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KV-Cohomology and Differential Geometry of Affinely Flat Manifolds. Information Geometry

Michel NGUIFFO BOYOM*
UMR CNRS 5149. 3M Département de Mathématiques et de Modélisation
Université Montpellier 2, Montpellier, France
Ferdinand NGAKEU†
UMR CNRS 5149. 3M Département de Mathématiques Université Montpellier2, France
and Université de Douala, Cameroun
Paul Mirabeau BYANDE ‡
UMR CNRS 5149. 3M Département de Mathématiques Université Montpellier2, France
and Université de Yaoundé I, Cameroun
Robert WOLAK§
Jagiellonian University, Poland

Abstract

This paper is devoted to the so-called twisted cohomology of Koszul-Vinberg algebras. We discuss relationships between the twisted cohomology of Koszul-Vinberg algebras and Chevalley-Eilenberg cohomology of the commutator algebra of these algebras. We also discuss some geometry applications of these relationships. For instance we obtain some homological criteria for hyberbolicity and for completeness of locally flat manifolds. We also discuss some topics which are related to twisted cohomology. In particular, we use some techniques of information geometry to discuss canonical representations of locally flat connections.

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1 Introduction

According to a conjecture of Gerstenhaber the cohomology of Koszul-Vinberg algebras (KV-cohomology of KV-algebras) is generated by the deformation theory of locally flat manifold,[7, 15]. From this viewpoint, the deformation of left invariant locally flat structures in a Lie group G generates the deformations of Koszul-Vinberg algebras whose commutator algebra is the Lie algebra of G. Deformations of bi-invariant locally flat structures

*E-mail address:boyom@math.univ-montp2.fr
†E-mail address: fngakeu@yahoo.fr
‡E-mail address: paul.byande@math.univ-montp2.fr and paulbyande@yahoo.fr
§E-mail address: robert.wolak@im.uj.edu.pl

in G will generate the classical Hochschild cohomology of associative algebras[2, 7, 9, 12, 32]

Recently, Nijenhuis has raised the question of relationships between the KV-cohomology of an associative algebra and the Hochschild cohomology of the same associative algebra. One may also raise the question of relationships between the KV-cohomology of a Koszul-Vinberg algebra $\mathcal{A}$ and the Chevalley-Eilenberg cohomology of its commutator Lie algebra,(viz the Lie algebra whose bracket is defined by $[a,b] = ab - ba$). Nowadays these relationships are still unclear. To every two-sided module of a KV-algebra $\mathcal{A}$ we assign a so-called twisted KV-module and twisted cohomology complex. The paper is devoted to the relationships between the cohomology of twisted complexes of KV-algebras and the Chevalley-Eilenberg cohomology of their commutator (Lie algebra). Our viewpoint is illustrated in locally flat manifolds. We show how closely related are the twisted cohomology, the de Rham cohomology and the cohomology of superorder differential forms [14].

The paper consists of five sections. This introduction is labelled Section 1. Section 2 is devoted to two-sided modules of KV-algebras and to their KV-cohomology. Main definitions and formulas are provided. Section 3 is devoted to the so-called twisted modules of two-sided KV-modules and to their twisted cohomology complexes. We show that the derived Chevalley-Eilenberg complexes of the commutator (Lie algebras[4]) are actually sub-complexes of twisted KV-complexes. We show that the canonical pairing between a twisted module and its dual vector space gives rise to a pairing between the twisted cohomology and the twisted homology. The aim of Section 4 consists in studying spectral sequences converging to the twisted cohomology. Our method is classical-like [10, 16]. Thus we obtain the analogue of the Hochschild-Serre spectral sequence [10]. Many relevant properties of the Hochschild-Serre spectral sequence admit their analogues in the spectral sequence of twisted KV-complexes. Section 5 is devoted to geometric interpretation of some properties of the twisted KV-cohomology of locally flat manifolds. We show that the second twisted cohomology space has some interesting properties. For instance, it is a global geometry invariant of the subset of gauge fields whose linear holonomy groups are (pseudo)-euclidean subgroups (see Theorem 5.6). Section 6 is devoted to discussions on information geometry. A few theorems whose proofs will be given elsewhere are stated.

2 KV-algebras and their modules

In this section we collect some useful facts on KV-algebras and we fix notation and terminology [24, 25, 31]. We work with the field $\mathbb{R}$ of real numbers but algebraic considerations we are concerned with are valid for any commutative field of characteristic zero. An algebra $\mathcal{A}$ is an $\mathbb{R}$-vector space endowed with a bilinear map $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. This map μ is the multiplication map of $\mathcal{A}$. For $a, b \in \mathcal{A}$, ab will stand for $\mu(a,b)$. Given an algebra $\mathcal{A}$, the Koszul-Vinberg anomaly (KV-anomaly) of $\mathcal{A}$ is the three-linear map $KV : \mathcal{A}^3 \to \mathcal{A}$ defined by

$$KV(a,b,c) = (ab)c - a(bc) - (ba)c + b(ac). \tag{2.1}$$

Definition 2.1. An algebra $\mathcal{A}$ is called a Koszul-Vinberg algebra (KV-algebra) if its KV-anomaly $\mathcal{A}$ vanishes identically.

KV-algebras are also called left symmetric algebras, and Pre-Lie algebras [3, 17]. Similarly, right symmetric algebras are algebras $(\mathcal{A},\mu)$ whose multiplication satisfies the following condition [6]:

$$\forall a,b,c\in\mathcal{A},\quad (ab)c-a(bc)=(ac)b-a(cb).$$

Example 2.2. a) Associative algebras are KV-algebras.
b) The opposite algebra of a right symmetric algebra is a KV-algebra.
c) Let (M,∇) be a locally flat manifold, viz ∇ is a torsion free linear connection whose curvature tensor vanishes identically. Then the space $\mathcal{X}(M)$ of smooth vector fields on M is a KV-algebra whose multiplication is given by

$$(X,Y)\mapsto X.Y=\nabla_X Y. \tag{2.2}$$

d) Let $\mathcal{F}$ be a Lagrangian foliation in a symplectic manifold (M,ω). The vector space $\mathcal{A}=\mathcal{X}(\mathcal{F})$ of smooth vector fields which are tangent to $\mathcal{F}$ is a KV-algebra[26, 27]. The multiplication $(X,Y)\mapsto X.Y$ of $\mathcal{A}$ is defined by :

$$i_{X.Y}\omega=L_X i_Y\omega. \tag{2.3}$$

e) Let $\mathcal{A}$ and $\mathcal{B}$ be two KV-algebras such that the commutator Lie algebra $\mathcal{A}_L$ acts in $\mathcal{B}$ as infinitesimal automorphisms of the algebra, viz $a.(bb')=(a.b)b'+b(a.b'),\forall a\in\mathcal{A},b,b'\in\mathcal{B}$. Then the vector space $\mathcal{A}\oplus\mathcal{B}$ is endowed with a structure of KV-algebra defined by

$$(a,b)(a',b')=(aa',a.b'+bb'). \tag{2.4}$$

For instance, let X be a smooth vector field on a smooth manifold M. The vector space $\mathcal{A}=C^\infty(M)X$ is a KV-algebra whose multiplication is $(fX).(gX)=f(Xg)X$, $\forall f,g\in C^\infty(M)$. Thereby, $C^\infty(M)X\oplus C^\infty(M)$ is a KV-algebra with the multiplication given by

$$(fX,h)(gX,l)=(f(Xg)X,f(Xl)+hl),\quad \forall f,g,h,l\in C^\infty(M). \tag{2.5}$$

Definition 2.3. Let $\mathcal{A}$ be a KV-algebra and let W be a vector space with two bilinear maps $\mathcal{A}\times W\to W$ and $W\times\mathcal{A}\to W$ which are denoted multiplicatively. The space W is called a two-sided KV-module over $\mathcal{A}$ if $KV(a,b,w)=0$ *and* $KV(a,w,b)=0$, $\forall a,b\in\mathcal{A},\forall w\in W$, where the polarised anomalies are defined by

$$KV(a,b,w)=(ab)w-a(bw)-(ba)w+b(aw),\ KV(a,w,b)=(aw)b-a(wb)-(wa)b+w(ab).$$

A KV-module W is called a left KV-module (respectively a right KV-module) if $wa=0$ (respectively $aw=0$) $\forall a\in\mathcal{A},w\in W$. An element w of the KV-module W is called a Jacobi element if $(ab)w-a(bw)=0$, $\forall a,b\in\mathcal{A}$. The set of Jacobi elements of W is denoted by $J(W)$. Here are some examples of KV-modules

Example 2.4. a) Any KV-algebra $\mathcal{A}$ is a two-sided KV-module over itself, the actions being left multiplication and right multiplication by elements of $\mathcal{A}$.
b) Let (M,∇) be a locally flat manifold. Then the vector space $C^\infty(M)$ of real smooth functions is a left KV-module over the KV-algebra $\mathcal{X}(M)$ by the covariant derivative, (the right action being the trivial action).

c) The tensor product $W \otimes V$ of two-sided modules over $\mathcal{A}$ is a two-sided module under the actions

$$a.(w \otimes v) = aw \otimes v + w \otimes av, \quad (w \otimes v).a = w \otimes va. \tag{2.6}$$

In the sequel, a W-valued q-multilinear map in $\mathcal{A}$,namely $f \in Hom(\mathcal{A}^q, W)$, will be identified with its linearized form $f \in Hom(\otimes^q \mathcal{A}, W)$, allowing us to write $f(a_1 \otimes a_2 ... \otimes a_q) := f(a_1, a_2, ..., a_q)$. We also set

$$(e_i(b)f)(a_1, ..., a_{q-1}) := f(a_1, ..., a_{i-1}, b, a_i ... a_{q-1}) \quad \text{and} \quad (e(b)f) := (e_1(b)f). \tag{2.7}$$

For $\zeta = a_1 \otimes ... \otimes a_q \in \otimes^q \mathcal{A}$, we set $\partial_i \zeta = a_1 \otimes ... \hat{a}_i \otimes ... \otimes a_q$, where $\hat{a}_i$ means that a_i is omitted.

d) If W is a two-sided KV-module over $\mathcal{A}$, then $Hom_{\mathbb{R}}(\otimes^q \mathcal{A}, W)$ is also a two-sided KV-module over $\mathcal{A}$ under the actions

$$(af)(\zeta) = af(\zeta) - f(a.\zeta), \quad (fa)(\zeta) = (f(\zeta))a, \quad \forall a \in \mathcal{A}, \zeta \in \otimes^q \mathcal{A}. \tag{2.8}$$

Combining the left action and the right action of $\mathcal{A}$ in W leads to a new left KV-module structure on $C^q(\mathcal{A}, W)$ given by:

$$(a.f)(\zeta) = a(f(\zeta)) - f(a.\zeta) - (f(\zeta))a \tag{2.9}$$

This action (2.9) will be very useful in subsequent sections.

e) Let $\mathcal{A} = M(n, \mathbb{R})$ be the algebra of real $n \times n$ matrices. We set $W = \mathbb{R}$ and we define the action of $\mathcal{A}$ on W as it follows: $\forall a \in \mathcal{A}, t \in \mathbb{R}, a.t = trace(a)t, \quad t.a = 0$. Then W is a left KV-module over $\mathcal{A}$. One easily checks that W needn't be a module over the associative algebra $\mathcal{A}$.

Let W be a two-sided KV-module over a KV-algebra $\mathcal{A}$. We are going to recall the complex of W- valued KV-cochain in $\mathcal{A}$ (we refer the reader to [25]). This cochain complex is the pair $(C_{KV}(\mathcal{A}, W) = \bigoplus_{q \in \mathbb{Z}} C^q(\mathcal{A}, W), d_{KV})$, where the vector space $C_{KV}(\mathcal{A}, W)$ is graded by the homogeneous subspaces $C^q(\mathcal{A}, W)$ given by: $C^q(\mathcal{A}, W) = \{0\}$, for $q < 0$, $C^0(\mathcal{A}, W) = J(W) = \{w \in W, a(bw) - (ab)w = 0, \forall a, b \in \mathcal{A}, w \in W\}$ and for $q > 0$, $C^q(\mathcal{A}, W) = Hom_{\mathbb{R}}(\otimes^q \mathcal{A}, W)$. The coboundary operator $d_{KV} : C^q(\mathcal{A}, W) \to C^{q+1}(\mathcal{A}, W)$ is defined as follows: $\forall a \in \mathcal{A}, w \in J(W), \forall \zeta \in \otimes^{q+1} \mathcal{A}, \forall f \in C^q(\mathcal{A}, W)$,

$$d_{KV} w(a) = -aw + wa, \quad d_{KV} f(\zeta) = \sum_{i=1}^{q} (-1)^i \{(a_i f)(\partial_i \zeta) + (f(\partial_i(\partial_{q+1} \zeta) \otimes a_i)) a_{q+1}\}. \tag{2.10}$$

We denote by $\mathcal{A}_L$ the commutator algebra of $\mathcal{A}$. Actually every two-sided KV-module over $\mathcal{A}$ is a left module over $\mathcal{A}_L$. These considerations yield two cochain complexes:

(1) The cochain complex $(C_{KV}(\mathcal{A}, W), d_{KV})$ we have just recalled [25].

(2) The W-valued Chevalley-Eilenberg complex $C_{CE}(\mathcal{A}_L, W)$, see also [4]. It is interesting to know the relationships between the cohomologies of the complexes from (1) and (2) .

We set $C^{q,1}(\mathcal{A}, W) = Hom(\Lambda^q \mathcal{A} \otimes \mathcal{A}, W)$. Obviously, $C_N(\mathcal{A}, W) = \bigoplus_{q \geq 0} C^{q,1}(\mathcal{A}, W)$ is a subcomplex of $(C_{KV}(\mathcal{A}, W), d_{KV})$. Furthermore $C_N(\mathcal{A}, W)$ can be viewed as the Chevalley-Eilenberg complex $C_{CE}(\mathcal{A}_L, Hom(\mathcal{A}, W))$.

