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EXISTENCE AND ATTRACTIVITY RESULTS FOR SOME FRACTIONAL ORDER PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY

SAÏD ABBAS*
Laboratory of Mathematics, University of Saïda
PO Box 138, 20000 Saïda, Algeria

MOUFFAK BENCHOHRA†
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie

TOKA DIAGANA‡
Department of Mathematics, Howard University
2441 6th Street NW
Washington, DC 20059, USA

Abstract

In this paper we study some existence, uniqueness, estimates and global asymptotic stability results for some functional integro-differential equations of fractional order with finite delay. To achieve our goals we make extensive use of some fixed point theorems as well as the so-called Pachpatte techniques.

AMS Subject Classification: 26A33; 45G05.

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1 Introduction

Fractional calculus is a generalization of the classical ordinary differentiation and integration of an arbitrary non-integer order. The subject is as old as differential calculus. This topic, from some speculations of G.W. Leibniz (1697) and L. Euler (1730) up to nowadays, has been progressing.

*E-mail address: abbasmsaid@yahoo.fr
†E-mail address: benchohra@yahoo.com
‡E-mail address: tdiagana@howard.edu

Fractional differential and integral equations have recently been applied to various areas of engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport and the kinetic theory of gases and others [6, 8, 9, 10, 12, 13]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see, e.g., the following monographs by Abbas *et al.* [5], Baleanu *et al.* [7], Diethelm [11], Kilbas *et al.* [14], Miller and Ross [15], Podlubny [17], Samko *et al.* [18].

Recently, some existence and attractivity results to various classes of integral equations of two variables have been obtained by Abbas *et al.* [2, 3, 4].

In [16], Pachpatte proved some results concerning the existence, uniqueness and other properties of solutions to certain Volterra integral and integro-differential equations in two variables. The tools utilized in the analysis are based upon the applications of the Banach fixed point theorem coupled with the so-called Bielecki type norm and certain integral inequalities with explicit estimates.

In this paper, by means of integral inequalities and fixed point approach, we improve some of the above-mentioned results and study the global attractivity of solutions for the system of partial integro-differential equations of fractional order of the form

$$^{c}D^{r}_{\theta}u(t,x) = f(t,x,u_{(t,x)},(Gu)(t,x)); \quad \text{for } (t,x) \in J := \mathbb{R}_+ \times [0,b], \tag{1}$$

$$u(t,x) = \phi(t,x); \text{ if } (t,x) \in \tilde{J} := [-\alpha,\infty)\times[-\beta,b]\backslash(0,\infty)\times(0,b], \tag{2}$$

$$\begin{cases} u(t,0) = \varphi(t); \ t \in \mathbb{R}_+, \\ u(0,x) = \psi(x); \ x \in [0,b], \end{cases} \tag{3}$$

where

$$(Gu)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} g(t,x,s,y,u_{(s,y)})dyds, \tag{4}$$

$\alpha,\beta,b > 0$, $\theta = (0,0)$, $r = (r_1,r_2) \in (0,1]\times(0,1]$, $\mathbb{R}_+ = [0,\infty)$, I^r_θ is the left-sided mixed Riemann-Liouville integral of order r, $^cD^r_\theta$ is the standard Caputo's fractional derivative of order r, $f : J\times C \to \mathbb{R}$, $g : J_1 \times C \to \mathbb{R}$ are given continuous functions, $J_1 := \{(t,x,s,y) : 0 \le s \le t < \infty,\ 0 \le y \le x \le b]\}$, $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\psi : [0,b] \to \mathbb{R}$ are absolutely continuous functions with $\lim_{t\to\infty}\varphi(t) = 0$, and $\psi(x) = \varphi(0)$ for each $x \in [0,b]$, $\Phi : \tilde{J} \to \mathbb{R}$ is continuous with $\varphi(t) = \Phi(t,0)$ for each $t \in \mathbb{R}_+$, and $\psi(x) = \Phi(0,x)$ for each $x \in [0,b]$, $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0,$$

and $C := C([-\alpha,0]\times[-\beta,0])$ is the space of continuous functions on $[-\alpha,0]\times[-\beta,0]$ with the standard norm

$$\|u\|_C = \sup_{(t,x)\in[-\alpha,0]\times[-\beta,0]} |u(t,x)|.$$

If $u \in C := C([-\alpha,\infty)\times[-\beta,b])$, then for any $(t,x) \in J$ define $u_{(t,x)}$ by

$$u_{(t,x)}(\tau,\xi) = u(t+\tau,x+\xi);\ \text{for } (\tau,\xi) \in [-\alpha,0]\times[-\beta,0].$$

