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Abstract

We describe the natural identification of $FH_*(X\times X,\triangle;\omega\oplus-\omega)$ with $FH_*(X,\omega)$. Under this identification, we show that the extra elements in $\mathrm{Ham}(X\times X,\omega\oplus-\omega)$ found in [3], for $X=(S^2\times S^2,\omega_0\oplus\lambda\omega_0)$ for $\lambda>1$, do not define new invertible elements in $FH_*(X,\omega)$.

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1 Introduction

Let M be a symplectic manifold with an anti-symplectic involution c, such that L is the Lagrangian submanifold fixed by c. For any map $u:(\Sigma,\partial\Sigma)\to(M,\triangle)$, where Σ is a manifold with boundary, we define $v:\Sigma\cup_\partial\overline{\Sigma}\to X$ by

$$v|_\Sigma = p_1\circ u \text{ and } v|_{\overline{\Sigma}} = p_2\circ u,$$

where $\overline{\Sigma}$ is Σ with the opposite orientation. For any map $v:\Sigma\cup_\partial\overline{\Sigma}\to X$ we obtain the corresponding map $u:(\Sigma,\partial\Sigma)\to(M,\triangle)$ by

$$u(x)=(v(x),v(\overline{x})),$$

where $\overline{x}$ denotes $x\in\overline{\Sigma}$. We use δ to denote the map $v\mapsto u$.

For $M=X\times X$ and involution switching the factors, then $L=\triangle$. Let $\delta_k:=p_k\circ\delta$ where p_k is the projection to the k-factor, then it induces a map of Floer homologies. We show in §2

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Lemma 1.1. *There is a commutative diagram of isomorphisms of the Floer homologies:*

$$\begin{array}{ccc} & FH_*(M,\triangle) & \\ {\scriptstyle \delta_1,\ \cong} \swarrow & & \searrow {\scriptstyle \delta_2,\ \cong} \\ FH_*(X) & \xrightarrow[\cong]{\tau} & FH_*(\overline{X}) \end{array}$$

In [2], with proper assumptions, we described a construction of Lagrangian Seidel element from a path of Hamiltonian diffeomorphisms. In particular, a loop γ in $\mathrm{Ham}(M)$ defines a Lagrangian Seidel element $\Psi^L_\gamma \in FH_*(M,L)$, where L is a Lagrangian submanifold. The Albers' map

$$\mathscr{A} : FH_*(M) \to FH_*(M,L)$$

whenever well-defined, for example when L is monotone, relates the Seidel elements $\Psi^M_\gamma \in FH_*(M)$ to Ψ^L_γ. Let S_M denote the image of Seidel map $\Psi^M : \pi_1\mathrm{Ham}(M) \to FH_*(M)$ and S_L that of $\Psi^L : \pi_1(\mathrm{Ham}(M), \mathrm{Ham}_L(M)) \to FH_*(M,L)$, where $\mathrm{Ham}_L(M)$ is the group of Hamiltonian diffeomorphisms preserving L which restrict to isotopies on L, then

$$\mathscr{A}(S_M) \subseteq S_L$$

Question 1.2. When all the terms involved is well defined, is the inclusion $\mathscr{A}(S_M) \subseteq S_L$ (in general) proper?

An affirmative answer to this question would imply an affirmative answer to the open question about the non-triviality of $\pi_0\mathrm{Ham}_L(M)$.

For the case $L = \triangle$, since δ_1 is an isomorphism, the inclusion is equivalent to $\delta_1\mathscr{A}(S_M) \subseteq \delta_1(S_\triangle)$ as subsets of $FH_*(X)$. In §3, we show that $S_X \subseteq \delta_1\mathscr{A}(S_M)$. More precisely,

Theorem 1.3. *[Corollary 3.2] Let $\gamma \in \pi_1\mathrm{Ham}(X)$. It naturally lifts to a* split *element $\gamma_+ \in \pi_1\mathrm{Ham}(M)$, and we have $\delta_1\mathscr{A}(\Psi^M_{\gamma_+}) = \Psi^X_\gamma$.*

As a corollary, it shows that the natural map $\pi_1\mathrm{Ham}(X) \times \pi_1\mathrm{Ham}(\overline{X}) \to \pi_1\mathrm{Ham}(M)$ is injective. In light of this result, we pose the following question, which is related to Question 1.2 for the special case of diagonal.

Question 1.4. Is any inclusion in the sequence $S_X \subseteq \delta_1\mathscr{A}(S_M) \subseteq \delta_1(S_\triangle)$ proper?

For $X = S^2 \times S^2$ as in [3], we show in §3 that the image under $\delta_1\mathscr{A}$ of the extra Seidel elements found in [3] is contained in S_X.

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2 Identification of Floer homologies

2.1 Notations

Let

$$D^2_+ = \{z \in \mathbb{C} : |z| \leqslant 1, \Im z \geqslant 0\},$$

∂_+ denote the part of boundary of D^2_+ on the unit circle, parametrized by $t \in [0,1]$ as $e^{i\pi t}$, and ∂_0 the part on the real line, parametrized by $t \in [0,1]$ as $2t - 1$.

Let (M,L) be a pair of symplectic manifold and a Lagrangian submanifold, and Ω is the symplectic form. For $\beta \in \pi_2(M,L)$, $\mu_L(\beta)$ denotes its Maslov number, and $\Omega(\beta)$ its symplectic area. The space of paths in M connecting points of L is

$$\mathcal{P}_L M = \{l : ([0,1],\partial[0,1]) \to (M,L), [l] = 0 \in \pi_1(M,L)\}$$

and the corresponding covering space with covering group $\Gamma_L = \pi_2(M,L)/(\ker\omega \cap \ker\mu_L)$ is

$$\widetilde{\mathcal{P}}_L M = \{[l,w] : w : (D^2_+;\partial_+,\partial_0) \to (M;l,L)\}$$

where $(l,w) \sim (l',w') \iff l = l'$ and $\omega(w\#(-w')) = \mu_L(w\#(-w'))$. The space of contractible loops in M parametrized by $\mathbb{R}/\mathbb{Z}$ is denoted $\Omega(M)$ and the corresponding covering space with covering group $\Gamma_\omega = \pi_2(M)/(\ker\omega \cap \ker c_1)$ is given by

$$\widetilde{\Omega}(M) = \{[\gamma,v] : v : (D^2,\partial D^2) \to (M,\gamma)\}$$

where $(\gamma,v) \sim (\gamma',v') \iff \gamma = \gamma'$ and $\omega(v\#(-v')) = c_1(v\#(-v'))$. Here, ∂D^2 is parametrized as the unit circle in $\mathbb{C}$ by $\{e^{2\pi it} : t \in [0,1]\}$, and $c_1 = c_1(TM)$ in some compatible almost complex structure. We denote the space of loops in M parametrized by $\mathbb{R}/T\mathbb{Z}$ and the corresponding covering space as $\Omega^{(T)}(M)$ and $\widetilde{\Omega}^{(T)}(M)$ respectively, thus $\Omega(M) = \Omega^{(1)}(M)$ and $\widetilde{\Omega}(M) = \widetilde{\Omega}^{(1)}(M)$.