Theorem 2.5. *[32, 25] If W is a left KV-module over a KV algebra $\mathcal{A}$, then*

$$H_{KV}^{q+1}(C_N(\mathcal{A}, W)) = H_{CE}^{q}(\mathcal{A}_L, Hom(\mathcal{A}, W)).$$

The cohomology $H_{CE}(\mathcal{A}_L, Hom(\mathcal{A}, W))$ is the pioneering KV-cohomology of Nijenhuis [32]. Indeed according to [32], the q^{th} cohomology space $H^q(A, W)$ of a KV-algebra $\mathcal{A}$ is defined by setting $H^q(\mathcal{A}, W) = H_{CE}^{q-1}(\mathcal{A}_L, Hom(\mathcal{A}, W))$. The KV-cohomology theory briefly described here is in fact a solution to the Gerstenhaber conjecture[7]. Its role in the extension theory and in the deformation theory of KV-algebras and their modules is very important [32, 25, 12, 13]. For instance:

Theorem 2.6. *[25] If $\mathcal{A}$ is a KV-algebra and if $H_{KV}^2(\mathcal{A}, \mathcal{A}) = 0$, then $\mathcal{A}$ is rigid.*

Theorem 2.7. *[25] If W is a two-sided KV-module over a KV-algebra $\mathcal{A}$, then $H_{KV}^2(\mathcal{A}, W)$ is isomorphic to the set of equivalence class of exact sequences of KV-algebras*

$$0 \to W \to \mathcal{G} \to \mathcal{A} \to 0$$

where W is equipped with its trivial KV-algebra structure $w.w' = 0$.

Theorem 2.8. *[12] Let $\mathcal{A}$ be a KV-algebra with $H_{KV}^3(\mathcal{A}, \mathcal{A}) = 0$. Then $\forall \zeta \in Z_{KV}^2(\mathcal{A}, \mathcal{A})$ there exists a deformation quantization $(\mathcal{A}, *)$ of $\mathcal{A}$ such that*

$$a * b = ab + t\zeta + \sum_{k>1} t^k \zeta_k, \quad \zeta_k \in C_{KV}^2(\mathcal{A}, \mathcal{A}).$$

Except Theorem 2.5 above, few things are known about the relationships between KV-cohomology $H_{KV}(\mathcal{A}, W)$ and the Chevalley-Eilenberg cohomology $H_{CE}(\mathcal{A}_L, W)$.

The notions of twisted KV-module W_τ associated to a two-sided KV-module W over $\mathcal{A}$ and the notion of twisted KV-complex $C(\mathcal{A}, W_\tau)$ to be introduced below aim to study the relationships between the twisted KV-cohomology of $C(\mathcal{A}, W_\tau)$ and the cohomology of the Chevalley-Eilenberg complex $C_{CE}(\mathcal{A}_L, W_\tau)$.

3 Twisted KV-cohomology and KV-homology

3.1 Twisted KV-cochain complex

Let $\mathcal{A}$ be a KV-algebra and let W be a two-sided KV-module over $\mathcal{A}$. We equip the vector space W with the left module structure $\mathcal{A} \times W \longrightarrow W$ defined by

$$a * w = aw - wa, \quad \forall a \in \mathcal{A}, w \in W. \tag{3.1}$$

One has $KV(a, b, w) = (a, b, w) - (b, a, w) = 0$, where $(a, b, w) = (ab) * w - a * (b * w)$.

Definition 3.1. The left KV-module structure defined by (3.1) is called the twisted KV-module structure derived from the two-sided KV-module W.

The vector space W endowed with the twisted module structure is denoted by W_τ.

As a consequence of Definition 3.1, the map $(a,w) \longrightarrow a * w$ defines on W_τ a left module structure over the Lie algebra $\mathcal{A}_L$. The complex $C_{CE}(\mathcal{A}_L, W_\tau)$ is called the Chevalley-Eilenberg complex of the twisted module. Note that when W is a left KV-module, its twisted KV-module structure coincides with its initial KV-module structure.

Let $\mathcal{A}$ be a KV-algebra and let W be a two-sided KV-module over $\mathcal{A}$. We consider the graded vector space $C_{KV}(\mathcal{A}, W_\tau) = \bigoplus_{q\in\mathbb{Z}} C^q_{KV}(\mathcal{A}, W_\tau)$ where $C^q_{KV}(\mathcal{A}, W_\tau) = \{0\}$ if $q < 0$, $C^0_{KV}(\mathcal{A}, W_\tau) = W_\tau$ and for $q \geq 1$, $C^q_{KV}(\mathcal{A}, W_\tau) = Hom_{\mathbb{R}}(\otimes^q \mathcal{A}, W_\tau)$. When there is no risk of confusion, $C(\mathcal{A}, W_\tau)$ will stand for $C_{KV}(\mathcal{A}, W_\tau)$. Let us define the linear mapping $d : C^q(\mathcal{A}, W_\tau) \longrightarrow C^{q+1}(\mathcal{A}, W_\tau)$ as follows: $\forall w \in W_\tau$, $f \in C^q(\mathcal{A}, W_\tau)$, $a \in \mathcal{A}$ and $\zeta = a_1 \otimes ... \otimes a_{q+1} \in \otimes^{q+1}\mathcal{A}$,

$$(dw)(a) = -aw + wa, \quad (df)(\zeta) = \sum_{i=1}^{q+1} (-1)^i \{a_i * (f(\partial_i\zeta)) - f(a_i.\partial_i\zeta)\} \tag{3.2}$$

where the action $a_i.\partial_i\zeta$ is defined by (2.6). One of the main results of this paper is:

Theorem 3.2. *(i) The pair $(C(\mathcal{A}, W_\tau), d)$ is a cochain complex. Its q^{th} cohomology space is denoted by $H^q_{KV}(\mathcal{A}, W_\tau)$.*

(ii) The graded space $C_N(\mathcal{A}, W_\tau) = W \oplus \sum_{q>0} Hom(\wedge^q \mathcal{A}, W_\tau)$ is a subcomplex of $(C(\mathcal{A}, W_\tau), d)$. Its cohomology coincides with the cohomology of the Chevalley-Eilenberg complex $C_{CE}(\mathcal{A}_L, W_\tau)$.

Proof: We first prove assertion (i). Let $a, b \in \mathcal{A}$ and $w \in W$.

$$\begin{aligned}
d \circ d(w)(a,b) &= -adw(b) + dw(ab) + (dw(b))a + bdw(a) - dw(ba) - dw(a)b \\
&= -a(-bw + wb) + (-(ab)w + w(ab)) + (-bw + wb)a \\
&+ b(-aw + wa) - (-(ba)w + w(ba)) - (-aw + wa)b \\
&= KV(b,a,w) + KV(a,w,b) + KV(w,b,a) \\
&= 0
\end{aligned}$$

Now let $f \in C^q(\mathcal{A}, W_\tau)$, $q \geq 1$. To show that $d^2 f(a_1 \otimes ... \otimes a_{q+2}) = 0$, we fix $i < j$. From the direct computation of $d^2 f(a_1 \otimes ... \otimes a_{q+2})$ we focus on the terms in which a_i and a_j have been removed from their initial positions. Indeed,

$d^2 f(a_1 \otimes ... \otimes a_{q+2}) =$

$$\begin{aligned}
& (-1)^i \left\{ \begin{array}{c} a_i.(df(a_1 \otimes ... \otimes \hat{a}_i \otimes ... \otimes a_j ... \otimes a_{q+2})) \\ -df(a_i.(a_1 \otimes ... \hat{a}_i \otimes ... \otimes a_j ... \otimes a_{q+2})) \\ -df(a_1 \otimes ... \hat{a}_i \otimes ... \otimes a_j ... \otimes a_{q+2}).a_i \end{array} \right\} \\
+ \quad & (-1)^j \left\{ \begin{array}{c} a_j.(df(a_1 \otimes ... \otimes a_i \otimes ... \otimes \hat{a}_j ... \otimes a_{q+2})) \\ -df(a_j.(a_1 \otimes ... \otimes a_i \otimes ... \otimes \hat{a}_j ... \otimes a_{q+2})) \\ -df(a_1 \otimes ... \otimes a_i \otimes ... \otimes \hat{a}_j ... \otimes a_{q+2}).a_j \end{array} \right\} \\
+ \quad & \textit{similar summands}
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
= \quad & (-1)^{i+j+1}\left\{\begin{array}{c} a_i.[a_j.(f(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))] \\ -a_i.f(a_j.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})) \\ -a_i[f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}).a_j] \end{array}\right\} \qquad (3.4)\\
+ \quad & (-1)^{i+j}\left\{\begin{array}{c} a_j.f(a_i.(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})) \\ -f(a_j.[ai.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})]) \\ -[f(a_i.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))].a_j \\ +(a_ia_j).f(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}) \\ -f((a_ia_j).[a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}]) \\ -f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}).(a_ia_j) \end{array}\right\}\\
+ \quad & (-1)^{i+j}\left\{\begin{array}{c} [a_j.(f(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))].a_i \\ -[f(a_j.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))].a_i \\ -[(f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})).a_j].a_i \end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & (-1)^{i+j}\left\{\begin{array}{c} a_j.[a_i.(f(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))] \\ -a_j.f(a_i.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})) \\ -a_j[f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}).a_i] \end{array}\right\}\\
+ \quad & (-1)^{i+j+1}\left\{\begin{array}{c} a_i.f(a_j.(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})) \\ -f(a_i.[a_j.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})]) \\ -[f(a_j.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))].a_i \\ +(a_ja_i).f(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}) \\ -f((a_ja_i).[a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}]) \\ -f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}).(a_ja_i) \end{array}\right\}\\
+ \quad & (-1)^{i+j+1}\left\{\begin{array}{c} [a_i.(f(a_1\otimes...\otimes\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))].a_j \\ -[f(a_i.(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}))].a_j \\ -[(f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})).a_i].a_j \end{array}\right\}\\
+ \quad & \textit{similar summands}
\end{aligned}
\qquad (3.5)
$$

where the first three braces of (3.4) and the last three braces of the same (3.4) come from the expressions in the braces of (3.3) involving $(-1)^i$ and $(-1)^j$ respectively. Now we express (3.4) in term of KV-anomaly. Thus we obtain

$$
\begin{aligned}
(d^2f)(a_1\otimes...\otimes a_{q+2}) \quad = \quad & (-1)^{i+j+1}\left\{\begin{array}{c} KV(a_j,a_i,f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})) \\ KV(a_i,f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}),a_j) \\ KV(f(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2}),a_j,a_i) \\ f(KV[a_i,a_j,(a_1\otimes...\hat{a}_i\otimes...\otimes\hat{a}_j...\otimes a_{q+2})]) \end{array}\right\}\\
+ \quad & \textit{similar summands}
\end{aligned}
$$

According to Definition 2.3 $(d^2f)(a_1\otimes...\otimes a_{q+2}) = 0$. This proves assertion (i). The fact that the complex $C_N(\mathcal{A}, W_\tau)$ is a subcomplex of $(C(\mathcal{A}, W_\tau), d)$ is a straightforward consequence of (3.2).

Let $f \in Hom(\Lambda^q \mathcal{A}, W)$. Taking into account the skew symmetric property of f one has

$$(df)(a_1,...,a_{q+1}) = \sum_{k=1}^{q+1}(-1)^k \left\{ \begin{array}{c} a_k.(f(a_1,...,\hat{a}_k,...,a_{q+1})) \\ -\sum\limits_{s=1,s\neq k}^{q+1} f(a_1,...\hat{a}_k,...a_k a_s, a_{s+1}...,a_{q+1}) \\ -(f(a_1,...\hat{a}_k,...,a_{q+1})).a_k \end{array} \right\} \tag{3.6}$$

$$\begin{aligned} &= \sum_{k=1}^{q+1}(-1)^k a_k * (f(a_1,...,\hat{a}_k,...,a_{q+1})) \\ &\quad + \sum_{k=1}^{q+1}\sum_{s=1,s\neq k}^{q+1} (-1)^{k+1} f(a_1,...\hat{a}_k,...a_k a_s, a_{s+1}...,a_{q+1}) \\ &= \sum_{k=1}^{q+1}(-1)^k a_k * (f(a_1,...,\hat{a}_k,...,a_{q+1})) \\ &\quad + \sum_{k<s}^{q+1}(-1)^{k+s+1} f([a_k,a_s],a_1,...\hat{a}_k,...\hat{a}_s...,a_{q+1}) \\ &= -d_{CE} f(a_1,...,a_{q+1}). \end{aligned}$$

Theorem 3.2 is proved. ◇

Definition 3.3. The cohomology of the complex $(C(\mathcal{A}, W_\tau), d)$ is called the twisted KV-cohomology of $\mathcal{A}$ with coefficients in W. It is denoted by $H_{KV}(\mathcal{A}, W_\tau)$ or simply $H(\mathcal{A}, W_\tau)$ when there is no risk of confusion.

Example 3.4. If $\mathcal{A}$ if an associative and commutative algebra, then we have

$$H^0(\mathcal{A},\mathcal{A}_\tau) = \mathcal{A}, \quad H^1(\mathcal{A},\mathcal{A}_\tau) = End(\mathcal{A}), \quad H^2(\mathcal{A},\mathcal{A}_\tau) = Hom(\mathcal{A}\wedge\mathcal{A},\mathcal{A}).$$

Example 3.5. Let (M,α) be a contact manifold whose Reeb vector field is denoted by $\mathcal{R}$. Let $\mathcal{A} = C^\infty(M)\mathcal{R} \rtimes C^\infty(M)$ be the semi-direct product KV-algebra whose multiplication is defined in (2.5). We consider its two-sided ideal $W = C^\infty(M)$. Actually the contact form α is a cocycle of the twisted KV-complex $C(\mathcal{A}, C^\infty(M))$ whose cohomology class $[\alpha]$ never vanishes when M is compact.

