)\dim V^{even} + \dim V^{odd}$.*

When $dV \subset \Lambda^2 V$ in $(\Lambda V, d)$, we say that $(\Lambda V, d)$ is quadratic.

Theorem 2.5. *If $M(X)$ is a pure elliptic quadratic model, then $\dim G_*(X)_{\mathbb{Q}} = v_0(X) = cat_0(X)$.*

Proof. In this case, $G_*(X)_{\mathbb{Q}} = V^{odd}$ from Proposition 2.2 and $v_0(X) = \dim V^{odd}$ from Lemma 2.3. It is also equal to $cat_0(X) = \dim V^{odd}$ from Theorem 2.4. □

Remark 2.6. Suppose that the minimal model of X is given by $M(X) = (\Lambda(x_1, .., x_n, y_1, .., y_n), d)$ with $|x_i|$ even, $|y_i|$ odd, $dx_i = 0$ and $dy_i \in \Lambda(x_1, .., x_n)$ for all i. When its cohomology is finite, X is called as an F_0-space. For a fibration $\xi : X \to E \to S^{2k+1}$, $cat_0(E) = cat_0(X) + 1$ [8, Theorem 4.7]. Also $\dim G_*(E)_{\mathbb{Q}} = n + 1$ if and only if ξ is rationally trivial [14, Corollary A]. There is an open problem that ξ is rationally trivial if $cup_0(E) = cup_0(X) + 1$ [8]. We know $v_0(E) = n + 1$ since $Dx_i \in (x_1, .., x_n)$ for all i in the KS-extension

$$M(S^{2k+1}) = (\Lambda z, 0) \to (\Lambda(z, x_1, .., x_n, y_1, .., y_n), D) \to M(X).$$

Here $(x_1, .., x_n)$ is the ideal generated by $x_1, .., x_n$. Indeed, then there is the DGA-projection map $M(X) \to (\Lambda(z, y_1, .., y_n), 0)$ and then we have it from Lemma 2.1.

3 Examples

Example 3.1. Let $cup_0(X)$ be the rational cup length of X, the largest integer n such that the n-product of $H^+(X; \mathbb{Q})$ is not zero. The following examples are useful for Theorem 3.3 below.

(1) $v_0(X) = 0$ if and only if $X \simeq_{\mathbb{Q}} *$.

(2) $\dim G_*(S^n)_{\mathbb{Q}} = v_0(S^n) = \dim G_*(\mathbb{C}P^n)_{\mathbb{Q}} = v_0(\mathbb{C}P^n) = 1$ but $cup_0(\mathbb{C}P^n) = cat_0(\mathbb{C}P^n) = n$.

(3) $\dim G_*(S^m \vee S^n)_{\mathbb{Q}} = 0$ [10] but $v_0(S^m \vee S^n) = cat_0(S^m \vee S^n) = 1$.

Example 3.2. Recall Theorem 2.5. Even if $M(X)$ is a quadratic model, $v_0(X)$ may not be equal to $cat_0(X)$. For example, let $M(X) = (\Lambda(x,y,z,a,b,c),d)$ with $|x| = 2$, $|y| = |z| = 3$, $|a| = 4$, $|b| = 5$, $|c| = 7$, $dx = dy = 0$, $dz = x^2$, $da = xy$, $db = xa + yz$ and $dc = a^2 + 2yb$, which is an elliptic model [3, p.439]. Then $v_0(X) = 3$ by the affiliated map $\mu : S^3 \times S^5 \times S^7 \to X_{\mathbb{Q}}$. It is given from Lemma 2.1 by the DGA-restriction map

$$(\Lambda(x,y,z,a,b,c),d) \to (\Lambda(z,b,c),0)$$

and since we can directly check $v_0(X) \neq 4$. On the other hand, $\dim G_*(X)_{\mathbb{Q}} = 1$ from Proposition 2.2 and $cat_0(X) = 4$ from Theorem 2.4.

