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THREE APPROACHES TO MORSE-BOTT HOMOLOGY

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Abstract

In this paper we survey three approaches to computing the homology of a finite dimensional compact smooth closed manifold using a Morse-Bott function and discuss relationships among the three approaches. The first approach is to perturb the function to a Morse function, the second approach is to use moduli spaces of cascades, and the third approach is to use the Morse-Bott multicomplex.

With respect to an explicit perturbation (which can be used to derived the Morse-Bott inequalities), the first two approaches yield the same chain complex up to sign. The third approach is fundamentally different. It combines singular cubical chains and Morse chains in the same multicomplex, which provides a way of interpolating between the singular cubical chain complex and the Morse-Smale-Witten chain complex.

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1 Introduction

Functions with non-isolated critical points often arise in situations where there is some sort of symmetry or a group action. In such situations the concept of a Morse-Bott function and the homology groups associated to a Morse-Bott function developed by Raoul Bott in the 1950's have proved to be extremely useful [9] [10] [13].

A Morse-Bott function $f : M \to \mathbb{R}$ on a finite dimensional compact smooth closed manifold M is a function that is nondegenerate in the directions normal to its critical submanifolds (Definition 3.1). Bott found several applications of relationships he discovered between the Betti numbers of the critical submanifolds of the Morse-Bott function f and the Betti numbers of the underlying manifold M. These relationships are encoded in the Morse-Bott inequalities (Theorem 3.5), which generalize the Morse inequalities (cf. Section 3.4 of [3]). A Morse function is a Morse-Bott function with isolated critical points, and the Morse inequalities give relationships between the critical points of the Morse function and the Betti numbers of the underlying manifold.

The theory developed by Marston Morse in the 1920's originally gave information about the Betti numbers of manifolds [35], but more modern versions of Morse theory determine

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CW-structures and chain complexes. In particular, the Morse Homology Theorem, which was proved several decades after the Morse inequalities, says that the critical points and gradient flow lines of a Morse function determine a chain complex that computes the homology of the underlying manifold. The Morse inequalities follow as a direct consequence of the Morse Homology Theorem (cf. Section 3 of [5]), and thus one would expect that the chain complex determined by a Morse function should contain more information and have more applications than the Morse inequalities. This is indeed the case. Numerous applications of the Morse chain complex and its infinite dimensional analogues, pioneered by Floer, have been found in many different contexts.

Correspondingly, one would expect that a chain complex determined by a Morse-Bott function that computes the homology of the underlying manifold should contain more information and have more applications than the Morse-Bott inequalities. However, to find these applications one must first understand how a Morse-Bott function determines a chain complex. This understanding is complicated by the fact that there are several different chain complexes associated to a Morse-Bott function, and there are multiple ways that a Morse-Bott function can be used to compute the homology of the underlying manifold.

In this paper we will survey three of the many approaches that have been discovered for using a Morse-Bott function to construct a chain complex that computes the homology of the underlying manifold. The following three approaches to Morse-Bott homology will be discussed, together with relationships among the three approaches.

1. Perturb the Morse-Bott function to a Morse function and use the Morse-Smale-Witten chain complex of the perturbed function.

2. Pick auxiliary Morse functions on the critical submanifolds and use the cascade chain complex defined using the auxiliary functions.

3. Use the Morse-Bott multicomplex, which provides a common framework for both Morse chains and singular cubical chains.

In Section 3 we will explain how the first approach can be used to prove the Morse-Bott inequalities following [5]. In Section 4 we will describe the construction of the cascade chain complex and explain how the cascade chain complex can be viewed as the Morse-Smale-Witten chain complex of a specific perturbation of the Morse-Bott function following [7]. Section 5 will discuss the Morse-Bott multicomplex developed by Banyaga and Hurtubise using singular cubical chains and fibered product constructions [6]. The Morse-Bott multicomplex is fundamentally different from other approaches to Morse-Bott homology. It provides a common framework for singular cubical chains and Morse chains, and thus it provides a way of interpolating between the singular cubical chain complex and the Morse-Smale-Witten chain complex.

Other approaches to Morse-Bott homology that will not be discussed in detail in this paper include: the spectral sequence associated with the filtration of the manifold determined by a Morse-Bott function, the de Rham cochain complex of Austin and Braam [2], and the Morse-Bott chain complex defined using currents due to Latschev [30]. One common distinguishing feature of both Morse and Morse-Bott homology is that the boundary operators that define the homology are expressed in terms of moduli spaces of gradient flow lines.

We will not discuss the spectral sequence associated with the filtration of the manifold because currently there is no known way to relate the differentials in that spectral sequence to moduli spaces of gradient flow lines.

The paper by Austin and Braam [2] was a source of inspiration for the results discussed in Sections 3 and 5, and connections with their work are discussed in those sections. (A paper by Fukaya [24] also served as a source of inspiration for the results discussed in Section 5.) However, the Austin-Braam approach uses differential forms to construct a comulticomplex that computes the de Rham cohomology of the manifold with real coefficients, whereas the three approaches discussed in detail in this paper all concern homology with integer coefficients. Finally, the Morse-Bott chain complex defined by Latschev [30] is part of the program of Harvey and Lawson [25] to approach Morse theory using the de Rham-Federer theory of currents. Although their approach to Morse theory is very interesting, it is also quite different from the other approaches discussed in this paper and will not be further reviewed.

2 The Morse-Smale-Witten chain complex

In this section we briefly recall the construction of the Morse-Smale-Witten chain complex and the Morse Homology Theorem. For more details see [3].

Let $Cr(f) = \{p \in M | df_p = 0\}$ denote the set of critical points of a smooth function $f : M \to \mathbb{R}$ on a finite dimensional smooth manifold M. A critical point $p \in Cr(f)$ is said to be **nondegenerate** if and only if the Hessian $H_p(f)$ is nondegenerate. The **index** λ_p of a nondegenerate critical point p is defined to be the index of the symmetric bilinear form $H_p(f)$, i.e. the dimension of the subspace of T_pM where $H_p(f)$ is negative definite. If all the critical points of f are non-degenerate, then f is called a **Morse function**.

If $f : M \to \mathbb{R}$ is a Morse function on an m-dimensional compact smooth Riemannian manifold (M,g), then the **stable manifold** $W^s_f(p)$ and the **unstable manifold** $W^u_f(p)$ of a critical point $p \in Cr(f)$ are defined to be

$$\begin{aligned} W^s_f(p) &= \{x \in M | \lim_{t\to\infty} \varphi_t(x) = p\} \\ W^u_f(p) &= \{x \in M | \lim_{t\to-\infty} \varphi_t(x) = p\} \end{aligned}$$

where φ_t is the 1-parameter group of diffeomorphisms generated by minus the gradient vector field, i.e. $-\nabla f$. The Stable/Unstable Manifold Theorem for a Morse Function says that the tangent space at p splits as

$$T_pM = T^s_pM \oplus T^u_pM$$

where the Hessian is positive definite on $T^s_pM \stackrel{def}{=} T_pW^s_f(p)$ and negative definite on $T^u_pM \stackrel{def}{=} T_pW^u_f(p)$. Moreover, the stable and unstable manifolds of p are surjective images of smooth embeddings

$$\begin{aligned} E^s : T^s_pM &\to W^s_f(p) \subseteq M \\ E^u : T^u_pM &\to W^u_f(p) \subseteq M. \end{aligned}$$

Hence, $W^s_f(p)$ is a smoothly embedded open disk of dimension $m-\lambda_p$, and $W^u_f(p)$ is a smoothly embedded open disk of dimension λ_p.

If the stable and unstable manifolds of a Morse function $f : M \to \mathbb{R}$ all intersect transversally, then the function f is called **Morse-Smale**. For any metric g on M the set of smooth Morse-Smale functions is dense in the space of all smooth functions on M by the Kupka-Smale Theorem (cf. Theorem 6.6 and Remark 6.7 of [3]), and for a given Morse function $f : M \to \mathbb{R}$ one can always find a Riemannian metric on M so that f is Morse-Smale with respect to the chosen metric (cf. Theorem 2.20 of [1]). Moreover, if f is Morse-Smale and $p, q \in Cr(f)$ then $W_f(q,p) = W^u_f(q) \cap W^s_f(p)$ is an embedded submanifold of M of dimension $\lambda_q - \lambda_p$, and when $\lambda_q - \lambda_p = 1$ the number of gradient flow lines from q to p is finite (cf. Corollary 6.29 of [3]).

If we choose an orientation for each of the unstable manifolds of f, then there is an induced orientation on the normal bundles of the stable manifolds. Thus, we can define an integer associated to any two critical points p and q of relative index one by counting the number of gradient flow lines from q to p with signs determined by the orientations. This integer is denoted by $n_f(q,p) = \#\mathcal{M}_f(q,p)$, where $\mathcal{M}_f(q,p) = W_f(q,p)/\mathbb{R}$ is the moduli space of gradient flow lines of f from q to p. The **Morse-Smale-Witten chain complex** is defined to be the chain complex $(C_*(f), \partial_*)$ where $C_k(f)$ is the free abelian group generated by the critical points q of index k and the boundary operator $\partial_k : C_k(f) \to C_{k-1}(f)$ is given by

$$\partial_k(q) \;=\; \sum_{p \in Cr_{k-1}(f)} n_f(q,p)\,p$$

where $Cr_{k-1}(f)$ denotes the set of critical points with index $k-1$.

Theorem 2.1 (Morse Homology Theorem)**.** *The pair $(C_*(f), \partial_*)$ is a chain complex, and the homology of $(C_*(f), \partial_*)$ is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.*

Note that the Morse Homology Theorem implies that the homology of $(C_*(f), \partial_*)$ is independent of the Morse-Smale function $f : M \to \mathbb{R}$, the Riemannian metric, and the chosen orientations.

There are many different ways to prove the Morse Homology Theorem. The approach in [3] uses the Conley index and Conley's connection matrix to give an explicit isomorphism between the Morse homology groups and the singular homology groups. Another approach is to first show that the unstable manifolds of a Morse-Smale function $f : M \to \mathbb{R}$ determine a CW-structure X on M, and then show that the chain complex $(\underline{C}_*(X), \partial^{CW}_*)$ determined by the CW-structure is the same as the Morse-Smale-Witten chain complex $(C_*(f), \partial_*)$. Both of these steps are nontrivial. For instance, ∂^{CW}_* is induced from a connecting homomorphism in the homology exact sequence of a triple, whereas ∂_* is defined by counting gradient flow lines. (See Section 7.1 of [3] for a more complete discussion of the technical details encountered when using this approach.)

However, in spite of the difficulties, it is possible to prove that the unstable manifolds of a Morse-Smale function $f : M \to \mathbb{R}$ determine a CW-structure X on M and the following

diagram commutes

$$\begin{array}{ccc} C_k(f) & \xrightarrow{\partial_k} & C_{k-1}(f) \\ \uparrow\downarrow \approx & & \uparrow\downarrow \approx \\ \underline{C}_k(X) & \xrightarrow{\partial_k^{CW}} & \underline{C}_{k-1}(X) \end{array}$$

for every $k \in \mathbb{Z}_+$, where the vertical maps are induced by identifying critical points with their unstable manifolds. For more details concerning this approach see the recent papers by Qin [36] [37] and the references therein. For a list of other approaches to proving the Morse Homology Theorem see the introduction to [3].

3 Perturbing a Morse-Bott function to a Morse function

The chain groups in the Morse-Smale-Witten chain complex are finitely generated because a Morse function $f : M \to \mathbb{R}$ on a finite dimensional compact manifold has a finite number of isolated critical points. If the critical points of f are not isolated, then f has an infinite number of critical points and the Morse Homology Theorem does not apply. In this case, some additional assumptions and/or auxiliary data are required to construct a chain complex, a multicomplex, or a spectral sequence.