Here are other examples with some vanishing twisted cohomology spaces.

Example 3.6. Let $\mathcal{A} = M(n,\mathbb{R})$ be the associative algebra of $n\times n$ matrices. One has $\mathcal{A}_L = \mathcal{G}l(n,\mathbb{R})$.

(1) For $W = \mathbb{R}$ equipped with its trivial KV-module structure over $\mathcal{A}$, we have $H^q(\mathcal{A},\mathbb{R}_\tau) = 0$, $1 \le q \le 2$.. Indeed, let f be a W-valued one dimensional cocycle. Then $f(ab) = 0$ $\forall a,b \in \mathcal{A}$. Therefore $f = 0$.

If f is W-valued two dimensional cocycle then $f(ab,c) + f(b,ac) = f(ba,c) + f(a,bc)$ $\forall a,b,c \in CA$. In particular taking $b = 1$ yields $f(a,c) = f(1,ac)$ $\forall a,c \in \mathcal{A}$. Thus every two dimensional cocycle is exact.

(2) For $W = \mathcal{A}_\tau$, one checks also that $H^2(\mathcal{A},\mathcal{A}_\tau) = 0$.

Indeed let f be a $\mathcal{A}_\tau$-valued two dimensional cocycle of $\mathcal{A}$. The action of $\mathcal{A}$ in $\mathcal{A}_\tau$ defined as $a.m = am - ma, a \in \mathcal{A}, m \in \mathcal{A}_\tau$. Thereby given $a,b,c \in \mathcal{A}$ one has

$$a.f(b,c) - f(ab,c) - f(b,ac) = b.f(a,c) - f(ba,c) - f(a,bc).$$

Therefore taking $b = 1$ yields $f(a,c) = -a.f(1,c) + f(1,ac)$. Thus f is exact.

(3) Let $\mathcal{A} = M(n,\mathbb{R})$ and let $W = \mathbb{R}$ with the KV-module structure defined as it follows: $\forall a \in \mathcal{A}, \forall t \in \mathbb{R}, \quad a * t = trace(a)t,\ t * a = 0$. Note that the vector space $\mathbb{R}$ equipped with these actions is not a module over the associative algebra $\mathcal{A}$ but rather a KV-module over the KV-algebra $\mathcal{A}$. One has $H^q(\mathcal{A},\mathbb{R}_\tau) = 0,\ 1 \le q \le 2$.

If f is a one dimensional cocycle, then it is easy to see that $f(a) = trace(a)f(1), \forall a \in \mathcal{A}$.

If f is a two dimensional cocycle then it is easy to see that $(n-1)f(a,c) = trace(a)f(1,c) - f(1,ac)$. Thus f is exact.

3.2 Twisted KV-homology

Let us consider the graded vector space $C_*(\mathcal{A},W_\tau) = \bigoplus_{q\in\mathbb{Z}} C_q(\mathcal{A},W_\tau)$ where $C_q(\mathcal{A},W_\tau) = \{0\}$ for $q < 0$, $C_0(\mathcal{A},W_\tau) = W_\tau$ and $C_q(\mathcal{A},W_\tau) = (\otimes^q\mathcal{A}) \otimes W_\tau$ for $q \ge 1$. The vector space $C_q(\mathcal{A},W_\tau)$ is a left KV-module over $\mathcal{A}$ under the following action

$$a \bullet (\zeta \otimes w) = a\zeta \otimes w + \zeta \otimes a * w \tag{3.7}$$

We define the linear map $\delta_q : C_q(\mathcal{A},W_\tau) \to C_{q-1}(\mathcal{A},W_\tau)$ as it follows: $\forall f \in C_q(\mathcal{A},W_\tau), \forall \zeta = a_1 \otimes ... \otimes a_q \in \otimes^q\mathcal{A}$ and $w \in W_\tau$

$$\delta_0(w) = 0 \quad and \quad \delta_q(\zeta \otimes w) = \sum_{i=1}^{q}(-1)^i a_i \bullet (\partial_i\zeta \otimes w),\ for\ q \ge 1. \tag{3.8}$$

Theorem 3.7. *The pair $(C_q(\mathcal{A},W_\tau),\delta)$ is a chain complex.*

Proof: Obviously for $q \le 1$ and $\eta \in C_q(\mathcal{A},W_\tau)$ we have $\delta^2\eta = 0$. For $\eta = a \otimes b \otimes w \in \mathcal{A} \otimes \mathcal{A} \otimes W$, on easily checks that $\delta^2\eta = KV(a,b,w) + KV(b,w,a) + KV(w,a,b) = 0$.

Now let $\eta = a_1 \otimes ... \otimes a_q \otimes w$ with $q \ge 3$. By virtue of (2.6), the expressions (3.8) yield

$$\delta(a_1 \otimes ... \otimes a_q \otimes w) = \sum_{i=1}^{q}(-1)^i \left\{ \begin{array}{c} a_1 \otimes ... \otimes \hat{a}_i \otimes ... \otimes a_q \otimes wa_i \\ -a_i.(a_1 \otimes ... \hat{a}_i \otimes ... \otimes a_q \otimes w) \end{array} \right\}.$$

To prove that $\delta^2\eta = 0$, we proceed as in the proof of Theorem 3.2. Thus we fix $i < j$ and we focus on the summands of $\delta^2\eta$ in which a_i and a_j have been removed from their initial positions i and j. So, we have

$$\begin{aligned} \delta^2\eta = \quad & (-1)^i\delta\left\{ \begin{array}{c} a_1 \otimes ... \otimes \hat{a}_i \otimes ... a_j ... \otimes a_q \otimes wa_i \\ -a_i.(a_1 \otimes ... \hat{a}_i \otimes ... a_j ... \otimes a_q \otimes w) \end{array} \right\} \\ & +(-1)^j\delta\left\{ \begin{array}{c} a_1 \otimes ... \otimes a_i \otimes ... \hat{a}_j ... \otimes a_q \otimes wa_j \\ -a_j.(a_1 \otimes ... a_i \otimes ... \hat{a}_j ... \otimes a_q \otimes w) \end{array} \right\}. \\ & +similar\ summands \end{aligned}$$

According to (2.6) we have

$$\begin{aligned} a_i.(a_1\otimes ...\hat{a}_i\otimes ...a_j...\otimes a_q\otimes w) &= a_1\otimes ...\hat{a}_i\otimes ...a_j...\otimes a_q\otimes a_i w \\ &+ a_1\otimes ...\hat{a}_i\otimes ...a_i a_j...\otimes a_q\otimes w \\ &+ \sum_{k\neq j}(a_1\otimes ...\hat{a}_i\otimes ...a_i a_k...a_j...\otimes a_q\otimes w) \end{aligned}$$

and $\sum_{k\neq i,j} a_1\otimes ...\hat{a}_i\otimes ...a_i a_k...\hat{a}_j...\otimes a_q\otimes w a_j = a_i.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes w a_j.$

Thus using the expressions above we obtain

$$\begin{aligned} \delta^2\eta = ... &+ (-1)^{i+j+1}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes (wa_i)a_j \\ -a_j.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes wa_i \\ -a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes a_j(wa_i) \end{array}\right\} \\ &+ (-1)^{i+j}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes w(a_i a_j) \\ -(a_i a_j).(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes w \\ -a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes (a_i a_j)w \end{array}\right\} \\ &+ (-1)^{i+j}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes (a_i w)a_j \\ -a_j.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes a_i w \\ -a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes a_j(a_i w) \end{array}\right\} \\ &+ (-1)^{i+j}\left\{ \begin{array}{c} a_i.(a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes wa_j \\ -a_j.[a_i.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)]\otimes w \\ -a_i.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes a_j w \end{array}\right\} \\ &+ (-1)^{i+j}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes (wa_j)a_i \\ -a_i.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes wa_j \\ -a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes a_i(wa_j) \end{array}\right\} \\ &+ (-1)^{i+j+1}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes w(a_j a_i) \\ -(a_j a_i).(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes w \\ -a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes (a_j a_i)w \end{array}\right\} \\ &+ (-1)^{i+j+1}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes (a_j w)a_i \\ -a_i.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes a_j w \\ -a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes a_i(a_j w) \end{array}\right\} \\ &+ (-1)^{i+j+1}\left\{ \begin{array}{c} a_j.(a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes wa_i \\ -a_i.[a_j.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)]\otimes w \\ -a_j.(a_1\otimes ...\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q)\otimes a_i w \end{array}\right\}. \end{aligned}$$

When we write the previous expression in term of anomaly, $\delta^2\eta$ reduces itself to

$$\begin{aligned} \delta^2\eta = \quad &+(-1)^{i+j+1}\left\{ \begin{array}{c} a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes KV(w,a_i,a_j) \\ +a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes KV(a_i,a_j,w) \\ +a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q\otimes KV(a_j,w,a_i) \\ +KV(a_i,a_j,(a_1\otimes ...\otimes\hat{a}_i\otimes ...\hat{a}_j...\otimes a_q))\otimes w \end{array}\right\} \\ &+similar\ summands \end{aligned}$$

So we finally have

$$\begin{aligned}\delta^2\eta &= \sum_{i<j}(-1)^{i+j+1}\left\{\begin{array}{l} a_1\otimes\ldots\otimes\hat{a}_i\otimes\ldots\hat{a}_j\ldots\otimes a_q\otimes KV_{cyc}(w,a_i,a_j)\\ +KV(a_i,a_j,(a_1\otimes\ldots\otimes\hat{a}_i\otimes\ldots\hat{a}_j\ldots\otimes a_q))\otimes w\end{array}\right\}\\ &= 0\end{aligned}$$

where $KV_{cyc}(w,a_i,a_j) = [KV(w,a_i,a_j)+KV(a_i,a_j,w)+KV(a_j,w,a_i)]$. Thereby $\delta^2 = 0$. Theorem 3.7 is proved. $\diamond$

Definition 3.8. The homology of the complex $(C_*(\mathcal{A},W_\tau),\delta)$ is called W_τ-valued twisted KV-homology on $\mathcal{A}$ and denoted by $H_*(\mathcal{A},W_\tau)$.

3.3 Pairing between $H^*(\mathcal{A},W^*_\tau)$ and $H_*(\mathcal{A},W_\tau)$

We denote by W^*_τ the dual vector space of W, endowed with the left action of $\mathcal{A}$ given by

$$(a\theta)(w) = -\theta(a*w), \quad \forall\theta\in W^*, \forall a\in\mathcal{A}, w\in W. \tag{3.9}$$

The canonical pairing $W\times W^*\longrightarrow\mathbb{R}$ induces a pairing $C^q(\mathcal{A},W^*_\tau)\times C_q(\mathcal{A},W_\tau)\longrightarrow\mathbb{R}$ which is defined as it follows: $\forall f\in C^q(\mathcal{A},W^*_\tau)$, $\forall\eta=\zeta\otimes w\in\otimes^q\mathcal{A}\otimes W$

$$< f,\eta > = < f(\zeta),w > . \tag{3.10}$$

This pairing satisfies $< df,\eta > = < f,\delta\eta >$, therefore it induces a pairing $H^q(\mathcal{A},W^*_\tau)\times H_q(\mathcal{A},W_\tau)\longrightarrow\mathbb{R}$.

Theorem 3.9. *The pairing $H^q(\mathcal{A},W^*_\tau)\times H_q(\mathcal{A},W_\tau)\longrightarrow\mathbb{R}$ is nondegenerate.*

Proof: Let us denote by $<,>$ the canonical duality pairing $W^*\times W\to\mathbb{R}$. $(\theta,w)\mapsto <\theta,w> = \theta(w)$. This bracket is extended to the bilinear mapping $<,>$: $C^q(\mathcal{A},W^*_\tau)\times C_q(\mathcal{A},W_\tau)\to\mathbb{R}$. Let $f\in C^q(\mathcal{A},W^*_\tau)$ and let $\eta = a_1\otimes\ldots\otimes a_q\otimes w\in C_q(\mathcal{A},W_\tau)$, we set

$$< f,\eta > = < f(a_1\otimes\ldots\otimes a_q),w > .$$

It is then easy to check that $d: C^q(\mathcal{A},W^*_\tau)\to C^{q+1}(\mathcal{A},W^*_\tau)$ is the transpose of the map $\delta: C_{q+1}(\mathcal{A},W_\tau)\to C_q(\mathcal{A},W_\tau)$, in the following meaning

$$< df,\eta > = < f,\delta\eta > \tag{3.11}$$

for all $f\in C^q(\mathcal{A},W^*_\tau)$, $\eta\in C_{q+1}(\mathcal{A},W_\tau)$. We deduce a pairing $H^q(\mathcal{A},W^*_\tau)\times H_q(\mathcal{A},W_\tau)\to\mathbb{R}$. Indeed let f be an element of the vector space $Z^q(\mathcal{A},W^*_\tau)$ of q- cocycles. For any cycle $\eta = \sum_{i_1,\ldots,i_{q+1}} a_{i_1}\otimes\ldots\otimes a_{i_q}\otimes w_{i_{q+1}}$ in the vector space $Z_q(\mathcal{A},W_\tau)$ of q-cycles, we set $<[f],[\eta]> = \sum_{i_1,\ldots,i_{q+1}} < f(a_{i_1}\otimes\ldots\otimes a_{i_q}),w_{i_{q+1}} > = < f,\eta >$. Property 3.11 shows that this last formula is well defined. This pairing is nondegenerate. Indeed, let $[f]\in H^q(\mathcal{A},W_\tau)$ such that $<[f],[\eta]> = < f,\eta > = 0$ for all $\eta\in Z_q(\mathcal{A},W_\tau)$. From the short exact sequence

$$0\longrightarrow Z_q(\mathcal{A},W_\tau)\xrightarrow{i} C_q(\mathcal{A},W_\tau)\xrightarrow{\delta} B_{q-1}(\mathcal{A},W_\tau)\longrightarrow 0,$$

we deduce that f is a linear form defined on $C_q(\mathcal{A}, W_\tau)$ which vanishes on the kernel $Z_q(\mathcal{A}, W_\tau)$ of δ. The map f gives rise to a linear map $\tilde{f} : B_{q-1}(\mathcal{A}, W_\tau) \to \mathbb{R}$. Furthermore there exists a linear map $\hat{f} : C_{q-1}(\mathcal{A}, W_\tau) \to \mathbb{R}$ whose restriction to $B_{q-1}(\mathcal{A}, W_\tau)$ is $\tilde{f}$. Then we have $< \hat{f}, \delta\eta >=< d\hat{f}, \eta >$, hence $< d\hat{f}, \eta >=< f, \eta >$, $\forall \eta \in C_q(\mathcal{A}, W_\tau)$ i.e. $f = d\hat{f}$. ⋄