Let $H : [0,1] \times M \to \mathbb{R}$ be a time-dependent Hamiltonian function, which defines on $\widetilde{\mathcal{P}}_L M$ the action functional

$$a_H([l,w]) = -\int_{D^2_+} w^*\omega + \int_{[0,1]} H_t(l(t))dt,$$

where we use the convention $dH = -\iota_{X_H}\omega$ for the Hamiltonian vector fields. Similarly, a time dependent Hamiltonian function K for $t \in \mathbb{R}/T\mathbb{Z}$ defines an action functional a_K on $\widetilde{\Omega}^{(T)}(M)$. We will not distinguish notations for the two types of action functionals when it is clear from the context which one is under discussion.

Given the time dependent Hamiltonian function H, let $\widetilde{l} \in \widetilde{\mathcal{P}}_L M$ such that l is a connecting orbit for H, then $\mu_H(\widetilde{l})$ denotes the corresponding Conley-Zehnder index. Similarly, for the time dependent Hamiltonian functino K, let $\widetilde{\gamma} \in \widetilde{\Omega}^{(T)}(M)$ such that γ is a periodic orbit for K, then $\mu_K(\widetilde{\gamma})$ denotes the corresponding Conley-Zehnder index. The following relations hold

$$\mu_H(\widetilde{l}) - \mu_H(\widetilde{l}') = \mu_L(w\#(-w')) \text{ and } \mu_K(\widetilde{\gamma}) - \mu_K(\widetilde{\gamma}') = c_1(u\#(-v'))$$

where $l = l'$ and $\gamma = \gamma'$.

2.2 Doubling construction

First we describe the doubling construction when the Lagrangian submanifold is the fixed submanifold of an anti-symplectic involution. It applies in this case since the diagonal $\triangle$ is the fixed submanifold of the involution of switching the two factors.

Let $c : M \to M$ be an anti-symplectic involution and $L \subset M$ be the fixed submanifold of τ, then it is a Lagrangian submanifold. We'll use $(\mathbb{H},\mathbb{J})$ to denote a pair of 2-periodical Hamiltonian functions and compatible almost complex structures, i.e.

$$\mathbb{H} : \mathbb{R}/2\mathbb{Z} \times M \to \mathbb{R} \text{ and } \mathbb{J} = \{\mathbb{J}_t\}_{t\in\mathbb{R}/2\mathbb{Z}}.$$

Definition 2.1. The pair $(\mathbb{H},\mathbb{J})$ is *c-symmetric* if it satisfies

$$\mathbb{H}_t(x) = \mathbb{H}_{2-t}(c(x)) \text{ and } \mathbb{J}_t(x) = -dc \circ \mathbb{J}_{2-t} \circ dc.$$

For such a pair, we define the halves $(H,\mathbf{J}) := (\mathbb{H}_t,\mathbb{J}_t)_{t\in[0,1]}$ and

$$(H',\mathbf{J}') := (\mathbb{H}_{1-t} \circ c, -dc \circ \mathbb{J}_{1-t} \circ dc)_{t\in[0,1]} = (\mathbb{H}_{t+1},\mathbb{J}_{t+1})_{t\in[0,1]}.$$

The doubling map δ described in the introduction is a special case of the following construction for a symplectic manifold with an anti-symplectic involution:

Definition 2.2. Let $u : (\Sigma,\partial\Sigma) \to (M,L)$ be a map from a manifold Σ with boundary $\partial\Sigma$, the *doubled map* is given by:

$$v : \Sigma \cup_\partial \overline{\Sigma} \to M : v|_\Sigma = u \text{ and } v|_{\overline{\Sigma}} = c \circ u,$$

where $\overline{\Sigma}$ is Σ with the opposite orientation. We also write $\delta(u) := v$ which gives the *doubling map* between the spaces of continuous maps:

$$\delta : Map(\Sigma,\partial\Sigma; M,L) \to Map(\Sigma \cup_\partial \overline{\Sigma}; M).$$

In particular, we have the map between the space of paths in (M,L) and loops of period 2 in M, as well as their covering spaces:

$$\delta : \mathcal{P}_L M \to \Omega^{(2)}(M) \text{ and } \delta : \widetilde{\mathcal{P}}_L M \to \widetilde{\Omega}^{(2)}(M)$$

Let $(\mathbb{H},\mathbb{J})$ be a c-symmetric pair and $\{\phi_t\}_{t\in[0,2]}$ the Hamiltonian isotopy generated by $\mathbb{H}$, then

$$\phi_t = c \circ \phi_{2-t} \circ \phi_2^{-1} \circ c \Longrightarrow (c \circ \phi_2)^2 = \mathbb{1}. \tag{2.1}$$

Let $(H,\mathbf{J})$ and $(H',\mathbf{J}')$ be the two halves of $\mathbb{H}$, then

$$H_t = H'_{1-t} \circ c \text{ and } J_t = -dc \circ J'_{1-t} \circ dc,$$

Let ϕ'_t denote the Hamiltonian isotopy generated by H', then

$$\phi'_t = c \circ \phi_{1-t} \circ \phi_1^{-1} \circ c$$

It follows that if l is a Hamiltonian path generated by H connecting $x,y \in L$, then $l'(t) := c \circ l(1-t)$ is a Hamiltonian path generated by H' connecting $y,x \in L$, and the double $\gamma = \delta(l)$ is a periodic orbit for $\mathbb{H}$. This correspondence lifts to the covering spaces and the following holds.

Lemma 2.3. *For $\widetilde{l} \in \widetilde{\mathcal{P}}_L M$ let $\widetilde{\gamma} = \delta(\widetilde{l})$, then*

$$a_{\mathbb{H}}(\widetilde{\gamma}) = 2a_H(\widetilde{l}) = 2a_{H'}(\widetilde{l}').$$

Moreover, if $\widetilde{l}$ is a critical point of a_H then $\widetilde{\gamma}$ is a critical point of $a_{\mathbb{H}}$. If $\widetilde{\gamma}$ is non-degenerate, then $\widetilde{l}$ is as well. A Floer trajectory for a_H is taken to a Floer trajectory for $a_{\mathbb{H}}$ by δ, which converges to the corresponding critical points when the trajectory has finite energy. □

A result from [2] relates the Conley-Zehnder indices of connecting paths generated by H and H'.