Let $f : M \to \mathbb{R}$ be a smooth function whose critical set $\mathrm{Cr}(f)$ contains a submanifold C of positive dimension. Pick a Riemannian metric on M and use it to split $T_*M|_C$ as

$$T_*M|_C = T_*C \oplus \nu_*C$$

where T_*C is the tangent space of C and ν_*C is the normal bundle of C. Let $p \in C$, $V \in T_pC$, $W \in T_pM$, and let $H_p(f)$ be the Hessian of f at p. We have

$$H_p(f)(V, W) = V_p \cdot (\tilde{W} \cdot f) = 0$$

since $V_p \in T_pC$ and any extension of W to a vector field $\tilde{W}$ satisfies $df(\tilde{W})|_C = 0$. Therefore, the Hessian $H_p(f)$ induces a symmetric bilinear form $H_p^\nu(f)$ on ν_pC.

Definition 3.1. A smooth function $f : M \to \mathbb{R}$ on a smooth manifold M is called a **Morse-Bott function** if and only if the set of critical points $\mathrm{Cr}(f)$ is a disjoint union of connected submanifolds and for each connected submanifold $C \subseteq \mathrm{Cr}(f)$ the bilinear form $H_p^\nu(f)$ is non-degenerate for all $p \in C$. The Morse-Bott **index** of a critical submanifold $C \subseteq \mathrm{Cr}(f)$ is defined to be the index of $H_p^\nu(f)$ for any $p \in C$.

Note: The Morse-Bott index is well defined by the Morse-Bott Lemma (cf. Section 3.5 of [3]).

A well known theorem says that on a compact closed smooth manifold the space of smooth Morse functions is open and dense in the space of all smooth functions (cf. Theorem 5.31 of [3]). So, one approach to computing homology from a Morse-Bott function would be to perturb the Morse-Bott function to a Morse function and apply Theorem 2.1 (Morse Homology Theorem) using a metric on M such that the perturbed function is Morse-Smale with respect to the chosen metric.

Perturbing a Morse-Bott function to a Morse function using abstract perturbations and defining a Morse-like chain complex associated to the perturbed function is a standard technique used in gauge theory with respect to the Chern-Simons functional and in Floer theory with respect to the symplectic action functional. In the setting of a Morse-Bott function $f : M \to \mathbb{R}$ on a finite dimensional orientable compact smooth manifold M, Austin and Braam defined a more explicit perturbation of $f : M \to \mathbb{R}$ to a Morse function $h : M \to \mathbb{R}$ by introducing Morse functions on the critical submanifolds [2].

Austin and Braam used their perturbation technique to compare the homology of a filtered cochain complex (C^*, ∂^*) they defined using differential forms on the critical submanifolds of a Morse-Bott-Smale function $f : M \to \mathbb{R}$ (Definition 4.4) with the Morse-Smale-Witten cochain complex $(C^*(h) \otimes \mathbb{R}, \partial^*)$ of the perturbed function $h : M \to \mathbb{R}$ (Proposition 3.10 of [2]). Austin and Braam's cochain complex (C^*, ∂^*) has the structure of a comulticomplex (Definition 5.1), which determines a spectral sequence coming from the filtration [27]. By exhibiting a chain morphism between the filtered cochain complexes (C^*, ∂^*) and $(C^*(h) \otimes \mathbb{R}, \partial^*)$ that induces an isomorphism of the E^1 terms of the spectral sequences determined by the filtrations, Austin and Braam prove that there is also an isomorphism on the E^∞ terms, and hence an induced isomorphism on the homology of the filtered cochain complexes. This proves that both cochain complexes compute the de Rham cohomology of M.

Corollary 3.9 of [2] states without proof that the polynomial Morse-Bott inequalities follow from the fact that the comulticomplex (C^*, ∂^*) computes the de Rham cohomology of M. A proof of the Morse-Bott inequalities along those lines would most likely involve an analysis of the spectral sequence determined by the comulticomplex, and hence would not be as immediate as proving that the polynomial Morse inequalities follow from the existence of a CW-complex determined by the Morse function or the Morse Homology Theorem (cf. Section 3.4 of [3] and Section 3 of [5]). However, Banyaga and Hurtubise showed in [5] that it is possible to apply the perturbation technique used by Austin and Braam together with the polynomial Morse inequalities to prove the polynomial Morse-Bott inequalities without appealing to the full Morse-Bott multicomplex.

The polynomial Morse-Bott inequalities

Let M be a compact smooth manifold of dimension m, and define the k^{th} **Betti number** of M, denoted b_k, to be the rank of the k^{th} homology group $H_k(M;\mathbb{Z})$ modulo its torsion subgroup. Let $f : M \to \mathbb{R}$ be a smooth Morse function on M, and let ν_k denote the number of critical points of f of index k for all $k = 0, \ldots, m$.

Definition 3.2. The **Poincaré polynomial** of M is defined to be

$$P_t(M) = \sum_{k=0}^{m} b_k t^k,$$

and the **Morse polynomial** of f is defined to be

$$M_t(f) = \sum_{k=0}^{m} \nu_k t^k.$$

For a proof of the following theorem using the fact that $f : M \to \mathbb{R}$ determines a CW-complex homotopic to M with ν_k cells of dimension k for all $k = 0, \ldots, m$ see Section 3.4 of [3]. For a similar proof that uses the Morse Homology Theorem instead of the CW-complex see Section 3 of [6].

Theorem 3.3 (Polynomial Morse Inequalities)**.** *For any Morse function $f : M \to \mathbb{R}$ on a compact smooth manifold of dimension m we have*

$$M_t(f) = P_t(M) + (1+t)R(t)$$

where $R(t)$ is a polynomial with non-negative integer coefficients. That is, $R(t) = \sum_{k=0}^{m-1} r_k t^k$ where $r_k \in \mathbb{Z}$ satisfies $r_k \geq 0$ for all $k = 0, \ldots, m-1$.

Now let $f : M \to \mathbb{R}$ be a Morse-Bott function, and assume that

$$\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j,$$

where $C_1, \ldots, C_l$ are disjoint connected critical submanifolds.

Definition 3.4. The Morse-Bott polynomial of f is defined to be

$$MB_t(f) = \sum_{j=1}^{l} P_t(C_j) t^{\lambda_j}$$

where λ_j is the Morse-Bott index of the critical submanifold C_j and $P_t(C_j)$ is the Poincaré polynomial of C_j.

Bott proved a version of the following result stated in terms of Betti numbers of homology with local coefficients in an orientation bundle in place of any orientation assumptions [9] [11].

Theorem 3.5 (Morse-Bott Inequalities)**.** *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional orientable compact smooth manifold, and assume that all the critical submanifolds of f are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that*

$$MB_t(f) = P_t(M) + (1+t)R(t).$$

Bott proved his version of this theorem by studying how the homotopy type of the "half-spaces" $M^y = f^{-1}(-\infty, y]$ change as y crosses critical values. In [5], Banyaga and Hurtubise gave a proof of Theorem 3.5 via a dynamical systems approach by expanding on the perturbation technique used by Austin and Braam in [2].

Outline of the Banyaga-Hurtubise proof

Chose a small tubular neighborhood T_j around each connected component $C_j \subseteq \mathrm{Cr}(f)$ for all $j = 1,\ldots,l$ with local coordinates (u,v,w) consistent with those from the Morse-Bott Lemma (cf. Section 3.5 of [3] or [4]). Pick a Riemannian metric on M such that the charts from the Morse-Bott Lemma are isometries with respect to the standard Euclidean metric on $\mathbb{R}^m$, and then pick positive Morse functions $f_j : C_j \to \mathbb{R}$ that are Morse-Smale with respect to the restriction of the Riemannian metric to C_j for all $j = 1,\ldots,l$. The Morse-Smale functions $f_j : C_j \to \mathbb{R}$ exist by the Kupka-Smale Theorem.

For every $j = 1,\ldots,l$ extend f_j to a function on T_j by making f_j constant in the direction normal to C_j. Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j with the same coordinates as T_j, and let ρ_j be a smooth nonincreasing bump function that is constant in the direction parallel to C_j, equal to 1 on $\tilde{T}_j$, and equal to 0 outside of T_j. For a small $\varepsilon > 0$ the function $h : M \to \mathbb{R}$ given by

$$h = f + \varepsilon\left(\sum_{j=1}^{l} \rho_j f_j\right)$$

is a Morse function close to f, and the critical points of h are exactly the critical points of the f_j for $j = 1,\ldots,l$. Moreover, if $p \in C_j$ is a critical point of $f_j : C_j \to \mathbb{R}$ of index λ_p^j, then p is a critical point of h of index $\lambda_p^h = \lambda_j + \lambda_p^j$. A well-known folk theorem (cf. Section 2.12 of [1]) says that it is possible to perturb the Riemannian metric on M outside of the union of the tubular neighborhoods T_j for $j = 1,\ldots,l$ so that h satisfies the Morse-Smale transversality condition with respect to the perturbed metric.

This explicit perturbation and choice of metric makes it possible to compare the Morse-Smale-Witten chain complex of h with those of f_j for $j = 1,\ldots,l$. In particular, for every $n = 0,\ldots,m$ we have the following description of the n^{th} Morse-Smale-Witten chain group of h in terms of the Morse-Smale-Witten chain groups of the f_j for $j = 1,\ldots,l$.

$$C_n(h) = \bigoplus_{\lambda_j + k = n} C_k(f_j)$$

Now let $M_t(f_j)$ denote the Morse polynomial of $f_j : C_j \to \mathbb{R}$, and note that the relation $\lambda_p^h = \lambda_j + \lambda_p^j$ implies that

$$M_t(h) = \sum_{j=1}^{l} M_t(f_j) t^{\lambda_j}.$$

The polynomial Morse inequalities (Theorem 3.3) say that

$$M_t(h) = P_t(M) + (1+t)R_h(t)$$

and

$$M_t(f_j) = P_t(C_j) + (1+t)R_j(t)$$

where $R_h(t)$ and $R_j(t)$ are polynomials with non-negative integer coefficients for all $j =$

$1,\dots,l$. This leads to the following straightforward computation.

$$\begin{aligned}
MB_t(f) &= \sum_{j=1}^{l} P_t(C_j)t^{\lambda_j} \\
&= \sum_{j=1}^{l}\Big(M_t(f_j) - (1+t)R_j(t)\Big)t^{\lambda_j} \\
&= \sum_{j=1}^{l} M_t(f_j)t^{\lambda_j} - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\
&= M_t(h) - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\
&= P_t(M) + (1+t)R_h(t) - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\
&= P_t(M) + (1+t)\Big(R_h(t) - \sum_{j=1}^{l} R_j(t)t^{\lambda_j}\Big)
\end{aligned}$$

It remains to show that the polynomial multiplying $(1+t)$ in the last line has non-negative integer coefficients. This is accomplished by first noting that the proof of the polynomial Morse inequalities from the Morse Homology Theorem shows that the polynomial $R_j(t)$ is given by

$$R_j(t) = \sum_{k=1}^{c_j}(\nu_k^j - z_k^j)t^{k-1}$$

where $c_j = \dim C_j$, $\nu_k^j = \operatorname{rank} C_k(f_j)$, and z_k^j is the rank of the kernel of the boundary operator $\partial_k^{f_j} : C_k(f_j) \to C_{k-1}(f_j)$ in the Morse-Smale-Witten chain complex of $f_j : C_j \to \mathbb{R}$. The proof of Theorem 3.5 is then completed by using the close relationship between the dynamics of the gradient flow lines of f, f_j, and h to show that $\sum_{\lambda_j+k=n} z_k^j \geq z_n^h$ for all $n = 1,\dots,m$, where z_n^h is the rank of the kernel of the boundary operator $\partial_n^h : C_k(h) \to C_{k-1}(h)$ in the Morse-Smale-Witten chain complex of $h : M \to \mathbb{R}$.

