African Diaspora Journal of Mathematics
Special Volume in Honor of Prof. Augustin Banyaga ISSN 1539-854X
Volume 14, Number 2, pp. 57–75 (2012) www.math-res-pub.org/adjm

BONNET PAIRS OF SURFACES IN MINKOWSKI SPACE

GUEO GRANTCHAROV*
Department of Mathematics and Statistics, Florida International University,
Miami, FL 33199, USA

ROBERT SALOM†
Department of Mathematics and Statistics, Florida International University,
Miami, FL 33199, USA

Abstract

We review some results about Bonnet pairs in Minkowski space obtained using split quaternions and split complex numbers. We present also an example of Bonnet pairs of minimal immersed time-like tori with umbilical points. Such examples do not exists in the Euclidean space.

Dedicated to Professor Augustin Banyaga on the occasion of his 65th birthday.

AMS Subject Classification: 53A35; 53B25

Keywords: Bonnet pairs, split quaternions, split complex numbers.

1 Introduction

A pair of surfaces in Euclidean 3-dimensional space is called a Bonnet pair if they are isometric with the same mean curvature, but are not congruent via Euclidean motions. The problem of describing local Bonnet pairs had been solved first by Bianchi [1]. In [6] a modern solution in terms of quaternions and quaternionic analysis is given. The integrable systems approach [2] revealed many properties of the Bonnet pairs, but the question of existence of compact Bonnet pairs of non-constant mean curvatures is still open [12].

The main goal of present paper is to review some results about the Bonnet pairs in Minkowski space, obtained using the language of split quaternions and split complex numbers. Then we provide a simple example of Bonnet pairs of minimal time-like tori immersed in Minkowski space. Note that such examples could not exist in the Euclidean space. Moreover, the surfaces in any Bonnet pair of Euclidean tori have no umbilical points, a property which fails in the Minkowski space due to the example. The split quaternions form an algebra isomorphic to the algebra of 2×2 matrices and are related to 4-dimensional spaces equipped with indefinite metrics of signature $(2,2)$, just like the quaternions are related to

*Partially supported by a grant from Simons Foundation (♯246184). E-mail address: grantchg@fiu.edu
†E-mail address: rsalo001@fiu.edu

the 4-spaces with positive definite metric. One of the reasons behind many of the applications of split quaternions is their relation to theory of the integrable systems. The Bonnet pairs problem is one more instance of such relation, although we don't develop the integrable systems approach here. The split quaternions have often been used in the study of surfaces in Minkowski space, see [11], [4], [5] for recent results.

We start by describing the result of M.Magid [11] about the local classification of Bonnet pairs for space-like and time-like surfaces. The presentation follows closely [6] and the relation to isothermic surfaces is explained. We treat simultaneously the time-like and the space-like surfaces and notice that the results are complete analog to the ones in the Euclidean case.

In the last section we use split complex numbers to parametrize the domain of a time like surface and apply this description to construct a compact examples of immersed minimal Bonnet tori in Minkowski space. The split complex numbers are related to complex numbers in the same manner as the split quaternions are related to quaternions. Moreover the conformal class of a time-like surface defines a paracomplex structure, just like the conformal class of Riemannian metric defines a complex structure on an oriented surface. Previously such approach to time-like surfaces has been used in [8], [9], [3], [10]. Unlike the local theory, the example we present here suggests some essential differences from the Euclidean case.

2 Split Quaternions

The split quaternions $\mathbb{H}^{'}$ are a four dimensional algebra over the real numbers with a basis of elements $\{1, i, s, t\}$, satisfying the multiplicative relations

$$is = t, \quad st = -i, \quad ti = s,$$
$$i^2 = -1, \quad s^2 = t^2 = 1.$$

For $q = q_0 + q_1 i + q_2 s + q_2 t$ denote $Re(q) = q_0$ and $Im(q) = q_1 i + q_2 s + q_3 t$ the real and imaginary part of q and by $\bar{q} = Re(q) - Im(q)$ the conjugate of q. So in particular