Lemma 2.4 (Lemma 5.2 of [2])**.** *Let $\widetilde{l}$ and $\widetilde{l}'$ be respective critical points of a_H and $a_{H'}$ as above. Then $\mu_H(\widetilde{l}) = \mu_{H'}(\widetilde{l}')$.* □

2.3 Index comparison

We briefly recall the definition of Conley-Zehnder index using the Maslov index of paths of Lagrangian subspaces as in Robbin-Salamon [5]. Let $\widetilde{l} = [l, w]$ be a non-degenerate critical point of a_H. Then $w : D^2_+ \to M$ and $l = \partial w$ is a Hamiltonian path. There is a symplectic trivialization Φ of w^*TM given by $\Phi_z : T_{w(z)}M \to \mathbb{C}^n$ with the standard symplectic structure ω_0 on $\mathbb{C}^n$. Furthermore, we require that $\Phi_r(T_{w(r)}L) = \mathbb{R}^n$, for $r \in [-1,1] \subset D^2_+$. Then the linearized Hamiltonian flow $d\phi_t$ along l defines a path of symplectic matrices

$$E_t = \Phi_{e^{i\pi t}} \circ d\phi_t \circ \Phi_1^{-1} \in Sp(\mathbb{C}^n) \tag{2.2}$$

Then the Conley-Zehnder index of $\widetilde{l}$ is given by

$$\mu_H(\widetilde{l}) = \mu(E_t\mathbb{R}^n, \mathbb{R}^n)$$

where μ is the Maslov of paths of Lagrangian subspaces introduced in [5].

We continue with the notations of Lemma 2.3.

Proposition 2.5. *Suppose that all the critical points involved are non-degenerate, then*

$$\mu_H(\widetilde{l}) + \mu_{H'}(\widetilde{l'}) - \mu_{\mathbb{H}}(\widetilde{\gamma}) = \frac{1}{2} sign(Q), \tag{2.3}$$

where $Q(\bullet, *) = \Omega((\mathbb{1} - d\phi_2)\bullet, dc(*))$ *is a quadratic form on* $T_{l(0)}M$.

Proof: For notational convenience, we denote

$$\widetilde{l^+} = \widetilde{l}, \widetilde{l^-} = \widetilde{l'}, H^+ = H, H^- = H', \phi_t^+ = \phi_t \text{ and } \phi_t^- = c \circ \phi_{1-t} \circ \phi_1^{-1} \circ c \text{ for } t \in [0,1],$$

then $\phi^\pm$ is the flow generated by $H^\pm$. Assume that we can choose the trivialization $\Phi_z : T_{v(z)}M \to \mathbb{C}^n$ of v^*TM so that $\Phi_{\overline{z}} = c_z \circ \Phi_z \circ dc$, where $c_z : \mathbb{C}^n \to \mathbb{C}^n$ is the complex conjugation, which takes ω_0 to $-\omega_0$. In particular, $\Phi_r(T_{v(r)}L) = c_z \circ \Phi_r \circ dc(T_{v(r)}L) = \mathbb{R}^n$ for $r \in [-1,1]$. Define the following paths of symplectic matrices:

$$F_t = \Phi_{e^{i\pi t}} \circ d\phi_t \circ \Phi_1^{-1} \text{ for } t \in [0,2] \text{ and } F_t^\pm = \Phi_{\pm e^{i\pi t}} \circ d\phi_t^\pm \circ \Phi_{\pm 1}^{-1} \text{ for } t \in [0,1],$$

Then $F_t = c_z \circ F_{2-t} \circ F_2^{-1} \circ c_z$ and

$$\mu_{\mathbb{H}}(\widetilde{\gamma}) = \mu((F_t, \mathbb{1})\triangle, \triangle) \text{ and } \mu_{H^\pm}(\widetilde{l^\pm}) = \mu(F_t^\pm\mathbb{R}^n \oplus \mathbb{R}^n, \triangle)$$

where $\triangle : \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$ is the diagonal and the symplectic structure on $\mathbb{C}^n \oplus \mathbb{C}^n$ is given by $\Omega_0 = \omega_0 \oplus (-\omega_0)$. We have by additivity of Maslov index:

$$\mu_{\mathbb{H}}(\widetilde{\gamma}) = \mu((F_t^+, \mathbb{1})\triangle, \triangle) + \mu((F_t^- \circ F_1, \mathbb{1})\triangle, \triangle)$$

and the left hand side of (2.3) is the sum of the following differences:

$$\mu(F_t^+\mathbb{R}^n \oplus \mathbb{R}^n, \triangle) - \mu((F_t^+, \mathbb{1})\triangle, \triangle) \text{ and } \mu(F_t^-\mathbb{R}^n \oplus \mathbb{R}^n, \triangle) - \mu((F_t^- \circ F_1, \mathbb{1})\triangle, \triangle).$$

For $F \in Sp(\mathbb{C}^n)$, $(F, \mathbb{1})^{-1}\triangle = (\mathbb{1}, F)\triangle$, thus the first difference is

$$\begin{aligned}&\mu(F_t^+\mathbb{R}^n \oplus \mathbb{R}^n, \triangle) - \mu((F_t^+, \mathbb{1})\triangle, \triangle) = \mu((\mathbb{1}, F_t^+)\triangle, \triangle) - \mu((\mathbb{1}, F_t^+)\triangle, \mathbb{R}^n \oplus \mathbb{R}^n)\\ =&s(\mathbb{R}^n \oplus \mathbb{R}^n, \triangle; \triangle, (\mathbb{1}, F_1)\triangle) = s(\mathbb{R}^n \oplus \mathbb{R}^n, (\mathbb{1}, F_1)\triangle; \triangle, (\mathbb{1}, F_1)\triangle)\end{aligned}$$

where s is the Hömander index (cf. [5]) and the last equality follows from the following properties of Hörmander index for Lagrangian subspaces A,B,C,D,D':

$$\begin{aligned} s(A,B;A,C) = s(A,B;A,C) - s(A,C;A,C) = s(C,B;A,C) \\ s(A,B;C,D) - s(A,B;C,D') = s(A,B;D',D). \end{aligned} \tag{2.4}$$

Let $c' : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n : (z_1,z_2) \mapsto (z_2,z_1)$, then c' preserves $\triangle$ and $\mathbb{R}^n \oplus \mathbb{R}^n$ while reverses the sign of the symplectic structure, thus

$$s(\mathbb{R}^n \oplus \mathbb{R}^n, (\mathbb{1}, F_1)\triangle; \triangle, (\mathbb{1}, F_1)\triangle) = -s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, \mathbb{1})\triangle; \triangle, (F_1, \mathbb{1})\triangle).$$