Here is an interesting example of pairing:

Example 3.10. : Let $\lambda \in \mathbb{R}, \lambda > 1$ and $\Gamma = \{\lambda^m, m \in \mathbb{Z}\}$ be the subgroup of the multiplicative group $\mathbb{R}^*$ generated by λ. We consider the locally flat Hopf manifold (M, ∇) where $M = \Gamma \backslash (\mathbb{R}^n - \{0\})$ and ∇ is the linear connection induced on M by the canonical linear connection D on $\mathbb{R}^n$. Let $\mathcal{A} = (\mathcal{X}(M), \nabla)$ be the KV-algebra of (M, ∇). If we restrict our interest on the space $C_0(\mathcal{A}, C^\infty(M))$ of tensorial cochains (see in section 5 below), then we prove that $H_0^2(\mathcal{A}, C^\infty(M)) = H_{dR}^2(M)$. As consequence of the last equality, the twisted homology space $H_2(\mathcal{A}, C^\infty(M))$ is isomorphic the the singular homology space $H_2^s(M, \mathbb{R})$.

4 The analogue of the Hochschild-Serre Spectral Sequence for the Twisted KV-Complex

It is well known that spectral sequences are useful tools for computing the cohomology [10, 16]. For instance the Hochschild-Serre spectral sequence associated to a pair $(\mathcal{H}, \mathcal{G})$, where $\mathcal{H}$ is an ideal in a Lie algebra $\mathcal{G}$, is an efficient tool for computing the Chevalley-Eilenberg cohomology of $\mathcal{G}$, see also [15]. We plan to show that the twisted KV-complex admits an analog of the Hochschild-Serre spectral sequence with similar properties. The main results of this section are summarised in the following two theorems:

Theorem 4.1. *Let $C(\mathcal{A}, W_\tau)$ be the twisted KV-complex associated to a two-sided module W. To every ideal I of $\mathcal{A}$ is associated a spectral sequence $(E_r^{p,q})$ converging to $H_{KV}(\mathcal{A}, W_\tau)$ and such that the term E_1 is given by*

$$E_1^{p,q} = H^q(I, Hom(\otimes^p(A/I), W_\tau)$$

Theorem 4.2. *If $\mathcal{A}$ contains a two-sided KV-ideal I such that $I\mathcal{A} = \mathcal{A}I = 0$, then term E_2 of the spectral sequence in Theorem 4.1 is*

$$E_2^{p,q} = H^p(\mathcal{A}/I, H^q(I, W_\tau)).$$

Proofs of Theorem 4.1 and Theorem 4.2 are based on the techniques involving filtered graded complexes.

4.1 Filtration of the complex $(C(\mathcal{A}, W_\tau), d)$.

Definition 4.3. Let $\mathcal{A}$ be a KV-algebra, W a two-sided KV-module over $\mathcal{A}$ and let I be a KV-ideal of $\mathcal{A}$. We define the bounded filtration $(F^j)_{j\geq 0}$ of the complex $(C(\mathcal{A}, W_\tau), d)$ by:

$$F^0C = C(\mathcal{A}, W_\tau),\ F^jC = \bigoplus_{q\geq 1} F^jC \cap C^q(\mathcal{A}, W_\tau) \tag{4.1}$$

where $F^jC \cap C^q(\mathcal{A}, W_\tau) = 0$ for $j > q$ and for $j \leq q$, $f \in F^jC \cap C^q(\mathcal{A}, W_\tau)$ iff f is a q-cochain with $f(a_1, ..., a_q) = 0$ whenever $q - j + 1$ arguments a_i belong to the ideal I.

Often $F^jC \cap C^q$ will stand for $F^jC \cap C^q(\mathcal{A}, W_\tau)$. It is easy to verify that

$$F^{j+1}C \cap C^q \subset F^jC \cap C^q \quad \text{and} \quad d(F^jC \cap C^q) \subset F^jC \cap C^{q+1}. \tag{4.2}$$

Though W is not a module over $\mathcal{A}/I$, to simplify notation, $C^q(\mathcal{A}/I, W)$ will stand for $Hom((\mathcal{A}/I)^{\otimes q}, W)$. Therefore we will use the identification

$$F^qC \cap C^q(\mathcal{A}, W_\tau) = Hom((\mathcal{A}/I)^{\otimes q}, W) = C^q(\mathcal{A}/I, W). \tag{4.3}$$

The term E_0 of the spectral sequence $E_r^{p,q}$ associated to the filtration (F^jC) is

$$E_0^{j,q-j} = \frac{F^jC \cap C^q(\mathcal{A}, W_\tau)}{F^{j+1}C \cap C^q(\mathcal{A}, W_\tau)} \tag{4.4}$$

Considering (4.3), the left action of I in $C^q(\mathcal{A}/I, W)$ is given by:

$$(t.f)(\bar{b}_1, ..., \bar{b}_q) = t(f(\bar{b}_1, ..., \bar{b}_q)) - (f(\bar{b}_1, ..., \bar{b}_q))t, \tag{4.5}$$

Then it is relevant to consider the twisted complex $(\bigoplus_{q,j} C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau), d)$.

From now on, elements of I will be denoted $t, t_1, ..., t_i, etc.$ and those of $\mathcal{A}/I$ will be denoted by $\overline{b}, \overline{b_1}, ..., \overline{b_i}, etc$. Now we compute the term (E_1) of the spectral sequence $(E_r)_{r\in\mathbb{N}}$ associated to the filtration (F^jC).

Lemma 4.4. *The linear map* $r_{q,j} : F^j \cap C^q(\mathcal{A}, W_\tau) \to C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$ *defined by*

$$(r_{q,j}f)(t_1, ..., t_{q-j})(\overline{b_1}, ..., \overline{b_j}) = f(t_1, ..., t_{q-j}, b_1, ..., b_j) \tag{4.6}$$

is surjective and induces the isomorphism

$$\hat{r}_{q,j} : \frac{F^jC \cap C^q(\mathcal{A}, W_\tau)}{F^{j+1}C \cap C^q(\mathcal{A}, W_\tau)} = E_0^{j,q-j} \longrightarrow C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau). \tag{4.7}$$

Proof: The map $r_{q,j}$ is well defined. Indeed replacing b_k by $b'_k = b_k + t_k$ with $t_k \in I$, the right side of (4.6) does not change. To show that $r_{q,j}$ is surjective, let us consider $g \in C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$. It defines a q-multilinear map $\bar{g} \in C^q(I^{q-j} \times (\mathcal{A}/I)^j, W)$ given by

$$\bar{g}(t_1, ..., t_{q-j}, \bar{b}_1, ..., \bar{b}_j) := g(t_1, ..., t_{q-j})(\bar{b}_1, ..., \bar{b}_j).$$

Now we consider the linear map $\rho : I^{q-j} \times \mathcal{A}^j \longrightarrow I^{q-j} \times (\mathcal{A}/I)^j$ defined by

$$\rho(t_1, ..., t_{q-j}, b_1, ..., b_j) = (t_1, ..., t_{q-j}, \bar{b}_1, ..., \bar{b}_j).$$

We extend $\bar{g} \circ \rho$ to a q-multilinear map f defined in $\mathcal{A}$. This extension f belongs to $F^jC \cap C^q(\mathcal{A}, W_\tau)$. It is easy to check that $r_{q,j}(f) = g$. Of course $\ker(r_{q,j})$ is nothing but $F^{j+1}C \cap C^q(\mathcal{A}, W_\tau)$. ◇

4.2 The term $E_1^{j,q-j}$ of the spectral sequence $(E_r^{p,q})$.

From the discussions of the previous subsection we have

Lemma 4.5. *The following diagramm is commutative*

$$\begin{array}{ccccccccc}
0 \to & F^{j+1}C \cap C^{q+1} & \overset{i}{\to} & F^jC \cap C^{q+1} & \overset{\pi}{\to} & E_0^{j,q+1-j} & \overset{\hat{r}_{q+1,j}}{\to} & C^{q+1-j}(I, C^j(\mathcal{A}/I, W)_\tau) \to 0 \\
 & \uparrow d & & \uparrow d & & \uparrow d_0 & & \uparrow d_I \\
0 \to & F^{j+1}C \cap C^q & \overset{i}{\to} & F^jC \cap C^q & \overset{\pi}{\to} & E_0^{j,q-j} & \overset{\hat{r}_{q,j}}{\to} & C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau) \to 0
\end{array}$$

where i and π are respectively the canonical injection and the projection; d_0 is the differential operator deduced from d and d_I is the twisted coboundary operator.

Proof: From the definition of d, i, π and d_0, it is easy to check that the part of the diagramm involving these maps is commutative. What remains to be proved is $d_I \circ \hat{r}_{q,j} = \hat{r}_{q+1,j} \circ d_0$. which in turn is equivalent to

$$r_{q+1,j}(df) = d(r_{q,j}f), \quad \forall f \in F^jC \cap C^q. \tag{4.8}$$

Indeed, $\hat{r}_{q,j}([f]) = r_{q,j}(f)$. So let us set $T = t_1 \otimes ... \otimes t_{q-j+1} \in I^{\otimes^{q-j+1}}$, $\overline{B} = \overline{b_1} \otimes ... \otimes \overline{b_j} \in (A/I)^{\otimes j}$, $B = b_1 \otimes ... \otimes b_j \in A^{\otimes j}$ with $\overline{b_j} \in \mathcal{A}/I$, and $\partial_k T = t_1 \otimes ... \hat{t_k} ... \otimes t_{q-j+1}$. Then we have

$$\begin{aligned}
(r_{q+1,j}(df)(T))(\bar{B}) &= df(T \otimes B) \\
&= \sum_{k=1}^{q-j+1} (-1)^k \left\{ \begin{array}{l} t_k(f(\partial_k T \otimes B)) - f(t_k.(\partial_k T) \otimes B) \\ -f(\partial_k T \otimes t_k.B) - (f(\partial_k T \otimes B))t_k \end{array} \right\} \\
&+ \sum_{s=1}^{j} (-1)^{q-j+1+s} \left\{ \begin{array}{l} b_s(f(T \otimes \partial_s B)) - f(b_s.T \otimes \partial_s B) \\ -f(T \otimes b_s.\partial_s B) - (f(T \otimes \partial_s B))b_s \end{array} \right\}.
\end{aligned} \tag{4.9}$$

On the other hand we have

$$\begin{aligned}
(d(r_{q,j}f)(T))(\bar{B}) &= \sum_{k=1}^{q-j+1} (-1)^k \left\{ \begin{array}{l} (t_k(r_{q,i}f(\partial_k T))(\bar{B}) \\ -(r_{q,i}f(t_k.\partial_k T))(\bar{B}) \\ -((r_{q,i}f(\partial_k T))t_k)(\bar{B}) \end{array} \right\} \\
&= \sum_{k=1}^{q-j+1} (-1)^k \left\{ \begin{array}{l} t_k(f(\partial_k T \otimes B)) \\ -f(t_k.\partial_k T \otimes B) \\ -(f(\partial_k T \otimes B))t_k \end{array} \right\}
\end{aligned} \tag{4.10}$$

Now combining (4.9) and (4.10) gives

$$\begin{aligned}
(r_{q+1,j}(df)(T))(\bar{B}) &= (d(r_{q,j}f)(T))(\bar{B}) + (-1)^{q-j+1} d(e(T)f)(B) \\
&- \sum_{k=1}^{q-j+1} (-1)^k f(\partial_k T \otimes t_k.B) - \sum_{s=1}^{j} (-1)^{q-j+1+s} f(b_s.T \otimes \partial_s B).
\end{aligned} \tag{4.11}$$

with $(e(T)f)(B) = f(T \otimes B)$. Since $f \in F^jC \cap C^q$ we have $e(T)f = 0$, $f(\partial_k T \otimes t_k B) = 0$ and $f(b_s.T \otimes B) = 0$. Thereby we have

$$(r_{q+1,j}(df)(T))(\bar{B}) = (d(r_{q,j}f)(T))(\bar{B}).$$

This ends the proof of Lemma 4.5. ⋄

Theorem 4.6. *The term E_1 of the spectral sequence corresponding to the filtration (F^jC) of $C(\mathcal{A}, W_\tau)$ above is given by*

$$E_1^{j,q-j} \simeq H^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau). \tag{4.12}$$

The proof of Theorem 4.6 is based on Lemma 4.5

4.3 The term $E_2^{j,q-j}$ of the spectral sequence $(E_r^{p,q})$.

It is well known that each vector space $E_2^{q,0}$, $q \geq 0$ of the Hochschild-Serre spectral sequence associated to an ideal I of a Lie algebra $\mathcal{A}$ is isomorphic to the relative cohomology space $H^q_{CE}(\mathcal{A}, I, W_\tau)$, (see [10]). This fact has its analogue in the twisted cohomology of Koszul-Vinberg algebras.