We present our results for Eqs. (1)-(3) in the Banach space of bounded continuous functions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $L^1([0,a]\times[0,b])$; $a,b>0$ be the space of Lebesgue-integrable functions $u:[0,a]\times[0,b]\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^a\int_0^b|u(t,x)|dxdt.$$

As usual, by $C:=C(J)$ we denote the space of all continuous functions from J into $\mathbb{R}$.

By $BC:=BC([-\alpha,\infty)\times[-\beta,b])$ we denote the Banach space of all bounded and continuous functions from $[-\alpha,\infty)\times[-\beta,b]$ into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC}=\sup_{(t,x)\in[-\alpha,\infty)\times[-\beta,b]}|u(t,x)|.$$

For $u_0\in BC$ and $\eta\in(0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1. *[19] Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1([0,a]\times[0,b])$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-y)^{r_2-1}u(s,y)dyds.$$

In particular,

$$(I_\theta^\theta u)(t,x)=u(t,x),\ (I_\theta^\sigma u)(t,x)=\int_0^t\int_0^x u(s,y)dyds;\ \text{for almost all } (t,x)\in[0,a]\times[0,b],$$

where $\sigma=(1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1,r_2>0$, when $u\in L^1([0,a]\times[0,b])$. Moreover

$$(I_\theta^r u)(t,0)=(I_\theta^r u)(0,x)=0;\ t\in[0,a],\ x\in[0,b].$$

Example 2.2. *Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then*

$$I_\theta^r t^\lambda x^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2},\ \textit{for almost all } (t,x)\in[0,a]\times[0,b].$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D_{tx}^2:=\frac{\partial^2}{\partial t\partial x}$, the mixed second order partial derivative.

Definition 2.3. *[19] Let $r\in(0,1]\times(0,1]$ and $u\in L^1([0,a]\times[0,b])$. The Caputo fractional-order derivative of order r of u is defined by the expression*

$$^cD_\theta^r u(t,x)=(I_\theta^{1-r}D_{tx}^2 u)(t,x)=\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\int_0^t\int_0^x\frac{(D_{sy}^2 u)(s,y)}{(t-s)^{r_1}(x-y)^{r_2}}dyds.$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(t,x) = (D_{xy}^2 u)(t,x), \text{ for almost all } (t,x) \in [0,a]\times[0,b].$$

Example 2.4. *Let $\lambda,\omega \in (-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then*

$${}^cD_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1}x^{\omega-r_2}, \textit{ for almost all } (t,x)\in[0,a]\times[0,b].$$

In the sequel, we need the following lemma

Lemma 2.5. *[1] Let $f \in L^1([0,a]\times[0,b])$. A function $u \in AC([0,a]\times[0,b])$ is a solution of problem*

$$\begin{cases} ({}^cD_\theta^r u)(t,x) = f(t,x);\ (t,x)\in[0,a]\times[0,b],\\ u(t,0)=\varphi(t);\ t\in[0,a],\ u(0,x)=\psi(x);\ x\in[0,b],\\ \varphi(0)=\psi(0), \end{cases}$$

if and only if u satisfies

$$u(t,x) = \mu(t,x) + (I_\theta^r f)(t,x);\ (t,x)\in[0,a]\times[0,b],$$

where

$$\mu(t,x) = \varphi(t)+\psi(x)-\varphi(0).$$

Denote by $D_1 := \frac{\partial}{\partial t}$, the partial derivative of a function defined on J (or J_1) with respect to the first variable, $D_2 := \frac{\partial}{\partial x}$, $D_2D_1 := \frac{\partial^2}{\partial t\partial x}$. In the sequel we will make use of the following Lemma due to Pachpatte.