$$Im\mathbb{H}^{'} = \{q \in \mathbb{H}^{'} | Re(q) = 0\}.$$

When $q, p \in Im\mathbb{H}^{'}$ we have that

$$pq = Re(pq) + Im(pq) = -<p, q> + p \times_m q \tag{2.1}$$

Here $<p, q>:= a_1 b_1 - a_2 b_2 - a_3 b_3$ is minus the inner product in Minkowski space $\mathbb{R}^{2,1}$, and

$$p \times_m q = \begin{vmatrix} -i & s & t \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

is the corresponding cross product. For an imaginary quaternion $p \in Im\mathbb{H}'$, we have $-p^2 = |p|^2 = \bar{p}p$. Here $|p|^2$ is actually minus the norm of Minkowski three-space. Sometimes to denote this explicitly we write $-|p|^2 = |p|_m^2 = p^2$.

An $\mathbb{H}'$ valued k-form is the object

$$\alpha = \alpha_0 + \alpha_1 i + \alpha_2 s + \alpha_3 t$$

where each $\alpha_i \in \bigwedge^k(V^*, \mathbb{R})$ are usual real k-forms. For the purpose of this note we need $k = 1$ or 2.

The set of all $\mathbb{H}'$ valued 1-forms is denonted analogously by, $\bigwedge^1(V^*, \mathbb{H}')$. As in the real valued case, there is a wedge product defined on $\mathbb{H}'$ valued 1-forms given by

$$\alpha \wedge \beta(x,y) := \alpha(x)\beta(y) - \alpha(y)\beta(x) \tag{2.2}$$

satisfying the following identities:

$$\begin{aligned}
&\overline{\alpha \wedge \beta} = -\bar{\beta} \wedge \bar{\alpha} \\
&\alpha \wedge h\beta = \alpha h \wedge \beta \\
&d(h\alpha) = dh \wedge \alpha + h d\alpha \\
&d(\alpha h) = d\alpha h - \alpha \wedge d(h)
\end{aligned}$$

Denote by $\mathbb{H}'_*$ the set

$$\mathbb{H}' \backslash \{q \in \mathbb{H}' | q_o^2 - Im(q)^2 = 0\}.$$

Then every rotation ρ in $\mathbb{R}^{2,1}$ can be represented via split quaternions as

$$\rho(v) = \lambda^{-1} v \lambda$$

where $v \in Im(\mathbb{H}') \cong \mathbb{R}^{2,1}$ and $\lambda \in \mathbb{H}'_*$ such that

$$\lambda = cos(\theta) + u\ sin(\theta).$$

Then ρ is a rotation about a time-like axis given by the imaginary quaternion u, with $u^2 = -1$ and angle 2θ. Similarly

$$\lambda = cosh(\theta) + u\ sinh(\theta)$$

defines a rotation about a space-like axis u with $u^2 = 1$ and angle 2θ. In either case, $\lambda\bar{\lambda} = 1$.

3 Conformal immersions of surfaces in $\mathbb{H}'$

An immersion $f : M \to \mathbb{R}^3$ where M is a domain in $\mathbb{R}^2$, is said to be conformal if its first fundamental form is of the type;

$$I = \lambda(u,v) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or of the type;

$$I = \lambda(u,v) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for $\lambda : M \to \mathbb{R}^+$, where the first type of immersion is named space-like, and the second time-like. So for the partial derivatives of f, f_u, f_v at a point p∈M, we have the following;

If f is space-like,then

$$< f_u, f_u >= f_u^2 = |f_u|_m^2 = |f_v|_m^2 = f_v^2 < f_v, f_v >$$

If f is time-like then

$$< f_u, f_u >= f_u^2 = |f_u|_m^2 = -|f_v|_m^2 = -f_v^2 = -< f_v, f_v >$$

and for both space- and time-like f

$$< f_u, f_v >= 0.$$

On forms operating over a two dimensional space with a given volume 2-form dV, the Hodge star operator

$$* : \bigwedge^1 (V^*, Im\mathbb{H}') \to \bigwedge^1 (V^*, Im\mathbb{H}')$$

is defined by linearity $*(\alpha_0 + i\alpha_1 + s\alpha_2 + t\alpha_3) = *\alpha_0 + i * \alpha_1 + s * \alpha_2 + t * \alpha_3$ and for any real-valued 1-forms α, β,

$$\alpha \wedge \beta =< *\alpha, \beta > dV.$$

Then it is known that $** = * \circ * = -1$ in the space-like case (positive definite $<,>$) and $* \circ * = 1$ in the time-like case ($<,>$ has signature $(1,1)$). Using $*$ one can define an endomorphism $J \in End(V)$ via $\alpha(JX) = *\alpha(X)$ for any real-valued 1-form. Then in the space-like case J is a complex structure, while in the time-like $J^2 = 1$. For an immersed surface locally there is always a canonical choice of an orientation given by the normal vector.