For the second difference, we get

$$\begin{aligned} &\mu(F_t^- \mathbb{R}^n \oplus \mathbb{R}^n, \triangle) - \mu((F_t^- \circ F_1, \mathbb{1})\triangle, \triangle) \\ =&\mu((\mathbb{1}, F_t^-)\triangle, (F_1, \mathbb{1})\triangle) - \mu((\mathbb{1}, F_t^-)\triangle, \mathbb{R}^n \oplus \mathbb{R}^n) \\ =&s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, \mathbb{1})\triangle; \triangle, (\mathbb{1}, F_1^-)\triangle). \end{aligned}$$

It follows that the difference on the left side of (2.3) is

$$\begin{aligned} &s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, \mathbb{1})\triangle; \triangle, (\mathbb{1}, F_1^-)\triangle) - s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, \mathbb{1})\triangle; \triangle, (F_1, \mathbb{1})\triangle) \\ =&s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, \mathbb{1})\triangle; (F_1, \mathbb{1})\triangle, (\mathbb{1}, F_1^-)\triangle). \end{aligned}$$

We now identify the last Hömander index as the signature. Let $L = \mathbb{R}^n \oplus \mathbb{R}^n$, $K = (F_1, \mathbb{1})\triangle$ and $L' = (\mathbb{1}, F_1^-)\triangle$, then they are pairwisely transverse, by the non-degeneracy assumption. Thus in the splitting $\mathbb{C}^{2n} = L \oplus K$ we may write L' as the graph of an invertible linear map $f : K \to K^* \cong L$ and let $\overline{KL'} = graph(tf)$, $t \in [0,1]$ be the path of Lagrangian subspaces connecting K to L' then

$$s(L,K;K,L') = \mu(\overline{KL'}, K) - \mu(\overline{KL'}, L) = \mu(\overline{KL'}, K) = \frac{1}{2} sign(Q') \tag{2.5}$$

where $Q'(v) = \Omega_0(v, f(v))$ for $v \in K$ is a quadratic form on K. Choose the following coordinates

$$\begin{aligned} &L = \{(x,y) | x,y \in \mathbb{R}^n\}, K = \{(F_1(z), z) | z \in \mathbb{C}^n\} \text{ and} \\ &L' = \{(\overline{w}, F_1^-(\overline{w})) | w \in \mathbb{C}^n\} = \{(\overline{w}, \overline{F_1^{-1}(w)})\} = \{(\overline{F_1(w)}, \overline{w})\}, \end{aligned}$$

where we note $F_1^- = c \circ F_1^{-1} \circ c$, then it's easy to check that

$$f : K \to L : z \mapsto (x,y) = -(F_1(z) + \overline{F_1(z)}, z + \overline{z})$$

and for $v = (F_1(z), z)$

$$\begin{aligned} Q'(v) =& -\omega_0(F_1(z), \overline{F_1(z)}) + \omega_0(z, \overline{z}) \\ =& -\omega(F_1(z), F_1 \circ F_2^{-1}(\overline{z})) + \omega_0(z, \overline{z}) \\ =&\omega_0((\mathbb{1} - F_2)(z), \overline{z}) \\ =&Q(z). \end{aligned}$$

Together with (2.5), we are done.

We now show the existence of a trivialization Φ_z with $\Phi_{\overline{z}} = c_z \circ \Phi_z \circ dc$. Let $V^{\pm}$ be the ± 1 eigen-bundle of dc action on $v|^*_{[-1,1]}TM$, then they are transversal Lagrangian subbundles. Since $[-1,1]$ is contractible, we trivialize V^+ and choose a section $\{e^j_r\}^n_{j=1}$ for $r \in [-1,1]$ of the frame bundle. The induced trivialization of V^- is then given by $\{f^j_r\}^n_{j=1}$ where $\omega(e^j_r, f^k_r) = \delta_{kj}$. Then the trivialization Φ_r can be defined by $\{e^j_r, f^k_r\} \mapsto$ standard basis of $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. Then the trivialization Φ_r satisfies $\Phi_r = c_z \circ \Phi_r \circ dc$. Extend it to D^2_+ to obtain trivialization Φ_z for $z \in D^2_+$. Now define Φ_z for $z \in D^2_-$ by $\Phi_z = c_z \circ \Phi_{\overline{z}} \circ dc$ and $\Phi_{z \in D^2}$ gives a continuous trivialization of v^*TM with the desired property. □

2.4 Diagonal

For $(M,L) = (X \times X, \triangle)$, the doubling construction applies. Let $p_i : M \to X$, for $i = 1,2$, be the projection to the i-th factor, then we obtain the following maps

$$\delta_i = p_i \circ \delta : Map(\Sigma, \partial\Sigma; M, \triangle) \to Map(\Sigma \cup_\partial \overline{\Sigma}; X)$$

which are natural isomorphism between the spaces of continuous maps. As special cases, the doubling gives isomorphisms of the path / loop spaces and the respective covering spaces:

$$\delta_i : \mathcal{P}_\triangle(M) \to \Omega^{(2)}(X) \text{ and } \delta_i : \widetilde{\mathcal{P}}_\triangle(M) \to \widetilde{\Omega}^{(2)}(X)$$

More explicitly, for example, for $l \in \mathcal{P}_\triangle(M)$ we write $l(t) = (l_1(t), l_2(t))$ then

$$(\delta_1(l))(t) = \begin{cases} l_1(t) & \text{for } t \in [0,1] \\ l_2(2-t) & \text{for } t \in [1,2] \end{cases}$$

This isomorphism extends to their corresponding normed completions as well. They also induce the isomorphisms $\delta_i : \pi_2(M, \triangle) \to \pi_2(X)$. The exact sequence of homotopy groups gives

$$\ldots \to \pi_2(\triangle) \to \pi_2(M) \cong \pi_2(X) \times \pi_2(X) \xrightarrow{j} \pi_2(M, \triangle) \to \ldots$$

Then we have for $\beta \in \pi_2(X)$:

$$\delta_1 \circ j(\beta, 0) = \delta_2 \circ j(0, -\beta) = \beta$$

It's straight forward to see that for $\beta \in \pi_2(X)$, $\delta_2 \circ \delta_1^{-1}(\beta) = -\beta = \tau(\beta)$. The isomorphism of homotopy group gives rise the isomorphism $\delta_i : \Gamma_\triangle \cong \Gamma_\omega$ as well as the corresponding Novikov rings. More precisely, for $a_\beta e^\beta \in \Lambda_\triangle$, we have

$$\delta_1(a_\beta e^\beta) = a_\beta e^{\delta_1(\beta)} \in \Lambda_\omega \text{ and } \delta_2(a_\beta e^\beta) = (-1)^{\frac{1}{2}\mu_\triangle(\beta)} a_\beta e^{\delta_2(\beta)} \in \Lambda_{-\omega}$$

then $\delta_2 \circ \delta_1^{-1} : \Lambda_\omega \to \Lambda_{-\omega}$ coincides with the isomorphism induced by reversing the symplectic structure on (X, ω) (cf. [2] §4).