Lemma 2.6. *[16] Let $u,e,p \in C(J)$, $k, D_1k, D_2k, D_2D_1k \in C(J_1)$ be positive functions. If $e(t,x)$ is nondecreasing in each variable $(t,x)\in J$ and*

$$u(t,x) \le e(t,x) + \int_0^t\int_0^x p(s,y)$$
$$\times\left[u(s,y)+\int_0^s\int_0^y k(s,y,\tau,\xi)u(\tau,\xi)d\xi d\tau\right]dyds;\ (t,x)\in J, \tag{5}$$

then,

$$u(t,x) \le e(t,x)\left[1+\int_0^t\int_0^x p(s,y)\exp\left(\int_0^s\int_0^y [p(\tau,\xi)+A(\tau,\xi)]d\xi d\tau\right)dyds\right];\ (t,x)\in J, \tag{6}$$

where

$$A(t,x) = k(t,x,s,y)+\int_0^t D_1k(t,x,s,y)ds + \int_0^x D_2k(t,x,s,y)dy$$
$$+\int_0^t\int_0^x D_2D_1k(t,x,s,y)dyds;\ (t,x)\in J. \tag{7}$$

Let G be an operator from $\emptyset \neq \Omega \subset BC$ into itself and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{8}$$

Now we review the concept of attractivity of solutions for equation Eq. (8). For $u_0 \in BC$ and $\eta \in (0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.7. *[4] Solutions of Eq. (8) are locally attractive if there exist a ball $B(u_0,\eta)$ in the space BC such that for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of Eq. (8) belonging to $B(u_0,\eta) \cap \Omega$ we have that, for each $x \in [0,b]$,*

$$\lim_{t\to\infty}(v(t,x) - w(t,x)) = 0. \tag{9}$$

When the limit Eq. (9) is uniform with respect to $B(u_0,\eta)$, solutions of Eq. (8) are said to be locally attractive (or equivalently that solutions of Eq. (8) are asymptotically stable).

Definition 2.8. *[4] The solution $v = v(t,x)$ of equation Eq. (8) is said to be globally attractive if Eq. (9) hold for each solution $w = w(t,x)$ of Eq. (8). If condition Eq. (9) is satisfied uniformly with respect to the set Ω, solutions of Eq. (8) are said to be globally asymptotically stable (or uniformly globally attractive).*

3 Main Results

Let us start by defining what we mean by a solution to the system Eqs. (1)-(3).

Definition 3.1. *A function $u \in BC$ with its mixed derivative D^2_{tx} exists and is integrable is said to be a solution of the system Eqs. (1)-(3) if u satisfies equations (1) and (3) on J and the condition Eq. (2) on $\tilde{J}$.*

3.1 Existence and Uniqueness

Our first result is about the existence and uniqueness of a solution to Eqs. (1)-(3).

Theorem 3.2. *Assume that following assumptions hold,*

(H_1) *The function φ is continuous and bounded with*

$$\varphi^* = \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} |\varphi(t,x)|;$$

(H_2) *There exist positive functions $p_1, p_2 \in BC(J)$ such that*

$$|f(t,x,u_1,u_2) - f(t,x,v_1,v_2)| \leq p_1(t,x)\|u_1 - v_1\|_C + p_2(t,x)|u_2 - v_2|,$$

for each $(t,x) \in J$, $u_1, v_1 \in C$ and $u_2, v_2 \in \mathbb{R}$. Moreover, assume that the function $t \to \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} f(s,y,0,(G0)(s,y))dyds$ is bounded on J with

$$f^* = \sup_{(t,x)\in J} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,0,(G0)(s,y))|dyds;$$

(H_3) *There exists a positive function $q \in BC(J_1)$ such that*

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \leq q(t,x,s,y)|u-v|,$$

for each $(t,x,s,y) \in J_1$ and $u, v \in \mathbb{R}$.

If

$$p_1^* + p_2^* q^* < 1, \tag{10}$$

where

$$p_i^* = \sup_{(t,x)\in J}\Big[\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_i(s,y)dyds\Big];\ i = 1,2,$$

and

$$q^* = \sup_{(t,x)\in J}\Big[\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}q(t,x,s,y)dyds\Big],$$

then the system (1)-(3) has a unique solution on $[-\alpha,\infty)\times[-\beta,b]$.