A space X is said to be *formal* if there is a DGA-map from its minimal model to its rational cohomology with zero differential: $M(X) \xrightarrow{\sim} (H^*(X;\mathbb{Q}),0)$. For example, homogeneous spaces G/H with $rank(G) = rank(H)$ are formal.

Theorem 3.3. *Any triple* (a,b,c) *of* $0 < a \leq b \leq c$ *is realized as* $[X] := (\dim G_*(X)_{\mathbb{Q}}, v_0(X), cat_0X)$ *for a formal space* X.

Proof Notice that $[X \times Y] = [X] + [Y]$. For any triple (a,b,c) of $0 < a \leq b \leq c$, we have

$$(a,b,c) - [S^3_1 \times S^3_2 \times \cdots \times S^3_{a-1}] = (1, b-a+1, c-a+1),$$

$$(1, b-a+1, c-a+1) - [\Pi_{i=1}^{b-a}(S^3 \vee S^3)_i] = (1,1,c-b+1) \quad \text{and}$$

$$(1,1,c-b+1) = [\mathbb{C}P^{c-b+1}]$$

from the above example. Thus we have $[X] = (a,b,c)$ when

$$X \ = \ S^3_1 \times S^3_2 \times \cdots \times S^3_{a-1} \times \ \Pi_{i=1}^{b-a}(S^3 \vee S^3)_i \ \times \ \mathbb{C}P^{c-b+1},$$

for example. □

Example 3.4. Recall $cup_0(X) \leq cat_0(X)$ in general and the integer $cat_0(X) - cup_0(X)$ can be arbitrarily large for elliptic spaces [13]. If X is formal, it is known that $cup_0(X) = cat_0(X)$ [3]. Then we have $v_0(X) \leq cup_0(X)$ from Theorem 1.4. Consider non-formal cases:

(1) When X is the non-formal homogeneous space $SU(6)/SU(3)\times SU(3)$, $M(X) = (\Lambda(x,y,v_1,v_2,v_3),d) = (\Lambda V,d)$ with $|x| = 4$ and $|y| = 6$, $dx = dy = 0$, $dv_1 = x^2$, $dv_2 = xy$ and $dv_3 = y^2$. It satisfies the condition of Theorem 2.5. Then $[x]\cdot[yv_2 - xv_3]$ represents the fundamental class of $H^*(X;\mathbb{Q})$ and it is in $\Lambda^3 V$. Thus we have the inequality:

$$cup_0(X) = 2 < 3 = \dim G_*(X)_{\mathbb{Q}} = v_0(X) = cat_0(X).$$

(2) When $M(X) = (\Lambda(v_1,v_2,..,v_n),d)$ with $n > 4$ odd, $|v_i|$ odd and $dv_1 = dv_2 = 0$, $dv_3 = v_1v_2$, $dv_4 = v_1v_3, \cdots, dv_n = v_1v_{n-1}$, then $v_0(X) = n-1$ since there is the restriction map $M(X) \to (\Lambda(v_2,v_3,..,v_n),0)$. We see $cup_0(X) = (n+1)/2$ since there are cocycles v_1 and $v_2v_n - v_3v_{n-1} + \cdots + (-1)^{(n+1)/2}v_{(n+1)/2}v_{(n+3)/2}$ where

$$[v_1]\cdot[v_2v_n - v_3v_{n-1} + \cdots + (-1)^{(n+1)/2}v_{(n+1)/2}v_{(n+3)/2}]^{\frac{n-1}{2}} = c[v_1v_2\cdots v_n].$$

for a certain non-zero integer c. From Proposition 2.2, $\dim G_*(X)_{\mathbb{Q}} = \dim \mathbb{Q}\langle v_n\rangle = 1$. Also $cat_0(X) = e_0(X) = n$. It gives the inequalities:

$$\dim G_*(X)_{\mathbb{Q}} < cup_0(X) < v_0(X) < cat_0(X).$$

(3) Let X be the space of the above (2). From Example 3.1(2), we have the inequalities:

$$\dim G_*(X\times\mathbb{C}P^n)_{\mathbb{Q}} < v_0(X\times\mathbb{C}P^n) < cup_0(X\times\mathbb{C}P^n) < cat_0(X\times\mathbb{C}P^n)$$

for a sufficiently large n.