The Hodge star and the identification of $\mathbb{R}^{2,1}$ with $Im\mathbb{H}'$ allows to identify 2-forms on M with $Im\mathbb{H}'$-valued functions on T_pM. The identification for every wedge product between two 1-forms is

$$(\alpha \wedge \beta)(x) = \alpha(x) * \beta(x) - *\alpha(x)\beta(x) \tag{3.1}$$

Then bilinear forms are assumed to be expressed as their quadratic counterparts as

$$\omega(x) = \omega(x, Jx)$$

The differential of a conformal immersion at a point $p \in M$ can be thought of as the linear map

$$df : T_pM \to T_p\mathbb{R}^3 \simeq T_pIm\mathbb{H}'$$

or alternatively $df \in \bigwedge^1(T_pM, Im\mathbb{H}')$. For a tangent vector $x \in T_pM$, with $x = f_*(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v})$

$$df(x) = (f_u du + f_v dv)(x) = f_u a + f_v b$$

and by abuse of notations we can identify x with the vector $[a, b]^T$. In this notation we have: If f is space-like, then

$$*df(x) = df \circ J \circ x = -af_v + bf_u.$$

If f is time-like, then

$$*df(x) = df \circ J \circ x = af_v + bf_u.$$

It will be convenient to introduce the following notation in performing the calculations.

Notation: Let e be a number such that $e = -1$ for time-like f, and $e = 1$ for space-like f.

Then the above identity can be shortly written as

$$*df(x) = -aef_v + bf_u$$

The form $*df := -ef_v du + f_u dv$ and $df \circ J$ have the same domain and image so they are equal. Also since

$$df(f_*(\frac{\partial}{\partial u})) = f_u$$

and

$$df(f_*(\frac{\partial}{\partial v})) = f_v$$

then

$$*df(f_*(\tfrac{\partial}{\partial u})) = -ef_v \qquad \text{and} \qquad *df(f_*(\tfrac{\partial}{\partial v})) = f_u$$

So by an abuse of notation we will sometimes write

$$*f_u = ef_v \qquad \text{and} \qquad *f_v = f_u$$

with the $*$ having the property

$$** = -e$$

Given a conformal immersion $f : M \to \mathbb{R}^{2,1}$, pointwise $f_u, f_v \in Im\mathbb{H}'$ and together with (2.1), we have that the Gauss map is expressed as

$$N := \frac{f_u f_v}{|f_u|^2} = \frac{f_u f_v}{f_v^2} \tag{3.2}$$

It is easy to see now that for a space-like surface

$$|N|_m^2 = N^2 = -1$$

and for a time-like surface

$$|N|_m^2 = N^2 = 1,$$

so that the Gauss map simply has the property

$$|N|_m^2 = N^2 = -e.$$

With this in place we can characterize conformal immersions through the relationship between the $*$ map and the Gauss map N encoded in the following lemma:

Lemma 3.1 *$f : M \to \mathbb{R}^{2,1}$ is a conformal immersion if and only if there exits $N : M \to \mathbb{H}'$ such that*

$$*df = Ndf \tag{3.3}$$

To do this we first show that for a conformal immersion (3.3) characterizes the Gauss map, that is:

Lemma 3.2 *Let f be a conformal immersion for which (3.3) holds. Then $N : M \to Im\mathbb{H}'$ and it is the Gauss map.*