Comparison with Bott's proof

Bott's proof of the polynomial Morse-Bott inequalities was based on studying what he called "half-spaces" $M^y = f^{-1}(-\infty, y]$ [12]. The homotopy type of M^y is the same on any interval $a < y < b$ that doesn't contain a critical value, and when y crosses a critical value c the homotopy type changes by the attachment of disk bundles whose dimensions are given by the Morse-Bott indexes of the critical submanifolds in the level set $f^{-1}(c)$ (cf. Appendix B of [20]).

Bott's original version of the polynomial Morse-Bott inequalities avoided any orientation assumptions by using Betti numbers with local coefficients in an orientation bundle in place of the standard Betti numbers. The orientation bundles of the critical submanifolds he

considered are determined by disk bundles given by the unstable part of the gradient flow near the critical submanifolds. When these disk bundles are orientable the Betti numbers with local coefficients in the orientation bundles reduce to the standard Betti numbers of the critical submanifolds. For more details see the original papers by Bott or Appendix C of [20].

It is interesting to note that Bott's version of the Morse-Bott inequalities reduces to the conclusion of Theorem 3.5 when the disk bundles given by the negative part of the gradient flow near the submanifolds are orientable, whereas the proof given by Banyaga and Hurtubise assumes that M and the critical submanifolds are orientable. That is, the tangent space of M along a critical submanifold C has a decomposition

$$T_*M = T_*C \oplus \nu_*^- C \oplus \nu_*^+ C,$$

and Banyaga and Hurtubise assumed that T_*M and T_*C are orientable in order to prove Theorem 3.5. On the other hand, it is the assumption that the bundle $\nu_*^- C$ is orientable that allows one to conclude that the Betti numbers with local coefficients in the orientation bundle used by Bott reduce to the Betti numbers considered in Theorem 3.5. These two conditions are distinct when $\nu_*^+ C$ is not orientable.

4 Cascades

A second approach to computing homology using a Morse-Bott function involves introducing Morse functions on the critical submanifolds and defining chain groups generated by the critical points of the Morse functions that agree with those defined in the previous section. However, the boundary operator is defined by counting the number of "cascades" between two critical points of relative index one, which are defined without reference to the perturbed function. Roughly speaking, a cascade between two critical points is a concatenation of some gradient flow lines of the Morse-Bott function and pieces of the gradient flow lines of the Morse functions on the critical submanifolds. Cascades were introduced by Frauenfelder in [22] [23], and cascade-like objects were introduced independently in the context of holomorphic curves by Bourgeois in [14] [15]. Cascades have since been used by several authors studying symplectic and contact homology [15] [16] [17] [19]. We begin our discussion of cascades with the following definitions from [7].

Cascades and Morse-Bott-Smale transversality

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional compact smooth manifold, and let

$$\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j,$$

where $C_1, \ldots, C_l$ are disjoint connected critical submanifolds of Morse-Bott index $\lambda_1, \ldots, \lambda_l$ respectively. Let $f_j : C_j \to \mathbb{R}$ be a Morse function on the critical submanifold C_j for all $j = 1, \ldots, l$. For a critical point $q \in C_j$ of $f_j : C_j \to \mathbb{R}$ denote the Morse index of q relative to f_j by λ_q^j, the stable manifold of q relative to f_j by $W_{f_j}^s(q) \subseteq C_j$, and the unstable manifold of q relative to f_j by $W_{f_j}^u(q) \subseteq C_j$.

Definition 4.1. If $q \in C_j$ is a critical point of the Morse function $f_j : C_j \to \mathbb{R}$ for some $j = 1, \ldots, l$, then the **total index** of q, denoted λ_q, is defined to be the sum of the Morse-Bott index of C_j and the Morse index of q relative to f_j, i.e.

$$\lambda_q = \lambda_j + \lambda_q^j.$$

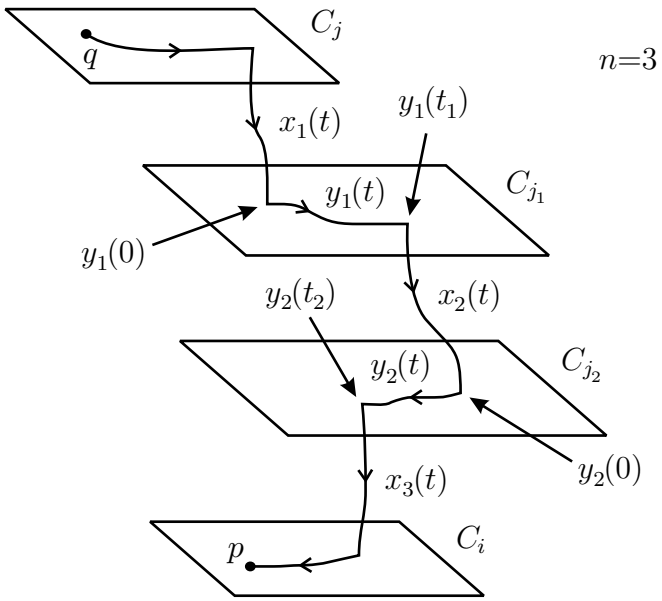
Definition 4.2. For $q \in \mathrm{Cr}(f_j)$, $p \in \mathrm{Cr}(f_i)$, and $n \in \mathbb{N}$, a **flow line with n cascades from q to p** is a $2n-1$-tuple:

$$((x_k)_{1\le k\le n}, (t_k)_{1\le k\le n-1})$$

where $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_+ = \{t \in \mathbb{R} |\ t \ge 0\}$ satisfy the following for all k.

1. Each x_k is a non-constant gradient flow line of f, i.e.
$$\frac{d}{dt}x_k(t) = -(\nabla f)(x_k(t)).$$
2. For the first cascade $x_1(t)$ we have
$$\lim_{t\to-\infty} x_1(t) \in W^u_{f_j}(q) \subseteq C_j,$$
and for the last cascade $x_n(t)$ we have
$$\lim_{t\to\infty} x_n(t) \in W^s_{f_i}(p) \subseteq C_i.$$
3. For $1 \le k \le n-1$ there are critical submanifolds C_{j_k} and gradient flow lines $y_k \in C^\infty(\mathbb{R}, C_{j_k})$ of f_{j_k}, i.e.
$$\frac{d}{dt}y_k(t) = -(\nabla f_{j_k})(y_k(t)),$$
such that $\lim_{t\to\infty} x_k(t) = y_k(0)$ and $\lim_{t\to-\infty} x_{k+1}(t) = y_k(t_k)$.

When $j = i$ a **flow line with zero cascades from q to p** is a gradient flow line of f_j from q to p.



Note: In the preceding definition the parameterizations of the gradient flow lines $y_k(t)$ of the Morse functions $f_{j_k} : C_{j_k} \to \mathbb{R}$ are fixed in (3) by $\lim_{t\to\infty} x_k(t) = y_k(0)$, and the entry t_k records the time spent flowing along the critical submanifold C_{j_k} (or resting at a critical point). However, the parameterizations of the cascades $x_1(t), \ldots, x_n(t)$ are not fixed. Hence, there is an action of $\mathbb{R}^n$ on a flow line with n cascades given by

$$((x_k(t))_{1\le k\le n}, (t_k)_{1\le k\le n-1}) \mapsto ((x_k(t+s_k))_{1\le k\le n}, (t_k)_{1\le k\le n-1})$$

for $(s_1, \ldots, s_n) \in \mathbb{R}^n$.

Definition 4.3. For $q \in \mathrm{Cr}(f_j)$, $p \in \mathrm{Cr}(f_i)$, and $n \in \mathbb{N}$ we denote the space of flow lines from q to p with n cascades by $W_n^c(q,p)$, and we denote the quotient of $W_n^c(q,p)$ by the action of $\mathbb{R}^n$ by

$$\mathcal{M}_n^c(q,p) = W_n^c(q,p)/\mathbb{R}^n.$$

The **set of unparameterized flow lines with cascades from** q **to** p is defined to be

$$\mathcal{M}^c(q,p) = \bigcup_{n\in\mathbb{Z}_+} \mathcal{M}_n^c(q,p)$$

where $\mathcal{M}_0^c(q,p) = W_0^c(q,p)/\mathbb{R}$. We will say that an element of $\mathcal{M}^c(q,p)$ **begins** at q and **ends** at p.

Somewhat surprisingly, under the right conditions moduli spaces of cascades have properties similar to moduli spaces of gradient flow lines of a Morse-Smale function. In particular, under various assumptions it is possible to prove that $\mathcal{M}^c(q,p)$ is a smooth manifold of dimension $\lambda_q - \lambda_p - 1$, and every sequence in $\mathcal{M}^c(q,p)$ has a subsequence that converges to a broken flow line with cascades. These two fundamental properties imply that there are only a finite number of cascades between any two critical points of relative index one, and hence it is possible to define a boundary operator ∂_*^c by counting cascades. (Proving directly that $\partial_*^c \circ \partial_*^c = 0$ requires a stronger result. Namely, that $\mathcal{M}^c(q,p)$ has a compactification consisting of broken flow lines with cascades when $\lambda_q - \lambda_p = 2$.)

These fundamental properties were proved by Frauenfelder in [22] under the assumptions that the Riemannian metric on M and the Riemannian metrics on the critical submanifolds meet certain generic conditions that imply that a particular Fredholm operator is surjective. They were also proved by Banyaga and Hurtubise in [7] under the assumptions that the Morse-Bott function satisfies the Morse-Bott-Smale transversality condition and the unstable and stable manifolds of the Morse functions on the critical submanifolds are transverse to certain beginning and endpoint maps.

Definition 4.4 (Morse-Bott-Smale Transversality)**.** A Morse-Bott function $f : M \to \mathbb{R}$ is said to satisfy the **Morse-Bott-Smale transversality** condition with respect to a given Riemannian metric g on M if and only if for any two connected critical submanifolds C and C', $W_f^u(q)$ intersects $W_f^s(C')$ transversely in M, i.e. $W_f^u(q) \pitchfork W_f^s(C') \subseteq M$, for all $q \in C$.

The Morse-Bott-Smale transversality condition depends on both the function and the Riemannian metric, and it may not be possible to perturb the metric to make a given Morse-Bott function satisfy the Morse-Bott-Smale transversality condition. For some interesting

examples see Section 2 of [30]. This is quite different from the situation encountered with the Morse-Smale transversality condition where it is always possible to perturb either the function or the metric to make the condition hold. (Of course, one can always perturb the Morse-Bott function to a Morse function to make the Morse-Bott-Smale transversality condition hold, but that approach was already discussed in the previous section.)

Moduli spaces of gradient flow lines and cascades

There are many technical consequences of the Morse-Bott-Smale transversality condition that have implications for moduli spaces of gradient flow lines. For instance, the moduli space of gradient flow lines between two critical submanifolds C_k and $C_{k'}$ of a Morse-Bott-Smale function $f : M \to \mathbb{R}$

$$\mathcal{M}_f(C_k, C_{k'}) = \left(W^u_f(C_k) \cap W^s_f(C_{k'})\right)/\mathbb{R}$$

is a manifold of dimension $\lambda_k - \lambda_{k'} + \dim C_k - 1$ (cf. Lemma 3.5 of [6]), and the beginning point map

$$\partial_- : \mathcal{M}_f(C_k, C_{k'}) \to C_k$$

sending a gradient flow line to its starting point is a submersion (cf. Lemma 5.19 of [6]). In fact, the moduli space of gradient flow lines $\mathcal{M}_f(C_k, C_{k'})$ has a compactification $\overline{\mathcal{M}}_f(C_k, C_{k'})$ consisting of broken gradient flow lines, which is a smooth manifold with corners, and the beginning point map

$$\partial_- : \overline{\mathcal{M}}_f(C_k, C_{k'}) \to C_k$$

is both a submersion and a stratum submersion (cf. Corollary 5.20 of [6]).