Example 3.5. (1) The space of Example 3.2 is the total space of a fibration $S^4\times S^5\to X\to S^2\times S^3$. Then $\dim G(X)_{\mathbb{Q}} = 1 < 2+2 = \dim G(S^4\times S^5)_{\mathbb{Q}} + \dim G(S^2\times S^3)_{\mathbb{Q}}$, $cat_0X+1 = 4+1 < 3\cdot 3 = (cat_0(S^4\times S^5)+1)(cat_0(S^2\times S^3)+1)$ in the fomula of Theorem 1.5 and $v_0(X) = 3 < 2+2 = v_0(S^4\times S^5)+v_0(S^2\times S^3)$ in the fomula of Theorem 1.7(1).

(2) The space of Example 3.4(1) is the total space of the fibration $S^9\to X\to S^4\times S^6$. It gives an example with $\dim G(X)_{\mathbb{Q}} = \dim G(S^9)_{\mathbb{Q}} + \dim G(S^4\times S^6)_{\mathbb{Q}}$ and

$$v_0(X) = 3 = 1+2 = v_0(S^9)+v_0(S^4\times S^6)$$

but $cat_0X+1 = 3+1 < 6 = 2\cdot 3 = (cat_0S^9+1)(cat_0(S^4\times S^6)+1)$ in the fomula of Theorem 1.5.

(3) Put $S^{4n-1}\to T\to S^{4n}$ the sphere bundle associated to the tangent bundle of S^{4n} where n is odd. Put the pull back fibration $S^{4n-1}\to Y\xrightarrow{f} S^n_1\times S^n_2\times S^n_3\times S^n_4$ along the map $S^n_1\times S^n_2\times S^n_3\times S^n_4\to S^{4n}$ collapsing the $(4n-1)$-skelton. Then Y is an $8n-1$-dimensional manifold with $M(Y) = (\Lambda(w_1,w_2,w_3,w_4,w),d_Y)$ with $|w_i| = n$, $|w| = 4n-1$, $d_Yw_i = 0$, $d_Yw = w_1w_2w_3w_4$. Then for the basis $A = \{w_1^*,w_2^*,w_3^*,w_4^*,w^*\}$ of $\pi_*(Y)\otimes\mathbb{Q}$, we see $v_0(Y) = 4$ by $S^n_{i_1}\times S^n_{i_2}\times S^n_{i_3}\times S^{4n-1}\to Y_{\mathbb{Q}}$ for $1\le i_1<i_2<i_3\le 4$. Consider the spherical fibration $S^{2n-1}\to E\to Y$ where $M(E) = (\Lambda(w_1,w_2,w_3,w_4,w,v),D)$ with $|v| = 2n-1$ and $Dw_1 = Dw_2 = Dw_3 = Dw_4 = 0$, $Dw = d_Yw$ and $Dv = w_1w_2$. Then there is a DGA-projection $M(E)\to(\Lambda(w_{i_1},w_{i_2},w_{i_3},w,v),0)$. Thus we have the equalities:

$$v_0(E) = 5 = 1+4 = v_0(S^{2n-1})+v_0(Y).$$

On the other hand, we have $G_*(E)_{\mathbb{Q}} = \mathbb{Q}\langle w_3,w_4,w,v\rangle$ and $G_*(Y)_{\mathbb{Q}} = \mathbb{Q}\langle w\rangle$ from Proposition 2.2. Thus $\dim G_*(E)_{\mathbb{Q}} = 4 > 1+1 = \dim G_*(S^{2n-1})_{\mathbb{Q}} + \dim G_*(Y)_{\mathbb{Q}}$ and $cat_0E+1 = 6+1 < 12 = 2\cdot 6 = (cat_0S^{2n-1}+1)(cat_0Y+1)$ in the fomula of Theorem 1.5.

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