Let $\{H_t, J_t\}_{t \in [0,2]}$ be a pair of periodic Hamiltonian functions and compatible almost complex structures on (X, ω), then

$$(\mathbb{H}_t, \mathbb{J}_t) = (H_t \oplus H_{2-t}, \mathbf{J}_t \oplus -\mathbf{J}_{2-t})$$

is a c-symmetric pair on $M = X\times X$, with symplectic form $\Omega = \omega\oplus(-\omega)$. Let $\{\phi_t\}_{t\in[0,2]}$ denote the Hamiltonian isotopy generated by H_t on X, then $\{\psi_t = (\phi_t, \phi_{2-t}\circ\phi_2^{-1})\}_{t\in[0,2]}$ is the Hamiltonian isotopy generated by $\mathbb{H}_t$ on M. It follows that $x\in X$ is a non-degenerate fixed point of ϕ_2 iff $(x,x)\in\triangle$ is a non-degenerate fixed point of ψ_2.

Let $(\mathbb{H}^1,\mathbb{J}^1)$ and $(\mathbb{H}^2,\mathbb{J}^2)$ be the two halves of $(\mathbb{H},\mathbb{J})$, i.e.

$$(\mathbb{H}^1,\mathbb{J}^1) = (\mathbb{H}_t,\mathbb{J}_t)_{t\in[0,1]} \text{ and } (\mathbb{H}^2,\mathbb{J}^2) = (\mathbb{H}_{t+1},\mathbb{J}_{t+1})_{t\in[0,1]}$$

Let $\widetilde{l}\in\widetilde{\mathcal{P}}_\triangle M$ be a critical point of $a_{\mathbb{H}^1}$, then Lemma 2.3 implies that $\widetilde{\gamma} = \delta(\widetilde{l})\in\widetilde{\Omega}^{(2)}(M)$ is a critical point of $a_{\mathbb{H}}$. Let $\widetilde{\gamma}_1 = p_1(\widetilde{\gamma}) = \delta_1(\widetilde{l})\in\widetilde{\Omega}(X)$, then it is a critical point of a_H. Similarly, $\widetilde{\gamma}_2 = p_2(\widetilde{\gamma})$ is a critical point of $a_{\underline{H}}$, with $\underline{H}_t = H_{2-t}$. Furthermore, the non-degeneracy of any one of these critical points implies that all the rest are also non-degenerate.

Lemma 2.6. *Suppose that all critical points involved are non-degenerate, then* $\mu_{\mathbb{H}}(\widetilde{\gamma}) = 2\mu_{\mathbb{H}}(\widetilde{l})$. *It follows that*

$$\mu_{\mathbb{H}}(\widetilde{l}) = \mu_H(\widetilde{\gamma}_1)$$

Proof: The critical point $\widetilde{\gamma}$ is determined by it projection to the two factors, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Notice that $\psi_t = (\phi_t,\phi_{2-t}\circ\phi_2^{-1})$, in (2.2), the identification Φ may chosen such that it respects the decomposition $TM = p_1^*TX\oplus p_2^*TX$. Then it's clear that

$$\mu_{\mathbb{H}}(\widetilde{\gamma}) = \mu_H(\widetilde{\gamma}_1) + \mu_{\underline{H}}(\widetilde{\gamma}_2)$$

Similar to Lemma 5.2 of [2], straight forward computation shows that

$$\mu_H(\widetilde{\gamma}_1) = \mu_{\underline{H}}(\widetilde{\gamma}_2) \Rightarrow \mu_{\mathbb{H}}(\widetilde{\gamma}) = 2\mu_H(\widetilde{\gamma}_1)$$

Now we only have to see that $\mu_{\mathbb{H}}(\widetilde{\gamma}) = 2\mu_{\mathbb{H}}(\widetilde{l})$. By Lemma 2.4 and Proposition 2.5, we only need to compute $sign(Q)$. Let $\gamma(0) = (x,x)\in\triangle$ and $\xi_1,\xi_2\in T_xX$, then $\xi = (\xi_1,\xi_2)\in T_{\gamma(0)}M$ and

$$\begin{aligned} Q(\xi,\xi) &= \Omega((\mathbb{1}-d\psi_2)(\xi_1,\xi_2),(\xi_2,\xi_1)) \\ &= \omega((\mathbb{1}-d\phi_2)\xi_1,\xi_2) - \omega((\mathbb{1}-d\phi_2)\xi_2,\xi_1) \\ &= 2\omega((\mathbb{1}-d\phi_2)\xi_1,\xi_2) \end{aligned}$$

It follows that $sign(Q) = 0$. □

2.5 Proof of the lemma

The lemma follows from the following proposition and Proposition 4.2 of [2] which relates the quantum homology of opposite symplectic structures.

Proposition 2.7. δ_1 *induces a natural isomorphism of the Floer theories*

$$\delta_1 : FH_*(M,\triangle;\Omega) \cong FH_*(X,\omega).$$

Proof: Using the notations from the last subsection, we first compare the action functionals. Let $\widetilde{l} = [l, w] \in \widetilde{\mathcal{P}}_{\triangle} M$ and $\widetilde{\gamma}_1 = [\gamma_1, v_1]$ so that $\widetilde{\gamma}_1 = \delta_1(\widetilde{l})$, then

$$a_H([\gamma_1, v_1]) = -\int_{D^2} v_1^* \omega + \int_{[0,2]} H_t(\gamma_1(t))dt = -\int_{D^2_+} w^* \Omega + \int_{[0,1]} \mathbb{H}_t(l(t))dt = a_{\mathbb{H}}([l, w]).$$

Let $\{\xi_t\}_{t\in[0,2]}$ be a vector field along γ_1, then $\{\eta_t = (\xi_t, \xi_{2-t})\}_{t\in[0,1]}$ is a vector field along l with $\eta_{0,1} \in T\triangle$ and vice versa. This gives the isomorphism on the tangent spaces:

$$D\delta_1 : T_l \mathcal{P}_{\triangle} M \to T_{\gamma_1} \Omega^{(2)}(X) : \eta \mapsto \xi.$$

It then follows that for $\eta, \eta' \in T_l \mathcal{P}_{\triangle} M$ and the corresponding ξ's:

$$(\xi, \xi')_{\mathbf{J}} = \int_{[0,2]} \omega(\xi_t, J_t(\xi'_t))dt = \int_{[0,1]} \omega(\xi_t, J_t(\xi'_t))dt + \omega(\xi_{2-t}, J_{2-t}(\xi'_{2-t}))dt = (\eta, \eta')_{\mathbb{J}}.$$

From these we see that the Floer equations for the two theories are identified by δ_1 and the moduli spaces of smooth solutions are isomorphic for the two theories.