These consequences of the Morse-Bott-Smale transversality condition were used by Banyaga and Hurtubise to construct smooth manifolds with corners defined in terms of iterated fibered products over the beginning and endpoint maps [6] and over the beginning point map and the endpoint map composed with the gradient flow along the critical submanifolds [7]. In the second case, the iterated fibered products can be viewed as spaces of cascades from one critical submanifold to another. This leads to the condition that the beginning and endpoint maps from the iterated fibered products are transverse to the unstable and stable manifolds of the Morse functions on the critical submanifolds; a condition that is always satisfied by an arbitrarily small perturbation of the Morse functions on the critical submanifolds [7].

Theorem 4.5. *Assume that f satisfies the Morse-Bott-Smale transversality condition with respect to the Riemannian metric g on M, $f_k : C_k \to \mathbb{R}$ satisfies the Morse-Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \ldots, l$, and the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps.*

1. *When $n = 0, 1$ the set $\mathcal{M}^c_n(q, p)$ is either empty or a smooth manifold without boundary.*
2. *For $n > 1$ the set $\mathcal{M}^c_n(q, p)$ is either empty or a smooth manifold with corners.*
3. *The set $\mathcal{M}^c(q, p)$ is either empty or a smooth manifold without boundary.*

In each case the dimension of the manifold is $\lambda_q - \lambda_p - 1$. When M is orientable and C_k is orientable for all $k = 1, \ldots, l$, the above manifolds are orientable.

At first glance, it may seem strange that $\mathcal{M}_n^c(q,p)$ is a smooth manifold with corners for $n > 1$ whereas

$$\mathcal{M}^c(q,p) = \bigcup_{n \in \mathbb{Z}_+} \mathcal{M}_n^c(q,p)$$

is a smooth manifold without boundary. However, the proof of the above theorem in [7] shows that the smooth manifolds with corners $\mathcal{M}_n^c(q,p)$ glue together to form the manifold without boundary $\mathcal{M}^c(q,p)$, similar to the way that the manifold with boundary $[0,\infty)$ can be glued to $(-\infty,0)$ to create the manifold without boundary $(-\infty,\infty)$.

Compactness for moduli spaces of cascades

In order to define a boundary operator by counting the number of cascades between two critical points, the compactness properties of the manifold $\mathcal{M}^c(q,p)$ must be addressed. As one might expect, $\mathcal{M}^c(q,p)$ in general won't be compact unless $\lambda_q - \lambda_p = 1$, because a sequence in $\mathcal{M}^c(q,p)$ may converge to a broken flow line with cascades from q to p. However, the precise definition of a "broken flow line with cascades" turns out to be more subtle than the definition of a "broken gradient flow line".

For a Morse-Bott function $f : M \to \mathbb{R}$, a broken gradient flow line is simply a concatenation of gradient flow lines. As such, a broken gradient flow line can be represented by an n-tuple $(x_1, \ldots, x_n)$ where x_k is a gradient flow line of f for all $1 \leq k \leq n$ and $\lim_{t\to\infty} x_k(t) = \lim_{t\to-\infty} x_{k+1}(t)$ for all $1 \leq k \leq n-1$. The second condition can be interpreted as saying that the time spent flowing along each intermediate critical submanifold is 0, and hence there is an obvious identification of the broken gradient flow line represented by $(x_1, \ldots, x_n)$ with the (non-broken) flow line with n cascades $((x_k)_{1\leq k\leq n}, (t_k)_{1\leq k\leq n-1})$ where $t_k = 0$ for all $1 \leq k \leq n-1$.

This identification is compatible with the topology of the space of cascades $\mathcal{M}^c(q,p)$ and the topology of the space of broken gradient flow lines $\overline{\mathcal{M}}_f(C_j, C_i)$. That is, suppose $\{\gamma_k\} \subset \mathcal{M}_f(C_j, C_i)$ is a sequence of unparameterized gradient flow lines of a Morse-Bott-Smale function $f : M \to \mathbb{R}$, with $\partial_-(\gamma_k) \in W^u_{f_j}(q) \subseteq C_j$ and $\partial_+(\gamma_k) \in W^s_{f_i}(p) \subset C_i$ for all k, that converges to a broken gradient flow line in $\overline{\mathcal{M}}_f(W^u_{f_j}(q), W^s_{f_i}(p))$ represented by $(x_1, \ldots, x_n)$. Then the proof of Theorem 4.5 shows that the sequence $\{\gamma_k\}$, viewed as a subset of $\mathcal{M}^c(q,p)$, converges to the unparameterized flow line with cascades represented by $((x_k)_{1\leq k\leq n}, (t_k)_{1\leq k\leq n-1})$ where $t_k = 0$ for all $1 \leq k \leq n-1$. Thus, the broken gradient flow lines from $W^u_{f_j}(q) \subseteq C_j$ to $W^s_{f_i}(p) \subseteq C_i$ are already included in the space of (unbroken) cascades $\mathcal{M}^c(q,p)$ in the sense that the above identification induces an embedding $\overline{\mathcal{M}}_f(W^u_{f_j}(q), W^s_{f_i}(p)) \hookrightarrow \mathcal{M}^c(q,p)$ making the following diagram commute.

$$\begin{array}{ccc} \mathcal{M}_f(W^u_{f_j}(q), W^s_{f_i}(p)) & \hookrightarrow & \overline{\mathcal{M}}_f(W^u_{f_j}(q), W^s_{f_i}(p)) \\ & \searrow & \downarrow \\ & & \mathcal{M}^c(q,p) \end{array}$$

Note: There are several equivalent ways of defining the topology on the space of broken gradient flow lines $\overline{\mathcal{M}}_f(W^u_{f_j}(q), W^s_{f_i}(p))$, see Section 2 of [26] or [31] for more details.

So, what is a "broken flow line with cascades"? Upon further analysis it turns out that a "broken flow line with cascades" should (roughly speaking) be a concatenation of unparameterized flow lines with cascades that either flows along an intermediate critical submanifold for infinite time or rests at an intermediate critical point of one of the Morse functions on the critical submanifolds for infinite time. This description of the space $\overline{\mathcal{M}}^c(q,p)$ of broken flow lines with cascades from q to p (and its topology) was made precise by Banyaga and Hurtubise in [7] by identifying the set of broken flow lines with cascades with a set of compact subsets of a compact metric space, whose topology is determined by the Hausdorff metric.

Definition 4.6. Let (X,d) be a compact metric space and let K_1 and K_2 be nonempty closed subsets of X. The **Hausdorff distance** between K_1 and K_2 is defined to be

$$\begin{aligned} d_H(K_1,K_2) &= \max\left\{\sup_{x_1\in K_1}\inf_{x_2\in K_2} d(x_1,x_2), \sup_{x_2\in K_2}\inf_{x_1\in K_1} d(x_1,x_2)\right\} \\ &= \inf\{\varepsilon>0 |\ K_1\subseteq N_\varepsilon(K_2) \text{ and } K_2\subseteq N_\varepsilon(K_1)\} \end{aligned}$$

where $N_\varepsilon(K)=\bigcup_{y\in K}\{x\in X |\ d(x,y)\le\varepsilon\}$.

An unparameterized gradient flow line of a Morse-Bott function $f: M\to\mathbb{R}$ can be identified with its image in M, and this image will be a compact subset of M diffeomorphic to $\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$ as long as we include the limits of the gradient flow in the image. However, an unparameterized flow line with cascades may "rest" at an intermediate critical point, and hence the map that sends an unparameterized flow line with cascades to its image might not be injective. In order to get an injective map one needs to keep track of the times t_k spent flowing along or resting on the intermediate critical submanifolds. This leads to a continuous injection

$$\mathcal{M}^c(q,p)\hookrightarrow\mathcal{P}^c(M)\times\overline{\mathbb{R}}^l$$

where $\mathcal{P}^c(M)$ denotes the space of all compact subsets of M and l is the number of critical submanifolds. All these ideas can then be extended to the space of broken flow lines with cascades $\overline{\mathcal{M}}^c(q,p)$ by considering the images of broken gradient flow lines of the Morse functions $f_k: C_k\to\mathbb{R}$ on the critical submanifolds and allowing the t_k to be ∞. From this point of view, the topology on the space of unparameterized broken flow lines with cascades is the topology the set inherits as a subspace of $\mathcal{P}^c(M)\times\overline{\mathbb{R}}^l$, i.e. the topology determined by the Hausdorff metric.

In [7] Banyaga and Hurtubise used these ideas to prove the following theorem.

Theorem 4.7. *The space $\overline{\mathcal{M}}^c(q,p)$ of broken flow lines with cascades is compact, and there is an injection that restricts to a continuous embedding*

$$\mathcal{M}^c(q,p)\hookrightarrow\overline{\mathcal{M}}^c(q,p)\subset\mathcal{P}^c(M)\times\overline{\mathbb{R}}^l.$$

Hence, every sequence of unparameterized flow lines with cascades from q to p has a subsequence that converges to a broken flow line with cascades from q to p.

The following fundamental property is a straightforward consequence of this theorem.

Corollary 4.8. *If $\lambda_q - \lambda_p = 1$, then $\mathcal{M}^c(q,p)$ is compact and hence a finite set.*

The cascade chain complex

We are now in a position to use the moduli spaces $\mathcal{M}^c(q,p)$ to define a cascade chain complex $(C^c_*(f), \partial^c_*)$ whose boundary operator is determined by counting cascades. Let $Cr = \bigcup_{j=1}^{l} Cr(f_j)$ be the collection of critical points of the Morse functions $f_j : C_j \to \mathbb{R}$, let $Cr_k \subseteq Cr$ be the collection of critical points whose total index is k, and let $C^c_k(f)$ be the free abelian group generated by the elements in Cr_k. We would like to define a boundary operator

$$\partial^c_k : C^c_k(f) \to C^c_{k-1}(f)$$

by counting the number of elements in $\mathcal{M}^c(q,p)$, where $q \in Cr_k$ and $p \in Cr_{k-1}$, either over $\mathbb{Z}_2$ or over $\mathbb{Z}$ with signs determined by some orientations.

The approach taken in [21] is to count the cascades over $\mathbb{Z}_2$, which gives a chain complex that computes the homology of M with coefficients in $\mathbb{Z}_2$. One reason for only counting the cascades mod 2 in [21] is that the approach used there to construct the moduli spaces $\mathcal{M}^c(q,p)$ doesn't readily yield orientations on the moduli spaces. In contrast, the approach used by Banyaga and Hurtubise to prove Theorem 4.5 shows that the moduli spaces are orientable when M and the critical submanifolds are orientable, and it is possible to define a coherent system of orientations for the moduli spaces.