Definition 4.7. Let A be a KV-algebra, let W be a two-sided KV-module over $\mathcal{A}$ and let I be a KV-ideal of $\mathcal{A}$. The relative cochain complex corresponding to the ideal I is defined by: $C(\mathcal{A}, I, W_\tau) = \bigoplus_{q\in\mathbb{Z}} C^q(\mathcal{A}, I, W_\tau)$ where

(i) $C^q(\mathcal{A}, I, W_\tau) = \{0\}$ for $q < 0$,

(ii) $C^0(\mathcal{A}, I, W_\tau) = W_\tau^I = \{w \in W, \forall t \in I, -tw + wt = 0\}$,

(iii) $C^q(\mathcal{A}, I, W_\tau) = \{f \in C^q(\mathcal{A}, W_\tau), \forall t \in I, t.f = 0 \text{ and } \forall i \leq q,\ e_i(t)f = 0\}$.

Remark. Taking into account the linearization of multilinear maps $f(a_1, ..., a_q) = f(a_1 \otimes ... \otimes a_q)$, the property (iii) in Definition 4.7 is equivalent to: $f \in C^q(\mathcal{A}, I, W_\tau)$ iff $t.f = 0$, $\forall t \in I$ and f vanishes identically in the ideal generated by I in the tensor algebra $T(A) = \bigoplus_q A^{\otimes q}$ and f is W_τ^I-valued mapping. Note the following identification

$$C^q(\mathcal{A}, I, W_\tau) = (F^qC \cap C^q(\mathcal{A}, W_\tau))^I = C^q(\mathcal{A}/I, W)^I. \tag{4.13}$$

The first identification $F^qC \cap C^q(\mathcal{A}, W) = C^q(\mathcal{A}/I, W)$ means that an element $\bar{f} \in C^q(\mathcal{A}/I, W)$ is identified with its image $f \in F^qC \cap C^q(\mathcal{A}, W_\tau)$ defined by $f(b_1, ..., b_q) = \bar{f}(\bar{b_1}, ..., \bar{b_q})$ with $b_j \in \overline{b_j}$.

Theorem 4.8. *The complex $(C(\mathcal{A}, I, W_\tau), d)$ is a subcomplex of the twisted KV-cochain complex $(C(\mathcal{A}, W_\tau), d)$, in other words, $d(C^q(\mathcal{A}, I, W_\tau)) \subset C^{q+1}(\mathcal{A}, I, W_\tau)$.*

Proof: Let $f \in C^q(\mathcal{A}/I, W)^I, t \in I, 1 \leq i \leq q+1$. We have to show that $e_i(t)(df) = 0$ and $t.(df) = 0$. Let $\overline{B} = \overline{b_1} \otimes ... \otimes \overline{b_{q+1}} \in (A/I)^{\otimes q+1}$, $B = b_1 \otimes ... \otimes b_j \in \mathcal{A}^{\otimes^j}$ with $b_i \in \overline{b_j}$.

$$\begin{aligned} df(b_1 \otimes ... \otimes b_{q+1}) &= \sum_{k=1}^{q+1} (-1)^k \{b_k f(\partial_k B) - f(b_k.\partial_k B) - f(\partial_k B) b_k\} \\ &= \sum_{k=1}^{q+1} (-1)^k (b_k f)(\partial_k B) \end{aligned}$$

Now suppose that $B = b_1 \otimes ... \otimes b_{q+1}$ contains a factor $b_i \in I$, then $df(B) = (-1)^j (b_j f)(\partial_j B) + \sum_{k \neq j} (-1)^k (b_k f)(\partial_k B)$. We have $(b_i f) = 0$ since $b_j \in I$ and $(b_k f)(\partial_k B) = b_k(f(\partial_k B)) - f(b_k.\partial_k B) - f(\partial_k B) b_k$. For $k \neq j$, $\partial_k B$ and $b_k.\partial_k B$ belong to the ideal of $T(A)$ generated by I, therefore $(b_k f)(\partial_k B) = 0$. So, $e_i(t)(df) = 0$. On the other hand , $\forall t \in I$, we have

$$\begin{aligned}
(tdf)(B) &= t(df(B)) - df(t.B) - (df(B))t \\
&= t(df(B)) - (df(B))t \\
&= \sum_{k=1}^{q+1} (-1)^k \left\{ \begin{array}{c} t[b_k(f(\partial_k B)) - f(b_k.\partial_k B) - (f(\partial_k B))b_k] \\ -[b_k(f(\partial_k B)) - f(b_k.\partial_k B) - (f(\partial_k B))b_k]t \end{array} \right\} \\
&= \sum_{k=1}^{q+1} (-1)^k \left\{ \begin{array}{c} t(b_k(f(\partial_k B))) - t((f(\partial_k B))b_k) \\ -(b_k(f(\partial_k B)))t + ((f(\partial_k B))b_k)t. \end{array} \right\}
\end{aligned} \tag{4.14}$$

Using the KV-anomaly in the KV-module W we write (4.14) as

$$\begin{aligned}
(tdf)(B) &= \sum_{k=1}^{q+1} (-1)^k \left\{ \begin{array}{c} (tb_k)f(\partial_k B) - (b_k t)f(\partial_k B) + b_k(tf(\partial_k B)) \\ -(tf(\partial_k B))b_k + (f(\partial_k B)t)b_k - (f(\partial_k B))(tb_k) \\ -b_k(f(\partial_k B)t) - ((f(\partial_k B))b_k)t + (f(\partial_k B))(b_k t) \\ +(f(\partial_k B)b_k)t \end{array} \right\} \\
&= \sum_{k=1}^{q+1} (-1)^k \left\{ b_k(tf(\partial_k B) - f(\partial_k B)t) - (tf(\partial_k B) - f(\partial_k B)t)b_k \right\} \\
&= 0
\end{aligned}$$

This ends the proof of Theorem 4.8. ⋄

Definition 4.9. The cohomology of the complex $(\bigoplus_q C^q(\mathcal{A}/I, W)^I, d)$ is called the W_τ-valued relative KV-cohomology of $\mathcal{A}$. Its q^{th} cohomology space is denoted by $H^q(\mathcal{A}, I, W_\tau)$.

It is worth noting that this complex contains the subcomplex $(C_{CE}(\mathcal{A}_L, I_L, W_\tau), d)$ of Chevalley-Eilenberg relative cochains on the Lie algebra $\mathcal{A}_L$.

Corollary 4.10. *The space $E_2^{j,0}$ is isomorphic to $H^j(\mathcal{A}, I, W_\tau)$.*

Proof: From Theorem 4.6 we have

$$E_1^{j,0} \simeq H^0(I, C^j(\mathcal{A}/I, W)_\tau) = C^j(\mathcal{A}/I, W)^I. \tag{4.15}$$

The right hand side of (4.15) is nothing else than the vector space $C^j(\mathcal{A}, I, W_\tau)$ of relative j-cochains. Now it is easy to check that the identification of $E_1^{*,0}$ with $C^*(\mathcal{A}, I, W_\tau)$ is really the identification of the complex $(E_1^{*,0}, d_1)$ with the complex $(C^*(\mathcal{A}, I, W_\tau), d)$.

Let us proceed with general considerations.

Theorem 4.11. *Let $\mathcal{A}$ be a KV-algebra and let W be a two-sided KV-module over $\mathcal{A}$. For any two-sided ideal I of $\mathcal{A}$, the vector space $H^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$ is isomorphic to the vector space $C^j(\mathcal{A}/I, H^{q-j}(I, W_\tau))$.*

Proof: Let ρ be the vector space isomorphism

$$\rho : C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau) \longrightarrow C^j(\mathcal{A}/I, C^{q-j}(I, W_\tau))$$

defined as follows: $\forall \bar{B} \in (\mathcal{A}/I)^{\otimes j}$, $\forall T \in I^{\otimes^{q-j}}$, $\forall f \in C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$

$$\rho(f)(\bar{B})(T) = f(T)(\bar{B}). \tag{4.16}$$

It is easy to check that

$$df(T)(\bar{B}) = d(\rho(f)(\bar{B}))(T). \tag{4.17}$$

Note that (4.17) is equivalent to

$$\rho(df)(\bar{B}) = d(\rho(f)(\bar{B})). \tag{4.18}$$

Let $\tilde{\rho} : H^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau) \longrightarrow C^j(\mathcal{A}/I, H^{q-j}(I, W_\tau))$ be the linear map derived from ρ and defined by

$$\tilde{\rho}([f])(\bar{B}) = [\rho(f)(\bar{B})] \tag{4.19}$$

We now show that the latter map is bijective and the theorem will follow.

For the surjection of $\tilde{\rho}$, let $\tilde{g} \in C^j(\mathcal{A}/I, Z^{q-j}(I, W_\tau))$. There exists a unique $g \in C^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$ such that $\rho(g) = \tilde{g}$ i.e. $\rho(g)(\bar{B}) = \tilde{g}(\bar{B}), \forall \bar{B} \in (\mathcal{A}/I)^j$ and hence $d(\rho(g)(\bar{B})) = d(\tilde{g}(\bar{B}))$ which by (4.18) gives $\rho(dg)(\bar{B}) = d(\tilde{g}(\bar{B})) = 0$, $\forall \bar{B}$ i.e. $\rho(dg) = 0$ and since ρ is an isomorphism we have $dg = 0$. Furthermore we get by construction $\tilde{\rho}[g](\bar{B}) = \tilde{g}(\bar{B})$, i.e. $\tilde{\rho}[g] = \tilde{g}$.

Let us show that $\tilde{\rho}$ is injective. For that end, let $f \in Z^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$ such that $[\rho(f)(\bar{B})] = 0$, $\forall \bar{B} \in (A/I)^j$. $[\rho(f)(\bar{B})] = 0$ implies that there exists $g_B \in C^{q-j-1}(I, W_\tau)$ such that

$$\rho(f)(\bar{B}) = dg_B \tag{4.20}$$

Note that the map $g : \bar{B} \mapsto g_B$ belongs to $C^j(\mathcal{A}/I, C^{q-j-1}(I, W_\tau))$, then there exists a unique $f_g \in C_\tau^{q-j-1}(I, C^j(\mathcal{A}/I, W)_\tau)$ such that $\rho(f_g) = g$ from which we deduce $d(\rho(f_g)(\bar{B})) = dg_B$. Using (4.20) we have $\rho(df_g)(\bar{B}) = dg_B = \rho(f)(\bar{B}), \forall \bar{B}$. Hence we finally have $\rho(df_g) = \rho(f)$ i.e. $f = df_g$ as expected. ◇

Remark The isomorphism $\tilde{\rho} : H^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau) \longrightarrow C^j(\mathcal{A}/I, H^{q-j}(I, W_\tau))$ sends the subspace $H_{CE}^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau)$ of Chevalley Eilenberg cohomology classes onto the space $C^j(\mathcal{A}/I, H^{q-j}(I, W_\tau))$ providing that the action of the Lie algebra I on $C^j(\mathcal{A}/I, W)$ is given by $(t.f)(\bar{b}) = t(f(\bar{b})) - (f(\bar{b}))t$. Hence $\tilde{\rho}$ is an extension of the isomorphism $H_{CE}^{q-j}(I, C^j(A/I, W)_\tau) \simeq C^j(A/I, H_{CE}^{q-j}(I, W_\tau))$ obtained in [10].

We are now in position to give the explicit form of $E_2^{j,q-j}$ in terms of $H^{q-j}(I, W_\tau)$-valued twisted KV-cohomology classes on $\mathcal{A}/I$.