Proof. Let us define the operator $N : BC \to BC$ by

$$(Nu)(t,x) = \begin{cases} \Phi(t,x), & (t,x) \in \tilde{J}, \\ \varphi(t) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s,y,u_{(s,y)},(Gu)(s,y))dyds, & (t,x) \in J. \end{cases} \tag{11}$$

It is clear that the function $(t,x) \mapsto (Nu)(t,x)$ is continuous on $[-\alpha,\infty)\times[-\beta,b]$. Now we prove that $N(u) \in BC$ for any $u \in BC$. For each $(t,x) \in \tilde{J}$ we have

$$|\Phi(t,x)| \leq \sup_{(t,x)\in\tilde{J}} |\Phi(t,x)| := \Phi^*,$$

then $\Phi \in BC$. From (H_2), and for arbitrarily fixed $(t,x) \in J$ we have

$$\begin{aligned}
|(Nu)(t,x)| &= \Big|\varphi(t) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad\times f(s,y,u_{(s,y)},(Gu)(s,y))dyds\Big| \\
&\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad\times |f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,0,(G0)(s,y))|dyds \\
&\quad+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,0,(G0)(s,y))|dyds \\
&\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad\times\Big(p_1(s,y)|u_{(s,y)}| + p_2(s,y)|(Gu)(s,y)|\Big)dyds \\
&\quad+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,0,(G0)(s,y))|dyds
\end{aligned}$$

$$\leq \varphi^* + f^* + p_1^*\|u\|_{BC} + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}$$
$$\times p_2(s,y)|(Gu)(s,y) - (G0)(s,y)|dyds. \quad (12)$$

But, (H_3) implies that

$$\begin{aligned}
|(Gu)(t,x) - (G0)(t,x)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad \times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}q(t,x,s,y)|u(s,y)|dyds \\
&\leq q^*\|u\|_{BC}.
\end{aligned}$$

Thus, by (12) we get

$$\begin{aligned}
|(Nu)(t,x)| &\leq \varphi^* + f^* + p_1^*\|u\|_{BC} \\
&\quad + \frac{q^*\|u\|_{BC}}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_2(s,y)dyds \\
&\leq \varphi^* + f^* + p_1^*\|u\|_{BC} + p_2^*q^*\|u\|_{BC} \\
&\leq \varphi^* + f^* + (p_1^* + p_2^*q^*)\|u\|_{BC}.
\end{aligned}$$

Hence $N(u) \in BC$. Let $u,\ v \in BC$. Using the hypotheses, for each $(t,x) \in J$, we have

$$\begin{aligned}
&|(Nu)(t,x) - (Nv)(t,x)| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times |f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,v_{(s,y)},(Gv)(s,y))|dyds \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times (p_1(s,y)\|u_{(s,y)} - v_{(s,y)}\|_C + p_2(s,y)|(Gu)(s,y) - (Gv)(s,y)|)dyds \\
&\leq \frac{\|u-v\|_{BC}}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_1(s,y)dyds \\
&+ \frac{\|u-v\|_{BC}}{\Gamma^2(r_1)\Gamma^2(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times p_2(s,y)\left(\int_0^s\int_0^y (s-\tau)^{r_1-1}(y-\xi)^{r_2-1}q(s,t,\tau,\xi)d\xi d\tau\right)dyds \\
&\leq (p_1^* + p_2^*q^*)\|u-v\|_{BC}.
\end{aligned}$$

From (10), it follows from the Banach contraction principle that N has a unique fixed point in BC which is a solution to Eqs. (1)-(3).

3.2 Estimates on the Solutions

Now, we shall prove the following theorem concerning the estimate on the solution to Eqs. (1)-(3).