By Lemma 2.6, the gradings of the two theories coincide via δ_1. We consider the orientations. Let's first orient the moduli spaces of holomorphic discs in $(M, \triangle)$. Here we may assume that the almost complex structures involved are generic. The map δ_1 induces

$$\delta_1 : H_*(M, \triangle) \to H_*(X)$$

as well as the maps between the moduli spaces of (parametrized) holomorphic objects (discs or spheres):

$$\delta_1 : \widetilde{\mathcal{M}}(M, \triangle; \mathbb{J}, B) \to \widetilde{\mathcal{M}}(X; J, \delta_1(B)).$$

The map δ_1 is an isomorphism. We the put the induced orientation on the moduli space of discs. The moduli spaces of caps are similarly related by δ_1 and the orientations for a preferred basis on either theory can be chosen to be compatible with respect to δ_1. It then follows that the orientations of the theories coincide under δ_1.

To identify the two theories in full, we study the compactifications of the moduli spaces, in particular the compactifications by bubbling off holomorphic discs/spheres. The partial compactification given by the broken trajectories is naturally identified by δ_1 and the identification of the Floer equations.

Consider next the moduli spaces of holomorpic discs in $(M, \triangle)$. The map δ_1 defined for the moduli spaces above extends to objects with marked points, which, for spheres, are along $\mathbb{RP}^1 \subset \mathbb{CP}^1$ while for the discs, are along the boundary:

$$\delta_1 : \widetilde{\mathcal{M}}_k(M, \triangle; \mathbb{J}_i, B) \to \widetilde{\mathcal{M}}_k(X; J_i, \delta_1(B)) \text{ for } i = 0, 1.$$

When we pass to the unparametrized moduli spaces, we also denote the induced map δ_1.

Next, we consider the evaluation maps from the moduli spaces of objects with 1-marked point:

$$ev^{\triangle} : \mathcal{M}_1(M, \triangle; \mathbb{J}_i, B) \to \triangle \text{ and } ev : \mathcal{M}_1(X; J_i, \delta_1(B)) \to X.$$

Let $p_1 : \triangle \to X$ be the natural projection, then we see that

$$p_1 \circ ev^{\triangle} = ev \circ \delta_1.$$

In particular, the image of the evaluation map $ev^{\triangle}$ has at most the same dimension as that of ev (in fact, they coincide via p_1):

$$\dim_{\mathbb{R}} = 2c_1(TX)(B) + 2n - 4.$$

The bubbling off of spheres are similar. The Floer theory $FH_*(X,\omega)$ is well defined and it follows that $FH_*(M,\triangle;\Omega)$ is well defined as well and they are isomorphic. □

Recall from [2] (Proposition 5.5) that the Lagrangian Floer theories of $(M,\triangle,\Omega)$ and $(M,\triangle,-\omega)$ are related by an isomorphism

$$\tau_* : FH_*(M,\triangle,\Omega;\mathbb{H},\mathbb{J}) \to FH_*(M,\triangle;-\Omega;\underline{\mathbb{H}},\underline{\mathbb{J}})$$

where $\underline{\mathbb{H}}_t = \mathbb{H}_{2-t}$ and $\underline{\mathbb{J}}_t = -\mathbb{J}_{2-t}$ here. We observe that the involution c on M identifies the tuples:

$$c : (M,\triangle,-\Omega;\underline{\mathbb{H}},\underline{\mathbb{J}}) \to (M,\triangle;\Omega;\mathbb{H},\mathbb{J})$$

and the induced map of c on Floer homology composed with τ_* is the identity map. The Lemma 1.1 is given by the following diagram

$$\begin{array}{ccccc}
FH_*(M,\triangle;\Omega) & \xrightarrow[\cong]{\tau_*} & FH_*(M,\triangle;-\Omega) & \xrightarrow{c_*} & FH_*(M,\triangle;\Omega) \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_1} & \swarrow{\scriptstyle \delta_2} & \\
FH_*(X,\omega) & \xrightarrow[\cong]{\tau} & FH_*(X,-\omega) & &
\end{array}$$

The commutativity of the left square follows from the discussion of reversing the symplectic structure in [2] (§4 – 5), while it's obvious that the right triangle commutes.

Corollary 2.8. *The half pair of pants product is well defined for $FH_*(M,\triangle)$ and it has a unit.*

Proof: Everything is induced from $FH_*(X,\omega)$ using the map δ_1. □

3 Seidel elements and the Albers map

Let $\Omega_0\mathrm{Ham}(M,\Omega)$ be the space of loops in $\mathrm{Ham}(M,\Omega)$ based at $\mathbb{1}$. It's a group under pointwise composition. In $\Omega_0\mathrm{Ham}(M,\Omega)$, a loop g is *split* if $g = (g_1, g_2)$ is in the image of the natural maps

$$\Omega_0\mathrm{Ham}(X,\omega) \times \Omega_0\mathrm{Ham}(X,-\omega) \to \Omega_0\mathrm{Ham}(M,\Omega)$$

Otherwise, it is *non-split*. Similarly, such notions are defined for the π_1 of the Hamiltonian groups.

3.1 Split loops

In Seidel [6], the covering space $\widetilde{\Omega}_0\mathrm{Ham}(M,\Omega)$ is defined as

$$\widetilde{\Omega}_0\mathrm{Ham}(M,\Omega) := \left\{ (g,\widetilde{g}) \in \Omega_0\mathrm{Ham}(M,\Omega) \times \mathrm{Homeo}(\widetilde{\Omega}(M)) \middle| \widetilde{g} \text{ lifts the action of } g \right\}$$

with covering group Γ_Ω. We use $\widetilde{g}$ to denote an element in $\widetilde{\Omega}_0\mathrm{Ham}(M,\Omega)$. The results in [2] imply that, similar to [6], $\widetilde{g}$ defines a homomorphism $FH_*(\widetilde{g})$ of $FH_*(M,\triangle)$ as a module over itself. Recall that $\delta_1 : \Gamma_\triangle \cong \Gamma_\omega$. Moreover, in the homotopy exact sequence

$$\ldots \to \pi_2(\triangle) \xrightarrow{i} \pi_2(M) \to \pi_2(M,\triangle) \to \ldots$$

we have $\mathrm{img}(i) \subset \ker c_1 \cap \ker \Omega$, from which it follows that $\Gamma_\Omega \cong \Gamma_\triangle$.