However, even though it would be possible to define a coherent system of orientations for the moduli spaces $\mathcal{M}^c(q,p)$, the main theorem in [7] is a correspondence theorem that says that when $\lambda_q - \lambda_p = 1$ there is a bijection

$$\mathcal{M}^c(q,p) \leftrightarrow \mathcal{M}_{h_\varepsilon}(q,p)$$

between the moduli space of cascades and the moduli space of gradient flow lines of the perturbed function

$$h_\varepsilon \stackrel{\text{def}}{=} f + \varepsilon\left(\sum_{j=1}^{l} \rho_j f_j\right)$$

discussed in Section 3 for $\varepsilon > 0$ sufficiently small. So, the approach taken in [7] is to use the Correspondence Theorem to transfer the orientations on $\mathcal{M}_{h_\varepsilon}(q,p)$ to $\mathcal{M}^c(q,p)$ and then define the boundary operator ∂^c_* over $\mathbb{Z}$ by counting cascades with signs given by the induced orientations.

This approach shows immediately that $\partial^c_* \circ \partial^c_* = 0$ and

$$H_*(C^c_*(f), \partial^c_*) \approx H_*(C_*(h_\varepsilon), \partial^{h_\varepsilon}_*) \approx H_*(M; \mathbb{Z}).$$

Moreover, it proves that the chain complex defined using cascades is the same as the Morse-Smale-Witten chain complex of the perturbed function $h_\varepsilon : M \to \mathbb{R}$, i.e. the generators of the two chain complexes are the same and the boundary operators agree up to sign. This is a much stronger result than the statement that the two chain complexes compute the same homology.

Remark. It should be noted that Bourgeois and Oancea used a similar approach to orienting moduli spaces of cascades in the context of symplectic homology [16] [17]. That is, they proved a correspondence theorem between moduli spaces of cascades and moduli spaces of flow lines of a perturbed function, and then they oriented the moduli spaces of cascades using their Correspondence Theorem. In particular, see Theorem 3.7 (Correspondence Theorem) in [17] and the discussion that follows.

Proving the Correspondence Theorem using the Exchange Lemma

The proof of the Correspondence Theorem in [7] has several steps. Starting with a Morse-Bott function $f : M \to \mathbb{R}$, a Riemannian metric g on M such that f satisfies the Morse-Bott-Smale transversality condition with respect to g, and an $\varepsilon > 0$ small enough so that a list of conditions are met, Banyaga and Hurtubise first show that there exists a small perturbation of the metric to a metric $\tilde{g}$ such that $h_{\varepsilon'} : M \to \mathbb{R}$ satisfies the Morse-Smale transversality condition with respect to $\tilde{g}$ for all $0 < \varepsilon' \leq \varepsilon$. The perturbation can be chosen small enough so that f satisfies the Morse-Bott-Smale transversality condition with respect to $\tilde{g}$ and the hypotheses of Theorem 4.5 still hold. Hence, there exists a metric $\tilde{g}$ such that moduli spaces of cascades are defined and $h_{\varepsilon'}$ satisfies the Morse-Smale transversality condition for all $0 < \varepsilon' \leq \varepsilon$.

Banyaga and Hurtubise then prove the following lemma.

Lemma 4.9. *Let $p, q \in Cr$ with $\lambda_q - \lambda_p = 1$, and let $0 < \varepsilon' \leq \varepsilon$. If $h_{\varepsilon'} : M \to \mathbb{R}$ and $h_\varepsilon : M \to \mathbb{R}$ are Morse-Smale with respect to the same Riemannian metric, then the number of gradient flow lines of $h_{\varepsilon'}$ from q to p is equal to the number of gradient flow lines of h_ε from q to p.*

This lemma shows that with respect to the perturbed metric $\tilde{g}$ from above there is a trivial cobordism

$$\mathcal{M}_{h_\varepsilon}(q,p) \times (0, \varepsilon]$$

such that

$$\mathcal{M}_{h_\varepsilon}(q,p) \times \{\varepsilon'\} \approx \mathcal{M}_{h_{\varepsilon'}}(q,p)$$

for all $0 < \varepsilon' \leq \varepsilon$. The next step is to analyze what happens as $\varepsilon' \to 0$. This is sometimes referred to as "degenerating the asymptotics".

Lemma 4.10. *Let $\{\varepsilon_\nu\}_{\nu=1}^{\infty}$ be a decreasing sequence such that $0 < \varepsilon_\nu \leq \varepsilon$ for all ν and $\lim_{\nu\to\infty} \varepsilon_\nu = 0$. Let $q, p \in Cr$, and suppose that $\gamma_{\varepsilon_\nu} \in \mathcal{M}_{h_{\varepsilon_\nu}}(q,p)$ for all ν. Then there exists a broken flow line with cascades $\gamma \in \overline{\mathcal{M}}^c(q,p)$ and a subsequence of $\{Im(\gamma_{\varepsilon_\nu})\}_{\nu=1}^{\infty}$ that converges to $Im(\gamma)$ in the Hausdorff topology.*

The proof of this lemma (which does not require $\lambda_q - \lambda_p = 1$) uses techniques similar to those used to prove that the space of broken flow lines with cascades is compact with respect to the Hausdorff topology (Theorem 4.7). If we use the Hausdorff topology on

$$\bigcup_{\varepsilon' \in (0,\varepsilon]} \mathcal{M}_{h_{\varepsilon'}}(q,p) \subset \mathcal{P}^c(M)$$

(where an element of $\mathcal{M}_{h_{\varepsilon'}}(q,p)$ is identified with its image, including q and p), then Lemma 4.10 says that the boundary of this space is contained in the union of the images of the

cascades in $\mathcal{M}^c(q,p)$. However, it is still possible that two distinct sequences of gradient flow lines from q to p might converge the same cascade as $\varepsilon' \to 0$ or some of the cascades in $\mathcal{M}^c(q,p)$ might not be near any of the gradient flow lines of $h_{\varepsilon'}$, even when $\varepsilon' > 0$ is extremely small. So, the proof of the Correspondence Theorem requires a much more detailed analysis than is provided by Lemma 4.10.

To conduct this more detailed analysis Banyaga and Hurtubise turned to the Exchange Lemma, which can be viewed as a generalization of Palis' λ-Lemma. The λ-Lemma applies to a critical point p of a Morse-Smale system, and it says (roughly speaking) that if N is an invariant submanifold that intersects $W^s(p)$ transversally then $W^u(p)$ must contain points that are close to N. The λ-Lemma is an essential tool for the dynamical systems approach to studying compactified moduli spaces of Morse-Smale flows (cf. Sections 6.2 and 6.3 of [3]), and the Exchange Lemma allows the dynamical systems approach to be extended to Morse-Bott-Smale systems.

The Exchange Lemma comes out of geometric singular perturbation theory, and it applies to "fast-slow" dynamical systems. Following the notation in [28], a fast-slow system of differential equations in local coordinates is of the form

$$\begin{aligned} x' &= f(x,y,\epsilon) \\ y' &= \epsilon g(x,y,\epsilon) \end{aligned}$$

where $' = \frac{d}{dt}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$, ϵ is a real parameter, and both f and g are C^∞ (and hence bounded) on some neighborhood of 0. The x coordinates are called the fast variables and the y coordinates are called the slow variables because in the limit as $\epsilon \to 0$ we have

$$\begin{aligned} x' &= f(x,y,0) \\ y' &= 0 \end{aligned}$$

where the x coordinates can vary but y remains constant. Alternately, when $\epsilon \neq 0$ is close to 0, y' is close to 0 and the y coordinates change slowly, whereas the x coordinates can change more quickly.

In the setup contained in [7], each critical submanifold has a neighborhood with coordinates (u,v,w) coming from the Morse-Bott Lemma, where the u coordinates are the coordinates along the critical submanifold and the (v,w) coordinates are the coordinates in the directions normal to the critical submanifold. The Morse function on the critical submanifold depends only on the u coordinates, which are the slow variables, and the Morse-Bott function depends only on the (v,w) coordinates, which are the fast variables. In fact, the Riemannian metric is chosen so that on a neighborhood of the critical submanifold

$$\nabla h_\varepsilon = \nabla f + \varepsilon \nabla f_j$$

where $\nabla f \perp \nabla f_j$. Thus, the gradient flow equation of the Morse-Smale function ∇h_ε in the local coordinates (u,v,w) near the critical submanifold is

$$\begin{aligned} (v',w') &= (\nabla f)(v,w) \\ u' &= \varepsilon(\nabla f_j)(u) \end{aligned}$$

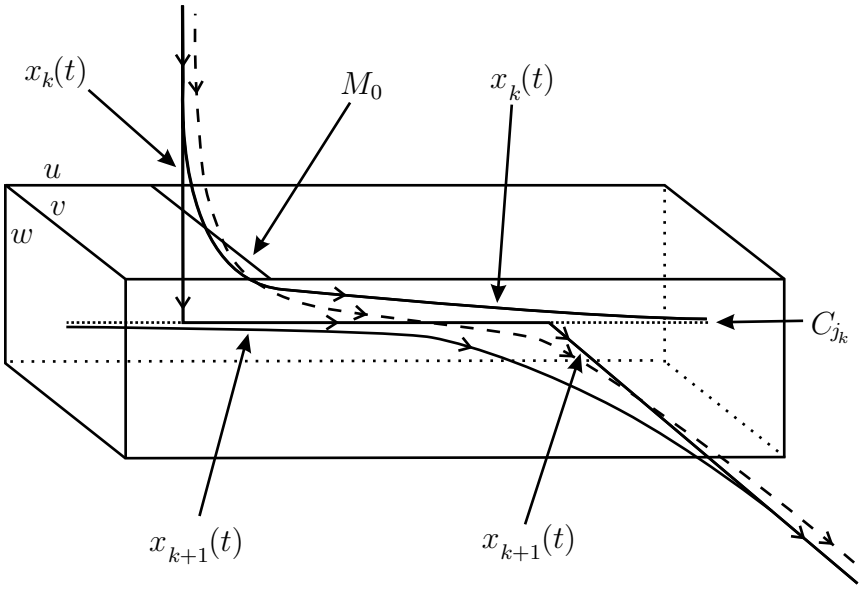
which is a fast-slow system.

Several versions of the Exchange Lemma with various levels of generality have been proved by many different authors, cf. [29] [33] [40] [41]. The lemma gives a relationship between the dynamics of a fast-slow system when $\varepsilon \neq 0$ and the dynamics of the system when $\varepsilon = 0$. Roughly speaking, the lemma says that a manifold M_0 that is transverse to the stable manifold $W_0^s(C)$ of a normally hyperbolic critical submanifold C of the system with $\epsilon = 0$ will have points that flow forward in time under the fast-slow system with $\varepsilon \neq 0$ to be near subsets of the unstable manifold $W_0^u(C)$ of the system with $\varepsilon = 0$.

With respect to the setup in [7], we have a flow line with n cascades

$$((x_k)_{1\leq k\leq n}, (t_k)_{1\leq k\leq n-1})$$

with intermediate critical submanifolds $C_{j_1}, \ldots, C_{j_n}$ and local coordinates (u,v,w) near an intermediate critical submanifold C_{j_k}. In the local coordinates the critical submanifold C_{j_k} consists of the u components $\{(u,0,0)\}$, the stable manifold $W_f^s(C_{j_k})$ is given by the (u,w) components $\{(u,0,w)\}$, and the unstable manifold $W_f^s(C_{j_k})$ is given by the (u,v) components $\{(u,v,0)\}$. Away from the critical submanifolds the gradient flow lines $x_k(t)$ of $f = h_0$ and $x_k^\varepsilon(t)$ of h_ε agree. However, near the critical submanifold C_{j_k} the gradient of h_ε with $\varepsilon \neq 0$ may be nonzero in the u components, whereas the gradient of f is zero in the u components. So, near the critical submanifold the gradient flow line $x_k^\varepsilon(t)$ of h_ε can diverge from the gradient flow line $x_k(t)$ of $f = h_0$.