Theorem 3.3. *Set*

$$d = \varphi^* + f^*. \tag{13}$$

Assume that $(H_1)-(H_3)$ *and the following hypotheses hold*

(H_4) $p_1 = p_2$ *and there exists a positive function* $p \in BC(J)$ *such that,*

$$p_1(s,y) \le \Gamma(r_1)\Gamma(r_2)(t-s)^{1-r_1}(x-y)^{1-r_2}p(s,y), \text{ for each } (t,x,s,y) \in J_1,$$

(H_5) $k, D_1k, D_2k, D_2D_1k \in BC(J_1)$, *where*

$$k(t,x,s,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}(t-s)^{r_1-1}(x-y)^{r_2-1}q(t,x,s,y).$$

If u is any solution to Eqs. (1)-(3) on $[-\alpha,\infty)\times[-\beta,b]$, *then for each* $(t,x) \in J$,

$$|u(t,x)| \le d\left[1 + \int_0^t\int_0^x p(s,y)\exp\left(\int_0^s\int_0^y [p(\tau,\xi)+A(\tau,\xi)]d\xi d\tau\right)dyds\right], \tag{14}$$

where $A(t,x)$ *is defined by Eq. (7).*

Proof. Using the fact that u is a solution to Eqs. (1)-(3) and from hypotheses, we have for each $(t,x) \in J$,

$$\begin{aligned}
|u(t,x)| &\le |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(t,x,0,(G0)(t,x))|dyds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times |f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,0,(G0)(s,y))|dyds \\
&\le \varphi^* + f^* + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_1(s,y)\Big[\|u_{(s,y)}\|_C \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^s\int_0^y (s-\tau)^{r_1-1}(y-\xi)^{r_2-1}q(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\Big]dyds \\
&\le d + \int_0^t\int_0^x p(s,y)\Big[\|u_{(s,y)}\|_C \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^s\int_0^y q(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\Big]dyds \\
&\le d + \int_0^t\int_0^x p(s,y)\left[\|u_{(s,y)}\|_C + \int_0^s\int_0^y k(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\right]dyds.
\end{aligned}$$

We consider the function w defined by

$$w(t,x) = \sup\{\|u(s,y)\| : -\alpha \le s \le t,\ -\beta \le y \le x\},\ 0 \le t < \infty,\ 0 \le x \le b.$$

Let $(t^*,x^*) \in [-\alpha,t]\times[-\beta,x]$ be such that $w(t,x) = |u(t^*,x^*)|$.
If $(t^*,x^*) \in \tilde{J}$, then $w(t,x) = \|\Phi\|_C$ and the previous inequality holds. If $(t^*,x^*) \in J$, then by the previous inequality, we have for $(t,x) \in J$,

$$w(t,x) \le d + \int_0^t\int_0^x p(s,y)\left[w(s,y) + \int_0^s\int_0^y k(s,y,\tau,\xi)w(\tau,\xi)d\xi d\tau\right]dyds.$$

From Lemma 2.6, we get

$$w(t,x) \le d\left[1+\int_0^t\int_0^x p(s,y)\exp\left(\int_0^s\int_0^y [p(\tau,\xi)+A(\tau,\xi)]d\xi d\tau\right)dyds\right];\ (t,x)\in J. \quad (15)$$

But, for every $(t,x)\in J$, $\|u_{(t,x)}\|_C \le w(t,x)$. Hence, Eq. (15) yields Eq. (14).

Theorem 3.4. *Set*

$$\overline{d} := f^* + \varphi^* p^*(1+q^*). \quad (16)$$

Assume that $(H_1)-(H_5)$ *hold. If* u *is any solution to Eq. (2) on* $[-\alpha,\infty)\times[-\beta,b]$, *then*

$$|u(t,x)-\varphi(t)| \le \overline{d}\left[1+\int_0^t\int_0^x p(s,y)\exp\left(\int_0^s\int_0^y [p(\tau,\xi)+A(\tau,\xi)]d\xi d\tau\right)dyds\right],\ (t,x)\in J, \quad (17)$$

where A *is given by Eq. (7).*

Proof. Let $h(t,x) = |u(t,x)-\varphi(t)|$. Using the fact that u is a solution to Eqs. (1)-(3) and hypotheses, for each $(t,x)\in J$, we have

$$\begin{aligned}
h(t,x) \le\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times|f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,\varphi(s),(G\varphi)(s))|dyds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,\varphi(s),(G\varphi)(s))|dyds \\
\le\ & \overline{d} + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times|f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,\varphi(s),(G\varphi)(s))|dyds \\
\le\ & \overline{d} + \int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p(s,y) \\
&\times\left[h(s,y)+\int_0^s\int_0^y k(s,y,\tau,\xi)h(\tau,\xi)d\xi d\tau\right]dyds. \quad (18)
\end{aligned}$$

Now from an application of Lemma 2.6, Eq. (18) yields Eq. (17).