In the following, we parametrize the loops in $\Omega_0\mathrm{Ham}(X,\omega)$ by $[0,2]$ and those in $\Omega_0\mathrm{Ham}(M,\Omega)$ by $[0,1]$. For $\alpha \in \Omega_0\mathrm{Ham}(X,\omega)$, define the reparametrization $\alpha^{(\frac{1}{2})}(t) = \alpha(2t)$ for $t \in [0,1]$. The natural injective map

$$i_+ : \Omega_0\mathrm{Ham}(X,\omega) \to \Omega_0\mathrm{Ham}(M,\Omega) : \alpha \mapsto \alpha_+ = (\alpha^{(\frac{1}{2})}, \mathbb{1})$$

lifts to an injective map $\widetilde{i}_+$ on the corresponding covering spaces (see the proof of Proposition 3.1). For $\widetilde{\alpha} \in \widetilde{\Omega}_0\mathrm{Ham}(X,\omega)$, let $\widetilde{\alpha}_+ = \widetilde{i}_+(\widetilde{\alpha}_+) \in \widetilde{\Omega}_0\mathrm{Ham}(M,\Omega)$ and $\widetilde{\alpha}_- = \widetilde{i}_-(\widetilde{\alpha})$ where $\widetilde{i}_-$ is the lifting of

$$i_- : \Omega_0\mathrm{Ham}(X,-\omega) \to \Omega_0\mathrm{Ham}(M,\Omega) : \alpha \mapsto \alpha_- = (\mathbb{1}, (\alpha^-)^{(\frac{1}{2})})$$

We note that $\widetilde{\alpha}_\bullet$ is determined by the image of any element in $\widetilde{\Omega}_0(M)$ by the unique lifting property of covering space. Take the trivial loop $p = (x,y) \in M$, then $x \in M$ is a trivial loop in $\Omega_0(X)$. Let $\widetilde{\alpha}(\widetilde{x}) = [\alpha(x), w] \in \widetilde{\Omega}_0(X)$, where $\widetilde{x} = [x,x] \in \widetilde{\Omega}_0(X)$. Then $\widetilde{\alpha}_+(\widetilde{p}) = [(\alpha^{(\frac{1}{2})}(x), y), w \times \{y\}]$.

Proposition 3.1. *The following diagram commutes*

$$\begin{array}{ccc} FH_*(M,\triangle;\Omega) & \xrightarrow{\delta_1} & FH_*(X,\omega) \\ \downarrow{\scriptstyle FH_*(\widetilde{\alpha}_\pm)} & & \downarrow{\scriptstyle FH_*(\widetilde{\alpha})} \\ FH_*(M,\triangle;\Omega) & \xrightarrow{\delta_1} & FH_*(X,\omega) \end{array}$$

A similar diagram is commutative with δ_2 and $FH_*(\widetilde{\alpha^-})$ in places of δ_1 and $FH_*(\widetilde{\alpha})$.

Proof: We describe the case for $\widetilde{\alpha}_+$ and $\widetilde{\alpha}_-$ is similar. Let $\widetilde{l} \in \widetilde{\mathcal{P}}_\triangle M$ and $\widetilde{\gamma} = \delta_1(\widetilde{l}) \in \widetilde{\Omega}^{(2)}(X)$. By definition we have $l(t) = (\gamma(t), \gamma(2-t))$ for $t \in [0,1]$ and h_1 acts on l by

$$(\alpha_+ \circ l)(t) = (\alpha_{2t} \circ \gamma(t), \gamma(2-t))$$

Then

$$(\delta_1(\alpha_+ \circ l))(t) = \begin{cases} \alpha_{2t}(\gamma(t)) & \text{for } t \in [0,1] \\ \gamma(t) & \text{for } t \in [1,2] \end{cases}$$

which implies that

$$\delta_1(\alpha_+ \circ l) = (\alpha^{(\frac{1}{2})} \# \mathbb{1}) \circ \gamma = (\alpha^{(\frac{1}{2})} \# \mathbb{1}) \circ \delta_1(l)$$

Notice that $\alpha^{(\frac{1}{2})} \# \mathbb{1}$ and α differ by a reparametrization. The equality above lifts to the covering of the loop spaces and gives a chain level identity for the respective Floer theories. In particular

$$\delta_1 \circ FH_*(\widetilde{\alpha}_+) = FH_*(\widetilde{\alpha}) \circ \delta_1$$

□

For $\alpha \in \Omega_0\mathrm{Ham}(X,\omega)$, let $\widetilde{\alpha}$ be a lifting to $\widetilde{\Omega}_0\mathrm{Ham}(X,\omega)$. The corresponding Seidel element is

$$\Psi^X(\widetilde{\alpha}) := FH_*(\widetilde{\alpha})(\mathbb{1}) \in FH_*(X,\omega)$$

where $\mathbb{1}$ is the unit of the pair of pants product. Moreover, for any other lifting $\widetilde{\alpha}'$ of α, there is $B \in \Gamma_\omega$ such that

$$\Psi^X(\widetilde{\alpha}') = e^B \Psi^X(\widetilde{\alpha})$$

Similarly, the Lagragian Seidel element is given by

$$\Psi^\triangle(\widetilde{\alpha}_+) = \Psi^\triangle(\widetilde{\alpha}_-) = FH_*(\widetilde{\alpha}_+)(\mathbb{1})$$

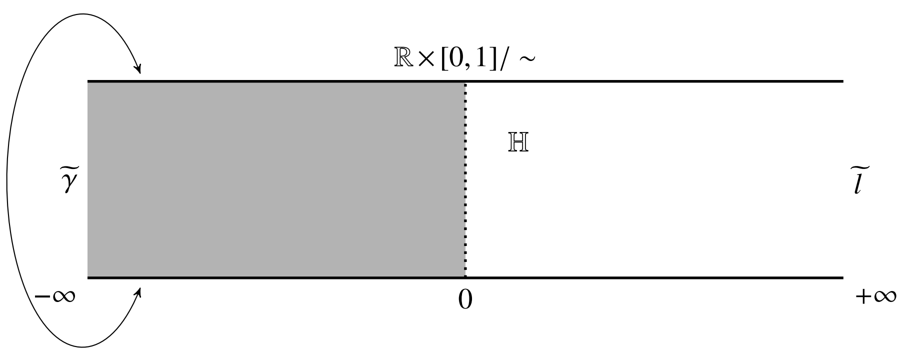
where $\mathbb{1}$ is the unit of the half pair of pants product.