Lemma 4.12. *Let $f \in C^{q-j}(I, W_\tau)$ and $T = t_1 \otimes ... t_{q-j+1} \in I^{\otimes^{q-j+1}}$. Then $\forall a \in \mathcal{A}$,*

$$(adf - d(af))(T) = \sum_{k=1}^{q-j+1} (-1)^{k+1}((t_k a)f)(\partial_k T). \tag{4.21}$$

Proof: By our convention above the action of $\mathcal{A}$ on $C^{q-j}(I,W_\tau)$ reads

$$(af)(T)=a(f(T))-f(a.T)-(f(T))a.$$

Let $f\in C^{q-j}(I,W_\tau)$ and $T=t_1\otimes ...t_{q-j+1}\in I^{\otimes^{q-j+1}}$. A direct computation using this action and the formula of the KV-coboundary operator in (3.2) gives

$$\begin{aligned}(adf)(T) &= a(df(T))-df(a.T)-(df(T))a\\ &= \sum_{k=1}^{q-j+1}(-1)^k\left\{\begin{array}{c} a(t_kf(\partial_kT))-af(t_k.\partial_kT)-a(f(\partial_kT)t_k)\\ -(t_kf(\partial_kT))a+(f(t_k.\partial_kT))a+(f(\partial_kT)t_k)a\\ -t_kf(a.\partial_kT)+f(t_k.(a.\partial_kT))+f(a.\partial_kT)t_k\\ -(at_k)f(\partial_kT)+f((at_k).\partial_kT)+(f(\partial_kT))(at_k)\end{array}\right\} \end{aligned}\tag{4.22}$$

On the other hand we have

$$d(af)(T)=\sum_{k=1}^{q-j+1}(-1)^k\left\{\begin{array}{c} t_k(af(\partial_kT))-t_kf(a.\partial_kT)-t_k(f(\partial_kT)a)\\ -a(f(t_k.\partial_kT))+f(a.(t_k.\partial_kT))+(f(t_k.\partial_kT))a\\ -(af(\partial_kT))t_k+(f(a.\partial_kT))t_k+(f(\partial_kT)a)t_k\end{array}\right\}\tag{4.23}$$

Next combining (4.22) and (4.23) and using KV-anomaly give

$$\begin{aligned}(adf-d(af))(T) &= \sum_{k=1}^{q-j+1}(-1)^k\left\{\begin{array}{c} -KV(a,t_k,f(\partial_kT))-KV(t_k,f(\partial_kT),a)\\ -KV(f(\partial_kT),a,t_k)+f(KV(a,t_k,\partial_kT))\\ -(t_ka)f(\partial_kT)+f(\partial_kT)(t_ka)+f(t_ka).\partial_kT\end{array}\right\}\\ &= \sum_{k=1}^{q-j+1}(-1)^{k+1}((t_ka)f)(\partial_kT)\end{aligned}\tag{4.24}$$

as stated. ◇

From now on, we make the assumption that the ideal I satisfies $I\mathcal{A}=\mathcal{A}I=\{0\}$. (It is the case when $\mathcal{A}$ is finitely nilpotent, i.e. the sequence $(I^{[k]})$ defined by $I^{[0]}=\mathcal{A}, I^{[1]}=\mathcal{A}^2$, $I^{[k+1]}=\mathcal{A}I^{[k]}+I^{[k]}\mathcal{A}$ converges to $\{0\}$,) Under this assumption, the term E_2 of the spectral sequence derived from (F^jC) may be computed using the same technique as in the Hochschild-Serre spectral sequence of a pair $(\mathcal{H},\mathcal{G})$ of Lie algebras, where $\mathcal{H}$ is an ideal of $\mathcal{G}$ (see [10]). Using (4.24), we see that the action of A on $C^{q-j}(I,W_\tau)$ descends to the cohomology space $H^{q-j}(I,W_\tau)$ as

$$a.[f]=[af],\quad \forall a\in\mathcal{A},\ f\in Z^{q-j}(I,W_\tau).\tag{4.25}$$

Using $I^2=\{0\}$, we have

$$t.f=-e_1(t)(df)-d(e_1(t)f),\forall t\in I,\ \forall f\in C^{q-j}(I,W_\tau).\tag{4.26}$$

Thus the action of I is homotopic to the trivial action in cohomology. Then the left action of the KV-algebra $\mathcal{A}$ in $H^{q-j}(I,W_\tau)$ in (4.25) gives rise to a left action of $\mathcal{A}/I$ in $H^{q-j}(I,W_\tau)$ given by

$$\bar{b}.[f]=b.[f]=[bf],\quad \forall b\in\mathcal{A},\ f\in Z^{q-j}(I,W_\tau).\tag{4.27}$$

We consider the complex $(C(\mathcal{A}/I, H^{q-j}(\mathcal{A}, W_\tau)_\tau), d)$ which is closely related to the vector space $E_2^{j,q-j}$. We first observe that the isomorphism

$\tilde{\rho} : H^{q-j}(I, C^j(\mathcal{A}/I, W)_\tau) \longrightarrow C^j(\mathcal{A}/I, H^{q-j}(I, W_\tau))$ induces by composition an isomorphism $\psi : E_1^{j,q-j} \longrightarrow C^j(\mathcal{A}/I, H^{q-j}(I, W_\tau))$ given by

$$\psi([f])(\bar{B}) = \tilde{\rho}([r_{q,j}(f)])(\bar{B}) = [\rho(r_{q,j}f)(\bar{B})]. \tag{4.28}$$

Proposition 4.13. *Let $d_1 : E_1^{j,q-j} \longrightarrow E_1^{j+1,q-j}$ be the coboundary operator from the spectral sequence (E_r) and let us denote by $d_{A/I} = d$ the coboundary operator of the twisted KV-complex $(C(\mathcal{A}/I, H^{q-j}(\mathcal{A}, W_\tau)_\tau), d)$. Then we have*

$$d_{A/I} \circ \psi = (-1)^{q-j} \psi \circ d_1 \tag{4.29}$$

Proof: For all $f \in F^jC \cap C^q(\mathcal{A}, W_\tau)$ and $\bar{B} \in (\mathcal{A}/I)^{j+1}$,

$$\begin{aligned}
d_{A/I}(\psi([f]))(\bar{B}) &= \sum_{k=1}^{j+1}(-1)^k\left\{ \bar{b_k}.(\psi([f])(\partial_k\bar{B})) - \psi([f])(\bar{b_k}.\partial_k\bar{B}) - (\psi([f])(\partial_k\bar{B}))\bar{b_k} \right\} \\
&= \sum_{k=1}^{j+1}(-1)^k\left\{ \begin{array}{c} \bar{b_k}.[\rho(r_{q,j}(f))(\partial_k\bar{B})] - [\rho(r_{q,j}(f))(\bar{b_k}.\partial_k\bar{B})] \\ -[\rho(r_{q,j}(f))(\partial_k\bar{B})]\bar{b_k} \end{array} \right\} \\
&= \sum_{k=1}^{j+1}(-1)^k\left\{ \bar{b_k}.[\rho(r_{q,j}(f))(\partial_k\bar{B})] - [\rho(r_{q,j}(f))(\bar{b_k}.\partial_k\bar{B})] \right\} \\
&= [d(\rho(r_{q,j}(f)))(\bar{B})].
\end{aligned} \tag{4.30}$$

On the other hand, we have $d_1([f]) = [df]$. Since $df \in F^{j+1}C \cap C^{q+1}$ for $[f] \in E_1^{j,q-j}$ and $\psi([df]) \in C^{j+1}(\mathcal{A}/I, H^{q-j}(I, W_\tau))$, we have

$$(\psi \circ d_1)([f])(\bar{B}) = \psi([df])(\bar{B}) = [\rho(r_{q+1,j+1}(df))(\bar{B})].$$

Now for all $T \in I^{\otimes^{q-j}}$, we have

$$\begin{aligned}
\rho(r_{q+1,j+1}(df))(\bar{B})(T) &= (r_{q+1,j+1}(df))(T)(\bar{B}) \\
&= df(T \otimes B)
\end{aligned} \tag{4.31}$$

Using the formula (3.2) and $I\mathcal{A} = \mathcal{A}I = \{0\}$, we have

$$\begin{aligned}
df(T \otimes B) &= \sum_{k=1}^{q-j}(-1)^k\left\{ \begin{array}{c} t_kf(\partial_kT \otimes B) - f(t_k.\partial_kT \otimes B) \\ -f(\partial_kT \otimes t_k.B) - f(\partial_kT \otimes B)t_k \end{array} \right\} \\
&+ (-1)^{q-j}\sum_{s=1}^{j+1}(-1)^s\left\{ \begin{array}{c} b_kf(T \otimes \partial_kB) - f(T \otimes \partial_kB)b_k \\ -f(b_k.T \otimes \partial_kB) - f(T \otimes b_k.\partial_kB) \end{array} \right\} \\
&= (d(r_{q,j}(f))(T)(\bar{B}) \\
&+ (-1)^{q-j}\sum_{s=1}^{j+1}(-1)^s\left\{ \begin{array}{c} b_k(r_{q,j}(f)(T)(\partial_k\bar{B})) - (r_{q,j}(f)(T)(\partial_k\bar{B}))b_k \\ -r_{q,j}(f)(T)(\bar{b_k}.\partial_k\bar{B}) \end{array} \right\} \\
&= (d(r_{q,j}(f))(T)(\bar{B}) + (-1)^{q-j}d(\rho(r_{q,j}(f)))(\bar{B})(T).
\end{aligned}$$

The calculation above implies

$$\rho(r_{q+1,j+1}(df))(\bar{B})(T) = d(r_{q,j}(f))(T)(\bar{B}) + (-1)^{q-j} d(\rho(r_{q,j}(f)))(\bar{B})(T) \tag{4.32}$$

Now observe that for any fixed $\bar{B}$, the cochain $T \mapsto (d(r_{q,j})(f))(T)(\bar{B})$ is exact in $C^{q-j}(I, W_\tau)$. In fact it is the coboundary of $T' \mapsto r_{q,j}(f)(T')(\bar{B})$. So, at the cohomology level, in $H_\tau^{q-j}(I, W)$ we have

$$[\rho(r_{q+1,j+1}(df))(\bar{B})] = [(-1)^{q-j} d(\rho(r_{q,j}(f)))(\bar{B})], \ \forall \bar{B} \in (A/I)^{\otimes} \tag{4.33}$$

That is nothing but the following equality

$$(\psi \circ d_1)([f])(\bar{B}) = (-1)^{q-j} d_{A/I} \circ \psi([f])(\bar{B}), \quad \forall \bar{B} \in (\mathcal{A}/I)^{\otimes^{j+1}}.$$

Since ψ is an isomorphism we have

$$E_2^{j,q-j} \simeq H^j(\mathcal{A}/I, H^{q-j}(I, W_\tau)_\tau).$$

5 Twisted cohomology of locally flat manifolds

In this section we apply the theoretical formalism of previous sections for studying the KV-algebras of a locally flat manifold. We point out some geometric invariants. The reader may regard a linear connection on a manifold M as a gauge field on the fiber bundle of linear frames over M. From this viewpoint the decomposition theorem below yields many relevant consequences.

We first observe that $C^2(\mathcal{A}, W_\tau) = Hom(\otimes^2 \mathcal{A}, W_\tau)$ splits into two KV-submodules as

$$Hom(\otimes^2 \mathcal{A}, W_\tau) = Hom(\wedge^2 \mathcal{A}, W_\tau) \oplus Hom(S^2 \mathcal{A}, W_\tau).$$

where $S^2\mathcal{A}$ is the vector space of symmetric 2-tensors.

Theorem 5.1. *The twisted KV-cohomology space $H^2_{KV}(\mathcal{A}, W_\tau)$ can be decomposed as follows:*

$$H^2_{KV}(\mathcal{A}, W_\tau) = H^2_{CE}(\mathcal{A}_L, W_\tau) \oplus H^0_{KV}(\mathcal{A}, Hom(S^2 A, W_\tau)). \tag{5.1}$$

Theorem 5.1 may be deduced from Theorem 5.2

Theorem 5.2. *below The skew-symmetric part and the symmetric part of any twisted 2-cocycle are also 2-cocyles.*

Proof: Let $f \in Z^2(\mathcal{A}, W_\tau)$ be a 2-cocycle and let λ and σ be the skew symmetric part and the symmetric part of f, respectively. Given $a, b, c, \in \mathcal{A}$, $df = 0$ implies $d\lambda(a, c, b) = d\sigma(a, b, c)$. Taking $c = b$ the right member $d\sigma(a, b, c)$ yields

$$a * (\sigma(b, b)) = 2\sigma(ab, b), \quad \forall a, b \in \mathcal{A}. \tag{5.2}$$

Therefore $a * (\sigma(b + c, b + c)) = 2\sigma(a(b + c), b + c)$ leads to

$$a * (\sigma(b, c)) = \sigma(ab, c) + \sigma(b, ac), \quad \forall a, b, c \in \mathcal{A}.$$

which means that $a.\sigma = 0, \forall a \in \mathcal{A}$. In other words $\sigma \in H^0_{KV}(\mathcal{A}, Hom(S^2 A, W_\tau))$. ⋄

We now focus on KV-algebras of locally flat manifolds.

Definition 5.3. A locally flat manifold (M,∇) whose universal covering $(\tilde{M},\tilde{\nabla})$ is diffeomorphic to a convex domain of $\mathbb{R}^n$ not containing any straight line is called hyperbolic locally flat manifold [13, 33].

Let $\mathcal{A} = \mathcal{X}(M)$ be the KV-algebra of a locally flat manifold (M,∇). The vector space $W_\tau = C^\infty(M)$ of real smooth functions defined in M is a left KV-module over $\mathcal{A}$ under the covariant derivative.

We are concerned with the twisted KV-complex $C(\mathcal{A},W_\tau)$ of $\mathcal{A}$ with coefficients in W_τ and with some of its relevant subcomplexes.

Definition 5.4. Given a non negative integer l, a cochain $f \in C^q(\mathcal{A},W_\tau)$ is of order $\leq l$ if $\forall x \in M$, and $\forall X_1,...,X_q \in \mathcal{A}$, $f(X_1,X_2,...,X_q)(x)$ depends on the l-jets $J^l_xX_1, J^l_xX_2,...,J^l_xX_q$ [14].

Let $C^q_l(\mathcal{A},W_\tau)$ be the vector space of q-cochains f of order $\leq l$. We set $C^q_\infty(\mathcal{A},W_\tau) = \cup_l C^q_l(\mathcal{A},W_\tau)$ and $C_\infty(\mathcal{A},W_\tau) = \bigoplus_{q\geq 0} C^q_\infty(\mathcal{A},W_\tau)$. The complex $C_\infty(\mathcal{A},W_\tau)$ is a subcomplex of $C_{KV}(\mathcal{A},W_\tau)$. Let us denote by $C_0(\mathcal{A},W_\tau)$ the vector subspace of $C_\infty(\mathcal{A},W_\tau)$ formed by cochains of order ≤ 0. Thus $C_0(\mathcal{A},W_\tau)$ consists of $C^\infty(M)-$ multilinear mappings.

Let (C,d) be a (co)chain complex which is computable using the spectral sequence techniques. Among spectral sequences that converge to $H(C,d)$, it would be nice to get one which collapses quickly. Fortunately this is the case for the twisted KV-cohomology of a locally flat manifold. Let (M,∇) be a locally flat manifold. Let $\mathcal{A} = (\mathcal{X}(M),\nabla)$ be the KV-algebra associated to (M,∇) and let $W_\tau = C^\infty(M)$ be the left KV-module over $\mathcal{A}$ under the covariant derivative.