3.3 Global Asymptotic Stability of Solutions

We next prove under more appropriate conditions on the functions involved in Eq. (1)-(3) that the solutions tends exponentially toward zero as $t\to\infty$.

Theorem 3.5. *Assume that* $(H_4),(H_5)$ *ant the following hypotheses hold*

(H_6) *There exist constants* $\lambda>0$ *and* $M\ge 0$ *such that*

$$|\varphi(t)| \le Me^{-\lambda t}; \quad (19)$$

$$|f(t,x,u_1,u_2)-f(t,x,v_1,v_2)| \le p_1(t,x)e^{-\lambda t}(\|u_1-v_1\|_C + |u_2-v_2|), \quad (20)$$

for each $(t,x) \in J,\ u_1, v_1 \in C,\ u_2, v_2 \in \mathbb{R}$,

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \leq q(t,x,s,y)|u-v|; \tag{21}$$

for each $(t,x,s,y) \in J_1,\ u,v \in \mathbb{R}$,
and $f(t,x,0,(G0)(t,x)) = 0$; *for each* $(t,x) \in J$ *and the functions* $p,\ q$ *be as in Theorem 3.3,*

(H_7) $\int_0^\infty \int_0^x [p(s,y) + A(s,y)]dyds < \infty$, *where A is given by Eq. (7).*

If u is any solution of Eq. (1)-(3) on $[-\alpha,\infty) \times [-\beta, b]$, *then all solutions to Eq. (1)-(3) are uniformly globally attractive on J.*

Proof. From the hypotheses, for each $(t,x) \in J$, we have that

$$\begin{aligned}
|u(t,x)| &\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad \times |f(s,y,u_{(s,y)},(Gu)(s,y)) - g(s,y,0,(G0)(s,y))|dyds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} |f(s,y,0,(G0)(s,y))|dyds \\
&\leq Me^{-\lambda t} + \int_0^t \int_0^x p(s,y)e^{-\lambda t}\Big[u_{(s,y)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \times \int_0^s \int_0^y (s-\tau)^{r_1-1}(y-\xi)^{r_2-1} q(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\Big]dyds.
\end{aligned} \tag{22}$$

From Eq. (22), we get

$$|u(t,x)|e^{\lambda t} \leq M + \int_0^t \int_0^x p(s,y)\left[u_{(s,y)} + k(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\right]dyds. \tag{23}$$

Now an application of Lemma 2.6 to Eq. (23) yields

$$|u(t,x)|e^{\lambda t} \leq M\left[1 + \int_0^t \int_0^x p(s,y)\exp\left(\int_0^s \int_0^y [p(\tau,\xi) + A(\tau,\xi)]d\xi d\tau\right)dyds\right];\ (t,x) \in J, \tag{24}$$

Multiplying both sides of Eq. (24) by $e^{-\lambda t}$ and in view of (H_6), we get

$$|u(t,x)| \leq M\left[e^{-\lambda t} + \int_0^t \int_0^x p(s,y)\exp\Big(-\lambda t + \int_0^s \int_0^y [p(\tau,\xi) + A(\tau,\xi)]d\xi d\tau\Big)dyds\right].$$

Thus, for each $x \in [0,b]$, we get

$$\lim_{t\to\infty} u(t,x) = 0.$$

Hence, the solution u tends to zero as $t \to \infty$. Consequently, all solutions to Eq. (1)-(3) are uniformly globally attractive on $[-\alpha,\infty) \times [-\beta, b]$.