The Exchange Lemma says that if M_0 intersects $W_f^s(C_{j_k})$ transversally near the image of $x_k(t)$, then there must be points in M_0 that flow forward in time under the gradient flow of h_ε with $\varepsilon \neq 0$ to be near the image of $x_{k+1}(t)$. Thus, there is a gradient flow line of h_ε passing through M_0 whose image is near the image of the cascade (x_k, x_{k+1}, t_k). This is shown in the diagram where the gradient flow line of h_ε near the image of the cascade is the dashed curve lying above the unstable manifold $\{(u,v,0)\}$.

Using these ideas, Banyaga and Hurtubise proved the main theorem in [7], which implies that the cascade chain complex is the same as the Morse-Smale-Witten chain complex of h_ε up to sign.

Theorem 4.11 (Correspondence of Moduli Spaces)**.** *Let $p, q \in Cr(h_\varepsilon)$ with $\lambda_q - \lambda_p = 1$. For any sufficiently small $\varepsilon > 0$ there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse-Smale function $h_\varepsilon : M \to \mathbb{R}$ between q and p,*

$$\mathcal{M}^c(q,p) \leftrightarrow \mathcal{M}_{h_\varepsilon}(q,p).$$

The Correspondence Theorem allows us to identify the space of cascades $\mathcal{M}^c(q,p)$ with the left side boundary of the trivial cobordism

$$\mathcal{M}_{h_\varepsilon}(q,p) \times [0,\varepsilon],$$

which will have the opposite orientation as the right side boundary.

Corollary 4.12 (Correspondence of Chain Complexes)**.** *For $\varepsilon > 0$ sufficiently small, the Morse-Smale-Witten chain complex $(C_*(h_\varepsilon), \partial_*^{h_\varepsilon})$ associated to the perturbation*

$$h_\varepsilon = f + \varepsilon \left(\sum_{j=1}^{l} \rho_j f_j \right)$$

of a Morse-Bott function $f : M \to \mathbb{R}$ is the same as the cascade chain complex $(C_*^c(f), \partial_*^c)$ up to sign. That is, the chain groups of both complexes have the same generators and $\partial_*^c = -\partial_*^{h_\varepsilon}$.

5 The Morse-Bott multicomplex

The approaches discussed in the previous sections require choosing auxiliary Morse functions on the critical submanifolds in order to define a chain complex generated by the critical points of the chosen Morse functions. The approach discussed in this section does not involve choosing any auxiliary Morse functions. Instead, the chain groups are generated by singular topological chains on the critical submanifolds. Keeping track of the degrees of the singular topological chains, the Morse-Bott indexes of the critical submanifolds, and homomorphisms defined using moduli spaces of gradient flow lines between the critical submanifolds leads to an algebraic structure known as a multicomplex, which generalizes the notion of a double complex.

Multicomplexes and assembled chain complexes

Definition 5.1. Let R be a principal ideal domain. A first quadrant **multicomplex** X is a bigraded R-module $\{X_{p,q}\}_{p,q \in \mathbb{Z}_+}$ with differentials

$$\mathsf{d}_j : X_{p,q} \to X_{p-j,q+j-1} \quad \text{for all } j = 0, 1, \ldots$$

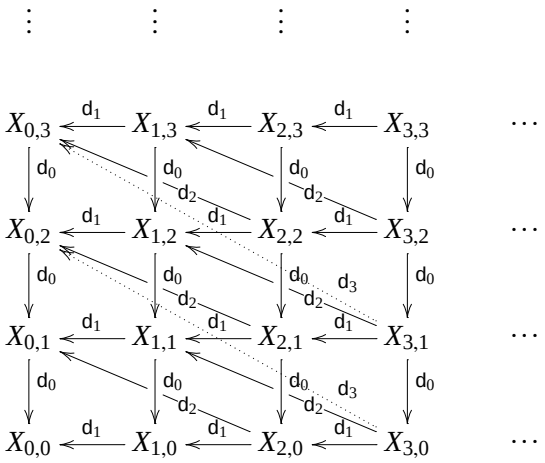
that satisfy

$$\sum_{i+j=n} \mathsf{d}_i \mathsf{d}_j = 0 \quad \text{for all } n.$$

A first quadrant multicomplex such that $\mathsf{d}_j = 0$ for all $j \geq 2$ is called a **double complex** (or a **bicomplex**).

A first quadrant multicomplex looks similar to a spectral sequence, but the differentials are all defined on the 0^{th} page and we may have $\mathsf{d}_j \circ \mathsf{d}_j \neq 0$ when $j > 0$.

A First Quadrant Multicomplex



A multicomplex can be **assembled** to form a filtered chain complex $((CX)_*, \partial_*)$ by summing along the diagonals. That is, if we define

$$(CX)_k \equiv \bigoplus_{p+q=k} X_{p,q}$$

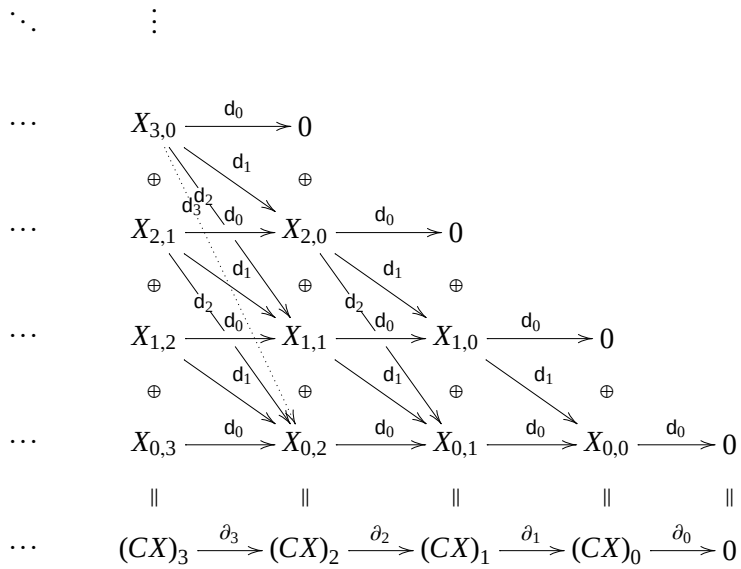
and $\partial_k = \mathsf{d}_0 \oplus \cdots \oplus \mathsf{d}_k$ for all $k \in \mathbb{Z}_+$, then the relations in Definition 5.1 imply that $\partial_k \circ \partial_{k+1} = 0$.

Note: The chain complex $((CX)_*, \partial_*)$ has a filtration given by

$$F_s(CX)_k \equiv \bigoplus_{\substack{p+q=k \\ p \leq s}} X_{p,q}$$

which determines a spectral sequence. However, the differentials in this spectral sequence are not necessarily induced from the differentials d_j when $j \geq 2$ [27].

The Assembled Chain Complex



A heuristic view of the Morse-Bott multicomplex

Let $f : M \to \mathbb{R}$ be a Morse-Bott-Smale function on an m-dimensional compact smooth closed Riemannian manifold M, and let $B_i \subseteq Cr(f)$ be the union of the critical submanifolds of Morse-Bott index i for $i = 0, \ldots, m$. The compactified moduli space $\overline{\mathcal{M}}(B_i, B_{i-j})$ of broken gradient flow lines of f from B_i to B_{i-j} for $j = 1, \ldots, i$ is a smooth manifold with corners and the beginning point map

$$\partial_- : \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_i$$

is a submersion and a stratum submersion (cf. Corollary 5.20 of [6]). Thus, every smooth map $\sigma : P \to B_i$ from a smooth manifold with corners P is transverse and stratum transverse to ∂_-, and the fibered product $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ of σ and ∂_- over B_i is a smooth manifold with corners (cf. Lemma 5.21 of [6]).

$$\begin{array}{ccc} P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) & \dashrightarrow & \overline{\mathcal{M}}(B_i, B_{i-j}) \\ \vdots & & \downarrow \partial_- \\ P & \xrightarrow{\sigma} & B_i \end{array}$$

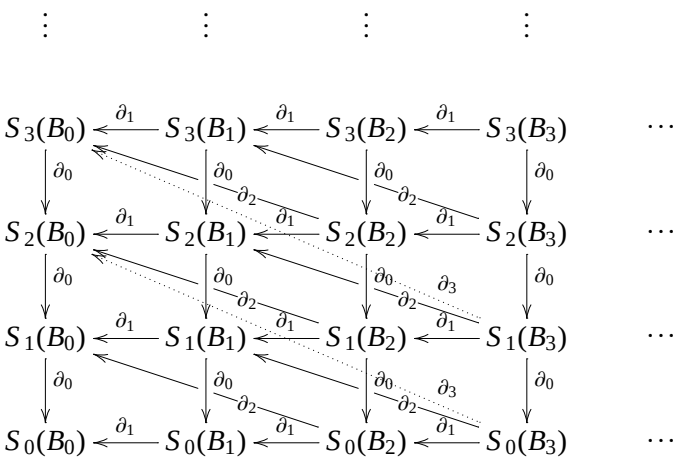
(Similar spaces were used in the proof of Theorem 4.5 on moduli spaces of cascades). Composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$

Moreover, if P has dimension p, then $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ has dimension $p + j - 1$, which is independent of the dimension of the connected components in B_i and B_{i-j}.

Up to this point, the discussion has been rigorous. We will now make explicit an unwarranted assumption that has been assumed implicitly by other authors (cf. [24] [32]). If the above fibered product had a preferred finite triangulation, then summing over the restrictions of the above map to the simplices making up the finite triangulation would define a singular chain $\partial_j(\sigma)$ in B_{i-j}. Moreover, if every smooth manifold with corners under consideration came with a preferred finite triangulation (or cubulation), then this fibered product construction would define a homomorphism $\partial_j : S_p(B_i) \to S_{p+j-1}(B_{i-j})$ from the singular p-chains on B_i to the singular $p+j-1$-chains on B_{i-j} (or singular cubical chains if we were given preferred finite cubulations). These maps would then yield the following, where ∂_0 is comes from the usual singular boundary operator.

Heuristic View of the Morse-Bott Multicomplex



Of course, smooth manifolds with corners don't usually come with preferred triangulations, and there is no preferred (or induced) finite triangulation on the fibered product of finitely triangulated spaces (cf. Example 5.17 of [6]). Still, it might be possible to pick finite triangulations on all the (uncountably many) spaces under consideration, prove that the relations in Definition 5.1 hold with respect to the chosen triangulations, and then show that the homology of the resulting assembled chain complex is independent of the chosen triangulations. However, there seem to be many technical difficulties involved with making this approach rigorous on the level of chains. Fortunately, by expanding the collection of allowed domains for the singular chains it is possible to construct a Morse-Bott multicomplex without choosing any triangulations.

The Banyaga-Hurtubise approach to the Morse-Bott multicomplex

Singular homology is usually defined using maps from the standard k-simplex Δ^k. However, other equivalent versions of singular homology have been defined using maps from domains other than Δ^k. For instance, there is singular cubical homology, which is based on maps from the unit k-cube I^k [34], and there is also a version of singular homology based on maps from permutahedra [39]. In order to create a singular homology theory that allows

for even more general domains Banyaga and Hurtubise make the following definitions in Section 4 of [6].

For each integer $p \geq 0$ fix a set C_p of topological spaces, and let S_p be the free abelian group generated by the elements of C_p, i.e. $S_p = \mathbb{Z}[C_p]$. Set $S_p = \{0\}$ if $p < 0$ or $C_p = \emptyset$.