Theorem 5.5. *There is a spectral sequence $(E^{p,q}_r)$ which collapses for $r = 3$ (i.e. $E_\infty = E_3$), and which converges to the twisted KV-cohomology $H(\mathcal{A},C^\infty(M))$ and such that*

$$E^{3,q-3}_\infty = H^q_{dR}(M,\mathbb{R}), \qquad q \geq 3.$$

Idea of proof: We recall that $C^q_\infty(\mathcal{A}_L,W_\tau) = \cup_l C^q_l(A_L,W_\tau)$, where $C^q_l(\mathcal{A}_L,W_\tau)$ is the space of skew symmetric q-cochains of order $\leq l$. Clearly $C_\infty(\mathcal{A}_L,W_\tau)$ is a subcomplex of the Chevalley-Eilenberg complex $C_{CE}(A_L,W_\tau)$. Let us consider the filtration of the vector space $C_{KV}(\mathcal{A},W_\tau)$ given by

$$\begin{aligned} F^0C(\mathcal{A},W_\tau) &= C_{KV}(\mathcal{A},W_\tau), \quad F^1C(\mathcal{A},W_\tau) = C_{CE}(\mathcal{A}_L,W_\tau), \quad F^2C(\mathcal{A},W_\tau) = C_\infty(\mathcal{A}_L,W_\tau),\\ F^3C(\mathcal{A},W_\tau) &= C_0(\mathcal{A},W_\tau),\ F^jC(\mathcal{A},W_\tau) = \{0\}\ for\ j \geq 4. \end{aligned}$$

$$\{0\} \hookrightarrow F^3C(\mathcal{A},W_\tau) \hookrightarrow F^2C(\mathcal{A},W_\tau) \hookrightarrow F^1C(\mathcal{A},W_\tau) \hookrightarrow F^0C(\mathcal{A},W_\tau).$$

Observe that $F^3C(\mathcal{A},W_\tau)$ is nothing but the $C^\infty(M)$-module of exterior differential forms on M. On the other hand each $F^jC(\mathcal{A},W_\tau)$ is a subcomplex of $F^0C(\mathcal{A},W_\tau)$. This filtration is bounded. Therefore the derived spectral sequence converges to the cohomology of $F^0C(\mathcal{A},W_\tau)$. (see [16]). Let $Z^{p,q}_r = \{f \in F^pC \cap C^{p+q}, df \in F^{p+r}C \cap C^{p+q+1}\}$. It is clear that $Z^{p,q}_r = \{0\}$ for $r > 3$. To end the proof, we observe that $E^{3,q-3}_3$ is the q^{th} cohomology space of $F^3C(\mathcal{A},W_\tau)$. $\diamond$

In the sequel, we particularly focus on the cohomology of the complex $C_0(\mathcal{A},W_\tau)$ whose q^{th} cohomology space is denoted by $H^q_0(\mathcal{A},W_\tau)$. Below there is a statement yielding some relevant gauge invariants.

Theorem 5.6. *The second cohomology space $H^2_0(\mathcal{A}, W_\tau)$ can be decomposed as follows:*

$$H^2_0(\mathcal{A}, W_\tau) = H^2_{dR}(M) \oplus H^0(\mathcal{A}, Hom(S^2A, W_\tau)) \tag{5.3}$$

where $H^2_{dR}(M)$ is the 2^{nd} de Rham cohomology space of M.

The proof of Theorem 5.6 is based on Theorem 5.1 above ⋄

Actually, the space $H(\mathcal{A}_\nabla, W_\tau) = \bigoplus_{q\geq 0} H^q(\mathcal{A}_\nabla, W_\tau)$ is a global geometric invariant of (M,∇), namely every $H^q(\mathcal{A}_\nabla, W_\tau)$ is invariant under the group $Diff_\nabla(M)$ of ∇-preserving diffeomorphisms of M.

We set $b_q(\nabla) = \dim H^q_0(\mathcal{A}_\nabla, C^\infty(M))$ and $b_q(M) = \dim H^q_{dR}(M,\mathbb{R})$. Then $b_q(\nabla)$ is called the q^{th} Betti number of (M,∇) while $b_q(M)$ is the classical q^{th} Betti number of M. Clearly $b_q(M) \leq b_q(\nabla)$.

Example 5.7. Let $\lambda \in \mathbb{R}, \lambda > 1$ and let $\Gamma = \{\lambda^m, m \in \mathbb{Z}\}$, be the subgroup of the multiplicative group $\mathbb{R}^*$ generated by λ. We consider the locally flat Hopf manifold (M,∇) where $M = \Gamma\backslash(\mathbb{R}^n - \{0\})$ and ∇ is the linear connection induced on M by the canonical linear connection D on $\mathbb{R}^n$. We have $b_q(M) = b_q(\nabla)$.

In Theorem 5.6 we are concerned with cochains of order ≤ 0. So the space $Hom_{C^\infty(M)}(\mathcal{A}, C^\infty(M))$ is nothing but the classical vector space $\Omega^1(M)$ of differential 1-forms. The algebra $\mathcal{A}$ acts on $\Omega^1(M)$ by the covariant derivative. Let us decompose $[\theta] \in H^2_0(\mathcal{A}_\nabla, C^\infty(M))$ as $[\theta] = [\lambda] + [\sigma] \in H^2_{dR}(M,\mathbb{R}) \oplus H^2_0(\mathcal{A}_\nabla, C^\infty(M))$, and let σ be the unique symmetric cocycle in $[\sigma]$, we assign to σ the cochain $\tilde{\sigma} \in C^1(\mathcal{A}, \Omega^1(M))$ such that $\tilde{\sigma}(a)(b) = \sigma(a,b)$. Let us denote now by d_∇ the coboundary operator for the twisted KV-cohomology of $\mathcal{A}$. It is easy to check that

$$(d_\nabla \sigma = 0) \Longrightarrow (d_\nabla \tilde{\sigma} = 0). \tag{5.4}$$

We set $\mathcal{P}([\theta]) = [\tilde{\sigma}] \in H^1_0(\mathcal{A}, \Omega^1(M))$. Thereby one easily sees that Theorem 5.6 yields the exact sequence

$$0 \to H^2_{dR}(M,\mathbb{R}) \to H^2_0(\mathcal{A}_\nabla, C^\infty(M)) \xrightarrow{\mathcal{P}} H^1_0(\mathcal{A}, \Omega^1(M)). \tag{5.5}$$

The map $\mathcal{P}$ is useful to the study of hyperbolicity of (M,∇), see Theorem 5.12 below.

Definition 5.8. [22, 23, 24] Let (M,∇) be a locally flat manifold. (i) A totally geodesic foliation $\mathcal{F}$ of (M,∇) is called affine foliation. (ii) A totally geodesic foliation $\mathcal{F}$ of M is tranversally euclidean if its normal bundle $TM/T\mathcal{F}$ is endowed with a ∇-parallel (pseudo) euclidean scalar product.

Let us define two numerical invariants. We set $Q(M) = Hom_{C^\infty(M)}(S^2\mathcal{A}, W_\tau)$, the vector space of tensorial quadratic forms on (sections of) TM. Given $\sigma \in H^0_{KV}(\mathcal{A}, Q(M))$, let $\bar{\sigma}$ be the quadratic form on $TM/\ker\sigma$ deduced from σ and let $sign(\sigma)$ be the Morse index of $\bar{\sigma}$. With this notations we define the following numerical invariants:

Definition 5.9. We set: $\rho_\nabla(M) = min\{\rho_\nabla(\sigma) = \dim\ker\sigma,\ \sigma \in H^0(\mathcal{A}, Q(M))\}$ and $S_\nabla(M) = min\{S_\nabla(\sigma) = \dim\ker\sigma + sign(\sigma),\ \sigma \in H^0(\mathcal{A}, Q(M))\}$.

Proposition 5.10. *Assume M to be a connected manifold. (i) Every element $\sigma \in H^0(\mathcal{A}, Q(M))$ defines a totally ∇-geodesic and transversally (pseudo) euclidean foliation . (ii) If $\rho_\nabla(M) = 0$, then ∇ is the Levi-Civita connection of a flat (pseudo)Riemannian metric on M.*

Proof : (i) Let $\sigma \in H^0(\mathcal{A}, Q(M))$. Since ∇ is torsion free, $\ker\sigma$ is involutive and of constant rank. According to the Frobenius Theorem , $\ker\sigma$ is completely integrable and totally $\nabla-$ geodesic. So the ∇-flat vector bundle $TM/\ker\sigma$ inherits the (pseudo)euclidean product $\bar{\sigma}$. (ii) Suppose $\rho_\nabla(M) = 0$. There exists a $\sigma \in H^0(\mathcal{A}, Q(M))$ with $\rho_\nabla(\sigma) = 0$. Therefore (M, σ) is a (pseudo) Riemannian manifold whose Levi-Civita connection is ∇. ◇

We have claimed that $H_0^2(\mathcal{A}, C^\infty(M))$ has some relevant gauge meaning. We intend to illustrate this viewpoint. Let $C(\mathcal{A}_\nabla, C^\infty(M))$ and $C(\mathcal{A}'_{\nabla'}, C^\infty(M'))$ be the twisted KV-complexes of locally flat manifolds (M, ∇) and (M', ∇'), respectively. Let $\tau(M)$ be the tensor algebra of T^*M i.e. $\tau(M) = \oplus_{k\geq 0}(T^*M)^{\otimes^k}$ with $(T^*M)^{\otimes^0} = C^\infty(M)$.

Let $\phi : M \longrightarrow M'$ be a diffeomorphism. We denote by ϕ^1 the pair (ϕ, ϕ^{*-1}). Then ϕ^1 is a diffeomorphism from T^*M to T^*M'. The algebra isomorphism from $\tau(M)$ to $\tau(M')$ derived from ϕ^1 is also denoted by ϕ^1. Let us denote by (M', ∇^ϕ) the locally flat structure defined in $\phi(M)$ by

$$\nabla^{\phi}{}^{Y'}_{X'} = d\phi(\nabla^{d\phi^{-1}Y'}_{d\phi^{-1}X'}). \tag{5.6}$$

Actually, the map ϕ^1 yields an isomorphism from $C(\mathcal{A}_\nabla, C^\infty(M))$ to $C(\mathcal{A}'_{\nabla^\phi}, C^\infty(M'))$ given by $\forall f \in C^q(\mathcal{A}_\nabla, C^\infty(M)), \forall X'_1, ..., X'_q \in \mathcal{A}'_{\nabla^\phi}$,

$$\phi^1(f)(X'_1, ..., X'_q) = f(d\phi^{-1}X'_1, ..., d\phi^{-1}X'_q) \circ \phi^{-1}. \tag{5.7}$$

Furthermore, it satisfies the relation $d_\nabla \circ \phi^1 = \phi^1 \circ d_\nabla$. Thus it induces an isomorphism from $H_{KV}(\mathcal{A}_\nabla, C^\infty(M))$ to $H_{KV}(\mathcal{A}_{\nabla^\phi}, C^\infty(M'))$, which is denoted by ϕ^1 as well. In particular $\phi^1(H^q_{KV}(\mathcal{A}_\nabla, C^\infty(M))) \subset H^q_{KV}(\mathcal{A}'_{\nabla^\phi}, C^\infty(M'))$, $\forall q \geq 0$. This picture yields:

Theorem 5.11. *Let us fix (M, ∇) and (M', ∇') such that M and M′ are connected and $\dim M = \dim M'$. We consider a diffeomorphism $\phi : M \to M'$ and we suppose that $\rho_\nabla(M) = 0$. If $\phi^1(H^2_{KV}(\mathcal{A}_\nabla, C^\infty(M))) \subset H^2_{KV}(\mathcal{A}'_{\nabla'}, C^\infty(M'))$, then $b_2(\nabla) > b_2(M)$ and $\nabla^\phi = \nabla'$.*

Proof: Let us equip $\phi(M)$ with the locally flat structure (M', ∇^ϕ). The map ϕ is a isomorphism from (M, ∇) to (M', ∇^ϕ). Since $\rho_\nabla(M) = 0$ there exists $\sigma \in H^0(\mathcal{A}_\nabla, Q(M))$ such that $\ker\sigma = \{0\}$. Thus Theorem 5.6 yields $b_2(\nabla) > b_2(M)$. The image $\phi^1(\sigma)$ of σ is a metric tensor whose Levi-Civita connection is ∇^ϕ. The assumption $\phi^1(H^2_{KV}(\mathcal{A}_\nabla, C^\infty(M))) \subset H^2_{KV}(\mathcal{A}'_{\nabla'}, C^\infty(M'))$ implies that $\phi^1(\sigma) \in H^0(A'_{\nabla'}, Q(M'))$. Thus $\phi^1(\sigma)$ is ∇'-parallel. Thereby $\nabla^\phi = \nabla'$. ◇

Theorem 5.12. *We suppose that (M, ∇) is a locally flat manifold with $S_\nabla(M) = 0$. Then, (i) TM admits a bundle-like Kahlerian metric. (ii) Furthermore let us suppose M to be compact and that there exists $\sigma \in H^0(\mathcal{A}_\nabla, Q(M))$ with $S_\nabla(\sigma) = 0$. If $\mathcal{P}([\sigma]) = 0$ in $H^1(\mathcal{A}, \Omega^1(M))$, then (M, ∇) is hyperbolic.*