Proof: First, suppose (3.3) holds. Then pointwise

$$**df = *(Ndf) = N(*df)$$
$$= NNdf = N^2 df = -edf$$

since df is pointwise injective, $** = -e$ implies $N^2 = -e$. In general for $N = n_0 + Im(N)$, $N^2 = (n_0^2 + 2n_o Im(N) + Im(N)^2) = -e$. So, $N \in \mathbb{R}$ or $N \in Im\mathbb{H}'$. If $N \in \mathbb{R}$ then, e must be -1 and $N = \pm 1$. But then we have

$$*df = \pm df$$

so $df \circ J = \pm df$ and $J = \pm Id$, a contradiction. Then $N \in Im\mathbb{H}'$. Now taking the conjugate of $*df = Ndf$ we have $- * df = \overline{Ndf}$ and

$$Ndf = *df = -\overline{Ndf}$$

so

$$2Re(Ndf) = Ndf + \overline{Ndf} = 0$$

Then

$$< N, df >= 0$$

By uniqueness of the Gauss map, N is the unit normal field to f. Notice that in this direction we did not require the assumption that f be conformal.

Now, assuming f to be conformal and N to be the Gauss map, we have

$$Nf_u = \frac{(f_u f_v)}{f_v^2} f_u$$

and anti-comutation of the cross product and associativity of quaternions yields

$$-\frac{f_v f_u^2}{f_v^2} = -ef_v$$

and similarly

$$Nf_v = \frac{(f_u f_v)}{f_v^2} f_v = \frac{f_u f_v^2}{f_v^2} = \frac{f_u f_v^2}{f_v^2} = f_u$$

Hence at a point p

$$Ndf = N(f_u du + f_v dv) = Nf_u du + Nf_v dv = -ef_v du + f_u dv = *df = df \circ J$$

Proof of Lemma 3.1:

Again suppose (3.3) holds. Since we know N is the Gauss map, point-wise we can consider $N, f_u, f_v \in Im\mathbb{H}'$, so we have on one hand that $Nf_u = N \times_m f_u$ by perpendicularity (same for f_v), so that

$$(Nf_u)(Nf_v) = N(f_u N)f_v = -N^2(f_u f_v) = ef_u f_v$$

Continuing through the use of (2.1) on the leftmost and rightmost sides of the previous equation

$$- < Nf_u, Nf_v > +Nf_u \times Nf_v = e(- < f_u, f_v > +f_u \times f_v)$$

Note also that $Nf_u = -ef_v$ and $Nf_v = f_u$ so that

$$- < -ef_v, f_u > + - ef_v \times f_u = e(- < f_u, f_v > +f_u \times f_v)$$

Then by bilinearity

$$e(< f_u, f_v > +f_u \times f_v) = e(- < f_u, f_v > +f_u \times f_v)$$

and

$$2 < f_u, f_v >= 0$$

which gives one property of the conformality. For the second, remembering the anti commutativity between N and f_u, f_v, we have

$$< f_u, f_u >= -eN^2 < f_u, f_u >= -e < -Nf_u, Nf_u >= -e < ef_v, -ef_v >$$

So

$$< f_u, f_u >= e^3 < f_v, f_v >= e < f_v, f_v >$$

and f is a conformal immersion. If we assume conformality we have by lemma 3.2 that indeed equation (3.3) holds.

4 Conformal decomposition and mean curvature

For any $\mathbb{H}'$ valued 1-form on M we can decompose it into its conformal and anti-conformal parts via

$$\alpha_+ = \frac{1}{2}(\alpha - eN * \alpha)$$

and

$$\alpha_- = \frac{1}{2}(\alpha + eN * \alpha)$$

For the conformal part we see

$$*\alpha_+ = \frac{1}{2}(*\alpha - eN * *\alpha)$$

$$= \frac{1}{2}(-eN^2 * \alpha - eN(-e)\alpha)$$

$$= N\frac{1}{2}((-e)^2\alpha - eN^2 * \alpha)$$
$$= N\alpha_+$$

and a similar calculation gives us, $*\alpha_- = -N\alpha_-$ for the anti-conformal part. So together we have

$$\alpha = \alpha_+ + \alpha_- \quad *\alpha_\pm = \pm N\alpha \tag{4.1}$$

Now from equation (3.3) we can see that

$$d * df = dNdf = dN \wedge df + Nddf$$

and by (3.1)