Corollary 3.2. *For $\widetilde{\alpha}$, $\widetilde{\alpha}_\pm$ as given above we have $\delta_1(\Psi^\triangle(\widetilde{\alpha}_+)) = \delta_1(\Psi^\triangle(\widetilde{\alpha}_-)) = \Psi^X(\widetilde{\alpha})$.* □

Since any split loop is the product of $(\alpha, \mathbb{1})$ and $(\mathbb{1}, \alpha')$, it follows that the split loops in $\Omega_0\mathrm{Ham}(M,\Omega)$ generate Seidel elements in $FH_*(X,\omega)$.

3.2 The Albers' map

Here we argue that the Albers' map is well defined for the example under consideration, where $X = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ with $\lambda \in (1,2]$. Recall that the map $\mathscr{A} : FH_*(M,\Omega) \to FH_*(M,\triangle;\Omega)$ is defined by counting of maps from the "chimney domain" $\mathbb{R}\times[0,1]/\sim$:



where $(s,0) \sim (s,1)$ when $s \leqslant 0$, and the conformal structure at $(0,0)$ is given by $\sqrt{z}$. In the figure above, the shaded left half of the strip has its two boundaries glued together forming a half infinite cylinder. At $-\infty$ it converges to $\widetilde{\gamma}$, a critical point for the Floer theory $FH_*(M,\Omega)$, while at $+\infty$ it converges to $\widetilde{l}$, a critical point for the Floer theory $FH_*(M,\triangle;\Omega)$.

In [1], the map $\mathscr{A}$ is defined for monotone Lagrangians. Here, $(M,\triangle)$ is not monotone because

$$c_1(TM)((01\overline{00}) - (10\overline{00})) = 0 \text{ while } \omega((01\overline{00}) - (10\overline{00})) = \lambda - 1 > 0$$

On the other hand, for generic ω-compatible J on X, the class $(01\overline{00}) - (10\overline{00})$ is not represented by J-holomorphic spheres. In fact, the space of non-generic J's has codimension

2. We choose such a generic pair $(\mathbf{H},\mathbf{J})$ (for the Floer theory $FH_*(X,\omega)$) then the corresponding c-symmetric pair $(\mathbb{H},\mathbb{J})$ on (M,Ω) is also generic for the Floer theories $FH_*(M,\Omega)$ and $FH_*(M,\triangle;\Omega)$. Since there is no holomorphic disc with non-positive Maslov number, the compactification of the 0-dimensional "chimney" moduli spaces would not contain disc bubblings. Similarly, we see that sphere bubblings can also be ruled out. It then follows that the map $\mathscr{A}$ is well-defined.

3.3 Non-split loops

We showed that in $\Omega_0\mathrm{Ham}(M,\Omega)$, there could be non-split loops, by computing directly the corresponding Seidel elements in $QH_*(M,\Omega)$. For such loops, Proposition 3.1 does not apply. On the other hand, let $g \in \Omega_0\mathrm{Ham}(M,\Omega)$ be a non-split loop and $\widetilde{g}$ be a lifting to $\widetilde{\Omega}_0\mathrm{Ham}(M,\Omega)$, then it defines a Seidel element $\Psi^M(\widetilde{g}) \in FH_*(M,\Omega)$. The Albers' map [1] $\mathscr{A}$ relates $FH_*(M,\Omega)$ and $FH_*(M,\triangle)$ when it's well defined, in which case, we have

$$\mathscr{A}\circ\Psi^M(\widetilde{g}) = \Psi^{\triangle}(\widetilde{g}) \in FH_*(M,\triangle) \text{ and } \delta_1\circ\mathscr{A}\circ\Psi^M(\widetilde{g}) \in FH_*(X,\omega)$$

Consider $(X,\omega) = (S^2\times S^2, \omega_0\oplus\lambda\omega_0)$ for $\lambda\in(1,2]$ and compute $\delta_1\circ\mathscr{A}\circ\Psi^M(\widetilde{g})$ for a non-split loop g. Also recall that McDuff [4] showed that liftings of the loops in the Hamiltonian group may be chosen such that

$$\Psi^X : \pi_1\mathrm{Ham}(X,\omega) \to QH_*(X,\omega) : \alpha \mapsto \widetilde{\alpha} \mapsto \Psi^X_\alpha := \Psi^X(\widetilde{\alpha})$$

is a group homomorphism. Let $\psi = \Psi^M_{S'}$, i.e.

$$\psi = \left[(01\overline{11}) - (11\overline{10})\right] e^{\frac{1}{2}(1000)+h[(0001)+(1000)]}$$

To compute $\delta_1\circ\mathscr{A}(\psi)$, we note first that $\delta_1\circ\mathscr{A}$ is linear with respect to the identifications of the Novikov rings. Consider the following Seidel elements of split loops:

$$\Psi^M_{R_1} = (01\overline{11})e^{\frac{1}{2}(1000)} \text{ and } \Psi^M_{\overline{R}_2} = -(11\overline{10})e^{-\frac{1}{2}(0001)}$$

then

$$\delta_1\circ\mathscr{A}\circ\Psi^M_{R_1} = \delta_1(\Psi^{\triangle}(R_1)) = \Psi^X(r_1) = (01)e^{\frac{1}{2}(10)}$$

$$\delta_1\circ\mathscr{A}\circ\Psi^M_{\overline{R}_2} = \delta_1(\Psi^{\triangle}(\overline{R}_2)) = \Psi^X(r_2) = (10)e^{\frac{1}{2}(01)}$$

where we use r_i to denote the rotation of the i-th S^2 factor of X. In particular, we recall that via the identifications of Novikov rings,

$$e^{\frac{1}{2}(1000)} \mapsto e^{\frac{1}{2}(10)} \text{ and } e^{-\frac{1}{2}(0001)} \mapsto e^{\frac{1}{2}(10)}$$

It follows that

$$\delta_1\circ\mathscr{A}(\psi) = [(01)+(10)]\, e^{\frac{1}{2}(10)+h[(10)-(01)]} = \Psi^X(s)$$

where s represents the element of infinite order in $\pi_1\mathrm{Ham}(X,\omega)$. In summary, we showed

Proposition 3.3. *Under $\delta_1\circ\mathscr{A}$ the Seidel elements of the non-split loops in* $\mathrm{Ham}(M,\Omega)$ *constructed in [3] map to the Seidel elements of the loops of infinite order in* $\mathrm{Ham}(X,\omega)$. □

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