Definition 5.2. A **boundary operator** on the collection S_* of groups $\{S_p\}$ is a homomorphism $\partial_p : S_p \to S_{p-1}$ such that

1. For $p \geq 1$ and $P \in C_p \subseteq S_p$, $\partial_p(P) = \sum_k n_k P_k$ where $n_k = \pm 1$ and $P_k \in C_{p-1}$ is a subspace of P for all k.
2. $\partial_{p-1} \circ \partial_p : S_p \to S_{p-2}$ is zero.

The pair (S_*, ∂_*) is called a **chain complex of abstract topological chains**, and elements of S_p are called **abstract topological chains** of **degree** p.

Definition 5.3. Let B be a topological space and $p \in \mathbb{Z}_+$. A **singular C_p-space** in B is a continuous map $\sigma : P \to B$ where $P \in C_p$, and the **singular C_p-chain group** $S_p(B)$ is the free abelian group generated by the singular C_p-spaces. Define $S_p(B) = \{0\}$ if $S_p = \{0\}$ or $B = \emptyset$. Elements of $S_p(B)$ are called **singular topological chains** of **degree** p.

For $p \geq 1$ there is a boundary operator $\partial_p : S_p(B) \to S_{p-1}(B)$ induced from the boundary operator $\partial_p : S_p \to S_{p-1}$. If $\sigma : P \to B$ is a singular C_p-space in B, then $\partial_p(\sigma)$ is given by the formula

$$\partial_p(\sigma) = \sum_k n_k \sigma|_{P_k}$$

where

$$\partial_p(P) = \sum_k n_k P_k.$$

The pair $(S_*(B), \partial_*)$ is called a **chain complex of singular topological chains**.

Example: Singular N-cube chains. Pick some large positive integer N and let $I^N = \{(x_1, \ldots, x_N) \in \mathbb{R}^N | \ 0 \leq x_j \leq 1, \ j = 1, \ldots, N\}$ denote the unit N-cube. For every $0 \leq p \leq N$ let C_p be the set consisting of the faces of I^N of dimension p, i.e. subsets of I^N where p of the coordinates are free and the rest of the coordinates are fixed to be either 0 or 1. For every $0 \leq p \leq N$ let S_p be the free abelian group generated by the elements of C_p. For $P \in C_p$ define

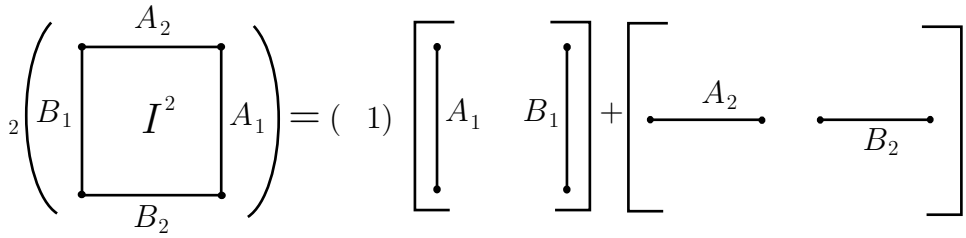
$$\partial_p(P) = \sum_{j=1}^{p} (-1)^j \left[P|_{x_j=1} - P|_{x_j=0} \right] \in S_{p-1}$$

where x_j denotes the j^{th} free coordinate of P. It is easy to show that $\partial_{p-1} \circ \partial_p = 0$, and hence the faces of I^N are abstract topological chains. Thus, a continuous map $\sigma_P : P \to B$ from a face of I^N of dimension p into a topological space B is a singular C_p-space in B, and the boundary operator applied to σ_P is

$$\partial_p(\sigma_P) = \sum_{j=1}^{p} (-1)^j \left[\sigma_P|_{x_j=1} - \sigma_P|_{x_j=0} \right] \in S_{p-1}(B)$$

where $\sigma_P|_{x_j=0}$ denotes the restriction $\sigma_P : P|_{x_j=0} \to B$ and $\sigma_P|_{x_j=1}$ denotes the restriction $\sigma_P : P|_{x_j=1} \to B$.

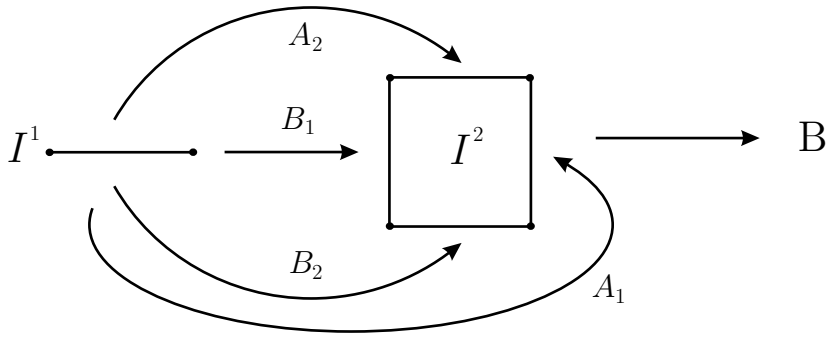
For instance, if $p = N = 2$ the abstract topological chain I^2 has boundary,



and the singular C_2-space $\sigma : I^2 \to B$ has boundary

$$\partial_2(\sigma) = (-1)[\sigma|_{A_1} - \sigma|_{B_1}] + [\sigma|_{A_2} - \sigma|_{B_2}].$$

Note that this differs from the usual boundary operator on singular cubical chains because there are several different domains of the same dimension. Normally, singular homology is defined by picking a unique domain in each dimension and then defining the boundary operator using inclusion maps. For instance, the boundary operator on singular cubes found in [34] is defined using the following inclusion maps when $p = 2$.



To account for the multiple domains in each dimension, Banyaga and Hurtubise define degeneracy relations in the form of a subgroup $D_p(B) \subseteq S_p(B)$ that identifies maps that are "essentially" the same. They then prove the following theorem.

Theorem 5.4 (Singular N-Cube Chain Theorem)**.** *The boundary operator for singular N-cube chains $\partial_p : S_p(B) \to S_{p-1}(B)$ descends to a homomorphism*

$$\partial_p : S_p(B)/D_p(B) \to S_{p-1}(B)/D_{p-1}(B),$$

and

$$H_p(S_*(B)/D_*(B), \partial_*) \approx H_p(B; \mathbb{Z})$$

for all $p < N$.

Fibered products and moduli spaces as abstract topological chains

Since most of the homomorphisms in the Morse-Bott multicomplex are defined using fibered products of compactified moduli space of gradient flow lines, the next step is to show that the compactified moduli spaces of gradient flow lines of a Morse-Bott-Smale function are abstract topological chains and a boundary operator on abstract topological chains extends to fibered products.

Let $f : M \to \mathbb{R}$ be a Morse-Bott-Smale function on an m-dimensional compact smooth closed Riemannian manifold M, and let $B_i \subseteq Cr(f)$ be the union of the critical submanifolds of Morse-Bott index i for $i = 0,\dots,m$. To simplify the notation in the following we will drop the subscript on ∂ and assume that for each $i = 0,\dots,m$ the components of B_i are all of the same dimension. In general one needs to group the components by their dimension and then define the degree and boundary operator on each group.

Definition 5.5. Let B_i be the set of critical points of index i. For any $j = 1,\dots,i$ the degree of $\overline{\mathcal{M}}(B_i, B_{i-j})$ is defined to be $j + b_i - 1$ and the boundary operator is defined to be

$$\partial\overline{\mathcal{M}}(B_i, B_{i-j}) = (-1)^{i+b_i} \sum_{i-j<n<i} \overline{\mathcal{M}}(B_i, B_n) \times_{B_n} \overline{\mathcal{M}}(B_n, B_{i-j})$$

where $b_i = \dim B_i$ and the fibered product is taken over the beginning and endpoint maps ∂_- and ∂_+. If $B_n = \emptyset$, then $\overline{\mathcal{M}}(B_i, B_n) = \overline{\mathcal{M}}(B_n, B_{i-j}) = 0$.

In order to check that $\partial \circ \partial = 0$ we need to know how ∂ extends to fibered products.

Definition 5.6. Suppose that $\{C_p\}_{p\geq 0}$ is a collection of topological spaces that is closed under the fibered product construction with respect to some collection of maps, and assume that (S_*, ∂_*) is a chain complex of abstract topological chains based on some subset of the collection $\{C_p\}_{p\geq 0}$. If $\sigma_i = \sum_k n_{i,k}\sigma_{i,k} \in S_{p_i}(B)$ is defined for $i = 1,2$ where $\sigma_{i,k} : P_{i,k} \to B$ is a singular C_{p_i}-space for all k, then the **fibered product** of σ_1 and σ_2 over B is defined to be

$$P_1 \times_B P_2 = \sum_{k,j} n_{1,k}n_{2,j}\ P_{1,k} \times_B P_{2,j}$$

where $P_1 = \sum_k n_{1,k}P_{1,k} \in S_{p_1}$ and $P_2 = \sum_j n_{2,j}P_{2,j} \in S_{p_2}$. The **boundary operator** applied to the fibered product is defined to be

$$\partial(P_1 \times_B P_2) = \partial P_1 \times_B P_2 + (-1)^{p_1+b} P_1 \times_B \partial P_2.$$

If $\sigma_i = 0$ for either $i = 1$ or 2, then we define $P_1 \times_B P_2 = 0$.

The following lemmas from Section 4 of [6] show that the fibered product of abstract topological chains is an abstract topological chain and the compactified moduli spaces of gradient flow lines of a Morse-Bott-Smale function are abstract topological chains. The signs $(-1)^{i+b_i}$ and $(-1)^{p_1+b}$ in Definitions 5.5 and 5.6 are essential to the proofs of these two lemmas.

Lemma 5.7. *The fibered product of two singular topological chains is an abstract topological chain, i.e. the boundary operator on fibered products is of degree -1 and satisfies $\partial \circ \partial = 0$. Moreover, the boundary operator on fibered products is associative, i.e.*

$$\partial((P_1 \times_{B_1} P_2) \times_{B_2} P_3) = \partial(P_1 \times_{B_1} (P_2 \times_{B_2} P_3)).$$

Lemma 5.8. *The degree and boundary operator for $\overline{\mathcal{M}}(B_i, B_{i-j})$ satisfy the axioms for abstract topological chains, i.e. the boundary operator on compactified moduli spaces of gradient flow lines from Definition 5.5 is of degree -1 and it satisfies $\partial \circ \partial = 0$.*

The Morse-Bott-Smale multicomplex

Fix some $N > \dim M$, and for any $p \geq 0$ let C_p be the set consisting of the faces of I^N of dimension p and the connected components of degree p of fibered products of the form

$$Q \times_{B_{i_1}} \overline{\mathcal{M}}(B_{i_1}, B_{i_2}) \times_{B_{i_2}} \overline{\mathcal{M}}(B_{i_2}, B_{i_3}) \times_{B_{i_3}} \cdots \times_{B_{i_{n-1}}} \overline{\mathcal{M}}(B_{i_{n-1}}, B_{i_n})$$

where $m \geq i_1 > i_2 > \cdots > i_n \geq 0$, Q is a face of I^N of dimension $q \leq p$, $\sigma : Q \to B_{i_1}$ is smooth, and the fibered products are taken with respect to σ and the beginning and endpoint maps ∂_- and ∂_+. Lemma 5.1 of [6] shows that the elements of C_p are all compact smooth manifolds with corners. Let S_p be the free abelian group generated by the elements of C_p, and let $S_p^\infty(B_i)$ denote the subgroup of the singular C_p-chain group $S_p(B_i)$ generated by those maps $\sigma : P \to B_i$ that satisfy the following two conditions:

1. The map σ is smooth.

2. If $P \in C_p$ is a connected component of a fibered product, then $\sigma = \partial_+ \circ \pi$, where π denotes projection onto the last component of the fibered product.