4 An Example

To illustrate our results, we consider the following system of partial integro-differential equations of fractional order of the form

$$^{c}D_{\theta}^{r}u(t,x) = f(t,x,u_{(t,x)},(Gu)(t,x)); \quad \text{for } (t,x) \in J := \mathbb{R}_{+} \times [0,1], \tag{25}$$

$$u(t,x) = \frac{1}{1+t^2}; \text{ if } (t,x) \in \tilde{J} := [-1,\infty)\times[-2,1]\backslash(0,\infty)\times(0,1], \tag{26}$$

$$\begin{cases} u(t,0) = \dfrac{1}{1+t^2}; \ t \in \mathbb{R}_{+}, \\ u(0,x) = 1; \ x \in [0,1], \end{cases} \tag{27}$$

where

$$(Gu)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} g(t,x,s,y,u_{(s,y)})dyds, \tag{28}$$

$r_1, r_2 \in (0,1]$,

$$\begin{cases} f(t,x,u,v) = \dfrac{x^2 t^{-r_1} \sin t}{2c(1+t^{-\frac{1}{2}})(1+|u(t+1,x+2)|+|v|)}; \\ \qquad for \ \ (t,x) \in J, \ t \neq 0 \ and \ \ u \in C, \ v \in \mathbb{R}, \\ f(0,x,u,v) = 0, \end{cases}$$

$$c := \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+r_1)} \left(1 + \frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)} \right),$$

$$\begin{cases} g(t,x,s,y,u) = \dfrac{t^{-r_1} s^{-\frac{1}{2}} e^{x-y-\frac{1}{s}-\frac{1}{t}}}{2c(1+t^{-\frac{1}{2}})(1+|u|)}; \ for \ \ (t,x,s,y) \in J_1, \ st \neq 0 \ and \ \ u \in \mathbb{R}, \\ \quad g(t,x,0,y,u) = g(0,x,s,y,u) = 0, \end{cases}$$

and

$$J_1 = \{(t,x,s,y) : 0 \le s \le t < \infty, \ 0 \le y \le x \le 1\}.$$

Set

$$\varphi(t) = \frac{1}{1+t^2}; \ t \in \mathbb{R}_{+}.$$

We can see that (H_1) is satisfied because the function φ is continuous and bounded with $\varphi^* = 1$. For each $u_1, v_1 \in C, \ u_2, v_2 \in \mathbb{R}$ and $(t,x) \in J$, we have

$$|f(t,x,u_1,u_2) - f(t,x,s,v_1,v_2)| \le \frac{1}{2c(1+t^{-\frac{1}{2}})} \left(x^2 t^{-r_1} |\sin t| \right) (|u_1 - v_1| + |u_2 - v_2|),$$

and for each $u, v \in \mathbb{R}$ and $(t,x,s,y) \in J_1$, we have

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \le \frac{1}{2c(1+t^{-\frac{1}{2}})} \left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}} \right) |u - v|.$$

Hence condition (H_2) is satisfied with

$$\begin{cases} p_1(t,x) = p_2(t,x) = \dfrac{x^2 t^{-r_1}|\sin t|}{2c(1+t^{-\frac{1}{2}})}; \ t \neq 0, \\ p_1(0,x) = p_2(0,x) = 0, \end{cases}$$

and condition (H_3) is satisfied with

$$\begin{cases} q(t,x,s,y) = \dfrac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right); \ st \neq 0, \\ q(t,x,0,y) = k(0,x,0,y) = 0. \end{cases}$$

We shall show that condition (10) holds with $b = 1$. Indeed

$$\begin{aligned} &\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_1(s,y)dyds \\ \leq\ & \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1}x^2t^{-r_1}dyds \\ \leq\ & \frac{\Gamma(\frac{1}{2})et^{-\frac{1}{2}}}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}, \end{aligned}$$

then

$$p_1^* = p_2^* \leq \frac{\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)}.$$

Also,

$$\begin{aligned} &\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}q(t,x,s,y)dyds \\ \leq\ & \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1}t^{-r_1}s^{-\frac{1}{2}}e^x dyds \\ \leq\ & e^x t^{-r_1} t^{-\frac{1}{2}+r_1}\frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)} \\ \leq\ & \frac{\Gamma(\frac{1}{2})et^{-\frac{1}{2}}}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}, \end{aligned}$$

then

$$q^* \leq \frac{e\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}.$$

Thus,

$$p_1^* + p_2^* q^* \leq \frac{\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)}\left(1 + \frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}\right) = \frac{1}{2} < 1,$$

which is satisfied for each $r_1, r_2 \in (0,\infty)$. Consequently Theorem 3.2 implies that the system Eq. (25)-(27) has a unique solution defined on $[-1,\infty)\times[-2,1]$.

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