Proof: The vanishing of $S_\nabla(M)$ implies that there exists $\sigma \in Q(M)$ which is invariant under the covariant derivative and such that $\ker\sigma = 0$ and $sign(\sigma) = 0$. Thus σ is a Riemannian metric whose Levi-Civita connection is ∇. To σ we associate $\tilde{\sigma} \in C_0^1(\mathcal{A}_\nabla, \Omega^1(M))$ as in (5.4). Let U_α be a domain of local affine coordinate functions $(x_i^\alpha), i = 1, ..., n = \dim M$. Locally, $\tilde{\sigma}$ may be regarded as a $T^*(U_\alpha)$-valued 1-form which is d_∇-closed. Let us suppose that U_α is small enough to admit the application of Poincaré Lemma. Then there exists a local section θ of $T^*(U_\alpha)$ such that $\tilde{\sigma} = d_\nabla\theta$. That is $\tilde{\sigma}(X) = \nabla_X^\theta, X \in \mathcal{X}(U_\alpha)$. So, $\sigma_\alpha(X,Y) = \tilde{\sigma}_\alpha(X)(Y) = (\nabla_X^\theta)(Y) = X\theta(Y) - \theta(XY) = Y\theta(X) - \theta(YX)$, which implies that θ is a real-valued closed 1-form. By Poincaré lemma there exists $h \in C^\infty(U_\alpha)$ such that $\theta = dh$. This means that on U_α the metric tensor σ coincides with the ∇-Hessian of $h \in C^\infty(U_\alpha)$. Following [5], TM admits a bundle-like Kahlerian metric. (ii) To end the proof of Theorem 5.12, one observes that $\mathcal{P}([\sigma]) = 0$ is equivalent to the existence of a de Rham closed 1-form θ such that $\sigma = \nabla^\theta$. Since M is compact, this condition is sufficient for the hyperbolicity of (M, ∇) (see[13]). ⋄

Affine dynamics together with the numerical invariant $\rho_\nabla(M)$ can be used to study the completeness of (M, ∇). In this perspective we have the following:

Theorem 5.13. *Let (M, ∇) be a connected locally flat manifold. Let $G \subset Diff(M)$ be the set of diffeomorphisms $\phi \in Diff(M)$ such that:*

(i) ϕ is isotopic to the identity of M through diffeomorphisms with property (ii) below,

(ii) $\phi^1(H_{KV}^2(\mathcal{A}_\nabla, C^\infty(M))) \subset H_{KV}^2(\mathcal{A}_\nabla, C^\infty(M))$.

If $\rho_\nabla(M) = 0$, then (a) G is a Lie group with $\dim G \leq n(n+1)$ where $n = \dim M$, (b) $\dim G = n(n+1)$ iff the connection ∇ is geodesically complete.

Proof: We suppose that $\rho_\nabla(M) = 0$ and that $\phi^1(H_{KV}^2(\mathcal{A}_\nabla, C^\infty(M))) \subset H_{KV}^2(\mathcal{A}_\nabla, C^\infty(M))$. Then by Theorem 5.11, $\nabla^\phi = \nabla$. Thus G is nothing but the group $Diff_\nabla(M)$ of ∇-preserving diffeomorphisms of M. It is a Lie group (see [11]).

Let $\tilde{\nabla}$ be the locally flat connection induced on the universal covering $\tilde{M}$ of M by the connection ∇. Let $\mathcal{D} : (\tilde{M}, \tilde{\nabla}) \to (\mathbb{R}^n, D)$ be the developing map of (M, ∇). The Lie algebra $\mathcal{G}$ of G is included in the Lie algebra $J(\mathcal{A}_\nabla)$ of Jacobi elements of $\mathcal{A}_\nabla$. Since $\mathcal{D}$ is a local diffeomorphism, $\mathcal{D}_*(\mathcal{G})$ is included in the Lie algebra $aff(\mathbb{R}^n)$. Therefore $\dim G \leq \dim Aff(\mathbb{R}^n) = n(n+1)$.

If $\dim G = n(n+1)$ then $\tilde{M}$ is homogeneous. Since $\tilde{M}$ and $\mathbb{R}^n$ are simply connected and are homogeneous under G and $Aff(\mathbb{R}^n)$ respectively, the developing map $\mathcal{D}$ is a diffeomorphism.

For other items regarding affine dynamics and the completeness of locally flat manifolds, we refer to [18, 19, 20, 34]. ⋄

6 Information geometry

We intend to show that the twisted KV cohomology yields some relevant geometrical invariants of statistical models for measurable sets.

6.1 Local statistical model

Let us start by recalling basic definitions in information geometry and some related topics.
Let (Ξ,Ω) a measurable set. Let $\Theta \subset \mathcal{R}^n$ be a connected subset.

Definition 6.1. A connected open subset $\Theta \subset \mathcal{R}^n$ is an n-dimensional statistical model for a measurable set (Ξ,Ω) if there exists a real valued positive function

$$p : \Theta \times \Xi \to \mathcal{R}$$

subject to the following requirements.
(i) For every fixed $\xi \in \Xi$ the function

$$\theta \to p(\theta,\xi)$$

is smooth.
(ii) For every fixed $\theta \in \Theta$ the function

$$\xi \to p(\theta,\xi)$$

is a probability density in (Ξ,Ω) viz

$$\int_{\Xi} p(\theta,\xi) d\xi = 1.$$

(iii) For every fixed $\xi \in \Xi$ there exists a couple (θ,θ') such that

$$p(\theta,\xi) \neq p(\theta',\xi)$$

Let ∇ a torsion free linear connection on the manifold Θ and let set

$$\updownarrow_n(\theta,\xi) = log(p(\theta,\xi)).$$

At each point $\theta \in \Theta$ we define the family (q_θ,ξ) of bilinear forms as follows. Let (X,Y) be a couple of smooth vector fields in Θ. We put

$$q_{(\theta,\xi)}(X,Y) = -(\nabla dl_n)(X,Y)(\theta,\xi).$$

Since ∇ is torsion free $q_{(\theta,\xi)}(X,Y)$ is symmetric w.r.t. the couple (X,Y).

Definition 6.2. The Fisher information g of the local model Θ, p is the mathematical expectation of the bilinear form $q_{(\theta,\xi)}$, viz

$$g(X,Y)(\theta) = \int_{\Xi} p(\theta,\xi) q_{(\theta,\xi)}(X,Y) d\xi.$$

It is to be noticed that the Fisher information g does not depend on the choice of the symmetric connection ∇, [27]. The Fisher information g is semi positive definite. When g is definite it is called a Fisher metric of the model (Θ,p). The Fisher metric is the fundamental source of the information geometry [1]. Another object we are interested in is the dualistic relation between linear connections.

Definition 6.3. On a Riemannian manifold (M,g) a pair $(\nabla,\nabla*)$ of linear connections are said to be dual if the identity

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla *_X Z)$$

holds true for all vector fields X,Y,Z on the manifold M.

Let T_∇ be the torsion tensor of ∇ and let R_∇ be its curvature tensor. Suppose $(\nabla,\nabla*)$ to be a dual pair in the Riemannian manifold (M,g). Let ∇^o be the Levi-Civita connection of (M,g). Then we have

$$T_\nabla + T_{\nabla *} = 0,$$

$$\frac{\nabla + \nabla*}{2} = \nabla^o$$

$$g(R_\nabla(X,Y).Z,T) + g(Z,R_\nabla *(X,Y).T) = 0.$$

The dualistic relation between linear connections play deep role in information geometry. The reader is referred to [1]. We are going to deduce from these information geometric tools some relevant statements. The complete discussion about these statements will appear elsewhere.

We are firstly concerned with the isomorphism classes of the so called canonical representations Lie algebroids.

Let us recall that a Lie algebroid over a smooth manifold M is a couple (E,a) where E is the total space of a vector bundle

$$E \to M$$

and anchor a is vector bundle morphism from E to the tangent bundle TM such that

(i) The real vector space $\Gamma(E)$ of smooth sections of E is endowed with a Lie algebra structure (whose bracket is denoted by [s,s']).

(ii) Given $s, s' \in \Gamma(E)$ and a real valued smooth function $f \in C^\infty(M)$ one has

$$[s, fs'] = df(a(s))s' + f[s,s'].$$

Let (E,a) be a Lie algebroid over the manifold M and let V be an vector bundle over M.

Definition 6.4. A structure E-module in V is a real bilinear map

$$\rho : (s,\sigma) \in \Gamma(E)\times\Gamma(V) \to \rho(s).\sigma \in \Gamma(V)$$

subject to the following identity

$$\rho(s).f\sigma = df(a(s))\sigma + f\rho(s).\sigma$$

$\forall f \in C^\infty(M)$.

Example 6.5. Given a Lie algebroid over M, (E,a) the tangent bundle TM is canonically a E-module under the map $\rho(s).X = [a(s),X]$. According to Koszul [14] cochains of Lie algebroids with coefficients in their modules are of order ≤ 1.

Example 6.6. We are particularly concerned with canonical module strutures defined by locally flat structures (M, D). The tangent bundle TM is equipped with the Lie algebroid structure whose anchor is the identity map ans the bracket is the Poisson bracket $[X, Y]$ of smooth vector fields.

Every locally flat structure (M, D) gives rise to the canonical structure of TM-module in itself which is defined by

$$\rho(X).Y = D_X Y$$

Of course morphisms of modules of Lie algebroids are morphisms of vector bundles. Thus given locally flat structures (M, D) and $(M, D*)$ a morphism from $\rho_D(X).Y = D_X Y$ to $\rho_{D*}(X).Y = D*_X Y$ is a vector bundle morphism $\phi \in End(TM)$ which satisfies the identity

$$\phi(D_X Y) = D*_X \phi(Y)$$

When ϕ is an isomorphism the representations ρ_D and ρ_D* are called conjugate representations. Now taking into account the decomposition

$$H_\tau(D) = H_{dR}(M) + Q^D$$

we see that if ρ_D and ρ_{D*} are conjugate by an isomorphism ϕ, viz

$$\phi(D_X Y) = D*_X \phi(Y)$$

the map

$$q \in Q^D* \to q\phi^{-1}$$

is an isomorphism onto Q^D. So the family of foliations

$$\mathcal{F}_q = ker(q), q \in Q^D$$

is isomorphic to

$$\mathcal{F}_q^* = ker(q*), q^* \in Q^{D^*}.$$

Let us sketch the discussion about dual pair of linear connections. Given a dual pair $(\nabla, \nabla*)$ in a Riemannian manifold (M, g). Let us assume that this given pair is flat. So both (M, ∇) and $(M, \nabla*)$ are locally flat structures. They the pair $([\rho_\nabla], [\rho_{\nabla*}])$ of conjugation class of canonical representations. Considerations we just presented yield two interesting statements.

Theorem 6.7. *The pair* $([\rho_\nabla], [\rho_{\nabla*}])$ *is does not depend on the choice of the Riemannian structure* (M, g).

Theorem 6.8. *Let* (M, D) *be a locally flat structure whose the twisted* 2*-cohomology space satisfies* $Q^D \neq 0$. *Let* $D*$ *be the dual of* D *w.r.t. a Riemannian structure* (M, g). *Then every non-zero undefinite* $q \in Q^D$ *defines the pair of* g*-orthogonal foliations* $(\mathcal{F}_q, \mathcal{F}*_q)$ *such that*

(i) $\mathcal{F}_q$ *is totally D-geodesic (or D-parallel in the sense of [1]. (ii)* $\mathcal{F}*_q$ *is totally D∗-geodesic.*

Our discussions about the dualistic relation yield the following. From the global differential topology viewpoint we obtain

Theorem 6.9. *Every locally flat manifold* (M,D) *whose* 2*-dimensional twisted cohomology* $H^2_\tau(D)$ *differs from the de Rham cohomology space* $H^2_{dR}(M)$ *is either a flat (pseudo)-Riemannian manifold or is foliated by a pair* $(\mathcal{F},\mathcal{F}*)$ *of g-orthogonal foliations for every Riemannian metric g. Moreover, these foliations are totally geodesic w.r.t. the g-dual pair* $(D,D*)$ *(respectively).*

According to the orthogonal decomposition

$$TM = T\mathcal{F} \oplus T\mathcal{F}*$$

we define a torsion torsion free linear connection by setting

$$\tilde{D}_{(X_1,X_2)}(Y_1,Y_2) = (D_{X_1}Y_1 + [X_2,Y_1], D*_{X_2}Y_2 + [X_1,Y_2])$$

$\forall (X_1,X_2),(Y_1,Y_2) \in \Gamma(T\mathcal{F}),\Gamma(T\mathcal{F}*)$.

Then $\tilde{D}$ is the unique torsion free linear connection which preserves $(\mathcal{F},\mathcal{F}*)$.

6.2 Singular Fisher information

Let (Θ,p) be a (local) statistical model for a measurable set (ξ,Ω). We assume that Fisher information g is undefinite and that it is parallel w.r.t. a locally flat structure (Θ,D).

We consider a Riemannian structure $(\Theta,g*)$. Let $(D,D*)$ be the $g*$-pair of linear connections. Of course $(\Theta,D*)$ is a locally flat structure as well.

Under this assumption

$$D_X g = O$$

we get $g*$-orthogonal foliations

$$\mathcal{F}_g, \mathcal{F}*$$

with

$$T\mathcal{F}_g = ker(g).$$

Now let F be a leaf of $\mathcal{F}*$. The couple (F,p) is a regular statistical model for (ξ,Ω), viz Fisher information of (F,p) is positive definite.

The discussion above yields

Theorem 6.10. *Let* (Θ,D,p) *be a triple such that* (Θ,D) *is a locally flat structure and* (Θ,p) *is a singular statistical model for a measurable set* (Ξ,Ω)*. If the Fisher information g of* (Θ,p) *is D-parallel then* (Θ,p) *is foliated by regular statistical model.*

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