Definition 5.9. Define the **Morse-Bott degree** of the singular topological chains in $S_p^\infty(B_i)$ to be $p + i$. For any $k = 0, \ldots, m$ the group of smooth singular topological chains of Morse-Bott degree k is defined to be

$$\tilde{C}_k(f) = \bigoplus_{i=0}^{m} S_{k-i}^\infty(B_i).$$

If $\sigma : P \to B_i$ is a singular C_p-space in $S_p^\infty(B_i)$, then for any $j = 1, \ldots, i$ composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$

Lemma 5.3 of [6] shows that restricting this map to the connected components of the fibered product $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ and adding these restrictions (with the sign determined by the orientation when the dimension of a component is zero) defines an element $\partial_j(\sigma) \in S_{p+j-1}^\infty(B_{i-j})$.

Definition 5.10. For $k = 1, \ldots, m$ define a homomorphism $\partial : \tilde{C}_k(f) \to \tilde{C}_{k-1}(f)$ as follows. If $\sigma \in S_p^\infty(B_i)$ is a singular S_p-space of B_i where $p = k - i$, then

$$\partial(\sigma) = \bigoplus_{j=0}^{m} \partial_j(\sigma)$$

where ∂_0 is $(-1)^k$ times the boundary operator on singular topological chains defined above, $\partial_j(\sigma) = \partial_+ \circ \pi_2 : P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_{i-j}$ for $j = 1, \ldots, i$, and $\partial_j(\sigma) = 0$ otherwise. The map ∂ extends to a homomorphism

$$\partial : \bigoplus_{i=0}^{m} S^{\infty}_{k-i}(B_i) \longrightarrow \bigoplus_{i=0}^{m} S^{\infty}_{k-1-i}(B_i).$$

The following is Proposition 5.5 of [6].

Proposition 5.11. *For every $j = 0, \ldots, m$ we have $\sum_{q=0}^{j} \partial_q \partial_{j-q} = 0$.*

Defining the Morse-Bott-Smale multicomplex over the integers requires a coherent system of orientations on the elements of C_p (cf. Section 5.2 of [6]) and a collection of degeneracy relations that identify maps from different domains that are "essentially" the same (cf. Section 5.3 of [6]). The degeneracy relations are expressed in the form of subgroups $D^{\infty}_p(B_i) \subseteq S^{\infty}_p(B_i)$, and the chain groups that make up the Morse-Bott-Smale multicomplex are defined to be $S^{\infty}_p(B_i)/D^{\infty}_p(B_i)$. Lemma 5.10 of [6] shows that the homomorphisms ∂_j on $S^{\infty}_p(B_i)$ induce homomorphisms on $S^{\infty}_p(B_i)/D^{\infty}_p(B_i)$, which we denote using the same notation.

Definition 5.12. Define

$$C_p(B_i) = S^{\infty}_p(B_i)/D^{\infty}_p(B_i)$$

to be the group of **non-degenerate** smooth singular topological chains in $S^{\infty}_p(B_i)$. The group $C_k(f)$ of k-**chains** in the Morse-Bott chain complex of f is defined to be the group of non-degenerate smooth singular topological chains of Morse-Bott degree k, i.e.

$$C_k(f) = \bigoplus_{i=0}^{m} C_{k-i}(B_i) = \bigoplus_{i=0}^{m} S^{\infty}_{k-i}(B_i)/D^{\infty}_{k-i}(B_i).$$

The boundary operator in the Morse-Bott-Smale chain complex

$$\partial : \bigoplus_{i=0}^{m} S^{\infty}_{k-i}(B_i)/D^{\infty}_{k-i}(B_i) \longrightarrow \bigoplus_{i=0}^{m} S^{\infty}_{k-1-i}(B_i)/D^{\infty}_{k-1-i}(B_i)$$

is defined to be $\partial = \oplus_{j=0}^{m} \partial_j$.

The Morse-Bott-Smale Multicomplex

$$\begin{array}{ccccccccc}
\ddots & & \vdots & & & & & & \\
 & & \oplus & & & & & & \\
\cdots & & C_1(B_2) & \xrightarrow{\partial_0} & C_0(B_2) & \xrightarrow{\partial_0} & 0 & & \\
 & & \oplus & \searrow^{\partial_1}\ \swarrow_{\partial_2} & \oplus & \searrow^{\partial_1}\ \swarrow_{\partial_2} & \oplus & & \\
\cdots & & C_2(B_1) & \xrightarrow{\partial_0} & C_1(B_1) & \xrightarrow{\partial_0} & C_0(B_1) & \xrightarrow{\partial_0} & 0 \\
 & & \oplus & \searrow^{\partial_1}\ \swarrow_{\partial_2} & \oplus & \searrow^{\partial_1}\ \swarrow_{\partial_2} & \oplus & \searrow^{\partial_1}\ \swarrow_{\partial_2} & \oplus \\
\cdots & & C_3(B_0) & \xrightarrow{\partial_0} & C_2(B_0) & \xrightarrow{\partial_0} & C_1(B_0) & \xrightarrow{\partial_0} & C_0(B_0) \xrightarrow{\partial_0} 0 \\
 & & \| & & \| & & \| & & \| \\
\cdots & & C_3(f) & \xrightarrow{\partial} & C_2(f) & \xrightarrow{\partial} & C_1(f) & \xrightarrow{\partial} & C_0(f) \xrightarrow{\partial} 0
\end{array}$$

Since the homomorphisms ∂_j are induced from the homomorphisms in Definition 5.10, Proposition 5.11 shows that the relations that define a multicomplex are satisfied.

Maps between Morse-Bott-Smale multicomplexes

The Banyaga-Hurtubise approach to constructing the Morse-Bott-Smale multicomplex has several advantages. For instance, it does not require picking any triangulations. Hence, all the maps in the multicomplex are well defined at the chain level and there is no need to prove that the homology of the multicomplex is independent of arbitrarily chosen triangulations. Moreover, Lemma 5.1 of [6] shows that all the fibered products used to define the multicomplex are compact smooth manifolds with corners, without having to perturb any maps used in the construction. Other approaches require perturbing the beginning and endpoint maps ∂_- and ∂_+, which would then necessitate proving that the homology of the resulting complex is independent of the chosen perturbations [24].

While the multicomplex constructed by Banyaga and Hurtubise does not depend on any extraneous choices, it obviously does depend on the Morse-Bott-Smale function $f : M \to \mathbb{R}$ and the Riemannian metric on M. However, Theorem 6.17 of [6] shows that the homology of the assembled chain complex does not depend on the Morse-Bott-Smale function or the Riemannian metric on M. The proof of Theorem 6.17 of [6] follows standard continuation arguments found in papers on Floer homology. In particular, given two Morse-Bott-Smale functions f_1 and f_2 on M a continuation map is defined between the multicomplexes determined by the two functions using moduli spaces of time dependent gradient flow lines, i.e. moduli spaces of gradient flow lines of a function $F_{21} : M \times \mathbb{R} \to \mathbb{R}$ where

$$\begin{aligned} \lim_{t\to-\infty} F_{21}(x,t) &= f_1(x)+1 \\ \lim_{t\to+\infty} F_{21}(x,t) &= f_2(x)-1 \end{aligned}$$

for all $x \in M$.

However, the time dependent moduli spaces of gradient flow lines are not allowed domains for the singular topological chains in Morse-Bott-Smale multicomplex. So, Banyaga and Hurtubise adapt the technique of **representing chain systems** from [8] in order to define their continuation maps. Roughly speaking, a representing chain system consists of singular topological chains (defined on the allowed domains) that represent the fundamental classes of the moduli spaces of time dependent gradient flow lines (which are compact smooth manifolds with corners). This means that the continuation maps are only defined at the chain level after choosing a representing chain system. However, Corollary 6.12 of [6] shows that the induced map between the homologies of the assembled chain complexes is independent of the representing chain systems. So, the continuation maps are well defined at the level of homology and independent of any of the choices made to define them at the chain level.

The following two corollaries proved in Section 6 of [6] show that standard arguments from Floer homology can be applied to the Morse-Bott-Smale multicomplex.

Corollary 5.13. *For any two Morse-Bott-Smale functions $f_1, f_2 : M \to \mathbb{R}$ the time-dependent gradient flow lines from f_1 to f_2 determine a canonical homomorphism*

$$(F_{21})_* : H_*(C_*(f_1),\partial) \to H_*(C_*(f_2),\partial),$$

i.e. the map $(F_{21})_*$ is independent of the choice of the function $F_{21} : M \times \mathbb{R} \to \mathbb{R}$ and the representing chain system used to define the chain map $(F_{21})_\square : C_*(f_1) \to C_*(f_2)$.

Corollary 5.14. *For any four Morse-Bott-Smale functions $f_k : M \to \mathbb{R}$, where $k = 1,2,3,4$, the canonical homomorphisms satisfy*

$$(F_{43})_* \circ (F_{31})_* = (F_{42})_* \circ (F_{21})_*$$

and

$$(F_{32})_* \circ (F_{21})_* = (F_{31})_*.$$

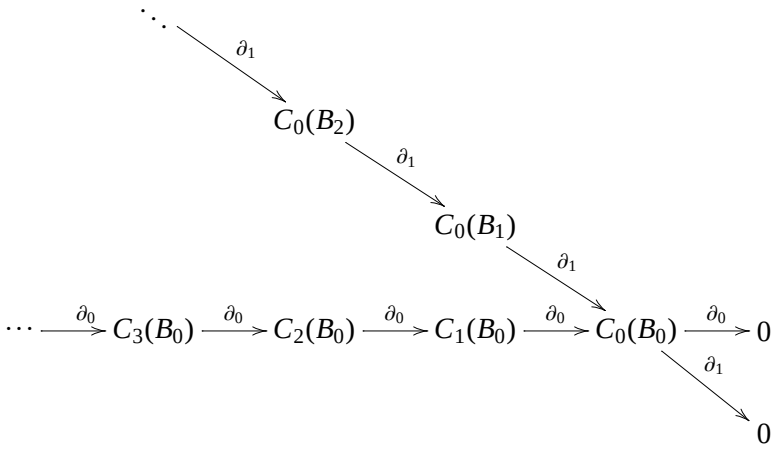
The preceding two corollaries and the Singular N-Cube Chain Theorem (Theorem 5.4) imply the following, which is Theorem 6.17 of [6].

Theorem 5.15. *The homology of the Morse-Bott chain complex $(C_*(f), \partial)$ is independent of the Morse-Bott-Smale function $f : M \to \mathbb{R}$. Therefore,*

$$H_*(C_*(f), \partial) \approx H_*(M; \mathbb{Z}).$$

Interpolating between singular N-cube chains and Morse chains

When the function $f : M \to \mathbb{R}$ is Morse-Smale the critical set B_i is a discrete set of points for all $i = 0, \ldots, m$, and the groups $C_p(B_i)$ are trivial for all $p > 0$. When the function is constant the entire manifold M is a critical submanifold of Morse-Bott index zero. In this case $B_i = \emptyset$ for all $i > 0$, and the groups $C_p(B_i)$ are trivial for all $i > 0$. These two cases appear in the diagram of a general Morse-Bott-Smale chain complex as follows.



In the first case the homomorphism ∂_1 is the Morse-Smale-Witten boundary operator, and in the second case we have the chain complex of singular N-cube chains, which computes the singular homology of M by Theorem 5.4. Thus, the Morse-Bott-Smale multicomplex provides a means of interpolating between the Morse-Smale-Witten chain complex and the chain complex of singular N-cube chains. Moreover, Theorem 5.15 shows that the homology of these two chain complexes are the same, and hence the results in [6] give a new proof of the Morse Homology Theorem (Theorem 2.1).

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