$$= dN * df - *dNdf = dNNdf - *dNdf = (dNN - *dN)df$$

$$= (dNN - *dN(-eN^2)df) = (dN + e * dNN)Ndf$$

Since N^2 is a real number then

$$d(N^2) = 0$$

hence

$$dNN = -NdN.$$

Continuing from above

$$d * df = (dN - e * NdN)Ndf = 2(dN_+)Ndf \tag{4.2}$$

The left hand side is $Im\mathbb{H}'$ valued, then so is the right, which implies

$$Re((dN_+)Ndf) = 0 =< dN_+, Ndf > \tag{4.3}$$

Now at a point p of M, and for a given vector $v \in T_pM$, we can think of df as simply an imaginary quaternion. We can represent it as $\alpha f_u + \beta f_v$ where $\alpha, \beta \in \mathbb{R}$ and depend on p and v. Noting this we observe

$$< df, Ndf >= -Re(dfNdf)$$
$$= -Re(Ndf^2)$$

and since df^2 is a real number and N is imaginary

$$< df, Ndf >= 0$$

At a point $p \in M$ and vector $v \in T_pM$, Ndf, dN_+ and df yield forms in the cotangent plane of f. Thus the previous calculation and (4.3) imply that dN_+ and df are parallel, or there exists a function $H : M \to \mathbb{R}$ such that

$$dN_+ = Hdf.$$

We can check that the function H is the mean curvature of f. To see this note that N_u and N_v are tangent vectors and

$$N_u = \alpha f_u + \beta f_v$$

$$N_v = \gamma f_u + \sigma f_v.$$

By standard arguments

$$< N_v, f_u >= - < N, f_{uv} >=< N_u, f_v >,$$

so together with

$$< N_u, f_v >= \beta < f_v, f_v >$$

$$< N_v, f_u >= \gamma < f_u, f_u >$$

we arrive at

$$\beta < f_u, f_u >= \gamma < f_v, f_v >$$

and

$$\beta f_u^2 = \gamma e f_u^2.$$

From here

$$\beta = e\gamma.$$

Applying this to the 1-form dN_+, component-wise we have

$$2dN_+ = dN - eN * dN$$

and since $*dN = *(N_u du + N_v dv) = (N_u du + N_v dv) \circ J = -eN_v du + N_u dv$ then

$$2dN_+ = (N_u + N_v)du - eN(-eN_v + N_u)dv$$

$$= (N_u + NN_v)du + (N_v - eNN_u)dv$$

which in terms of the vectors, $\alpha, \beta, \gamma, \delta$ from above simplifies to

$$2dN_+ = ((\alpha + \sigma) f_u + (\beta - e\gamma) f_v)du + ((\beta - e\gamma) f_u + (\alpha + \sigma) f_v)dv.$$

So from above we have

$$2dN_+ = ((\alpha + \sigma) f_u du + ((\alpha + \sigma) f_v dv$$

or

$$dN_+ = \frac{(\alpha + \sigma)}{2} df$$

and $H = \frac{(\alpha+\sigma)}{2}$.

The expression for mean curvature in terms of the partial derivatives of f and N for conformal immersion is ([8] pg 74):

$$H' = \frac{1}{2} \frac{< N_u, f_u >< f_v, f_v > + < N_v, f_v >< f_u, f_u >}{< f_u, f_u >< f_v, f_v >}$$

$$= -\frac{1}{2} \frac{< N_u, f_u > (f_v^2)+ < N_v, f_v > (f_u^2)}{f_u^2 f_v^2}.$$

Using

$$< N_u, f_u >= \alpha < f_u, f_u >$$

$$< N_v, f_v >= \sigma < f_v, f_v >$$

this will simplify to

$$H' = \frac{(\alpha+\sigma)}{2},$$

so $H = H'$ and

$$dN_+ = Hdf \tag{4.4}$$

where H is the mean curvature.

Finally we conclude that

$$dN = Hdf + \omega \tag{4.5}$$

where we set $\omega = dN_-$ the anti-conformal part of dN. Also recalling (4.2) we can write

$$d * df = 2(Hdf)Ndf = -2HN|df|^2 \tag{4.6}$$

Differentiating (4.5) we get

$$0 = dH \wedge df + dw$$

and by (3.1)

$$0 = dH * df - *dHdf + dw,$$

hence

$$dw = (*dH - dHN)df \tag{4.7}$$

In a similar way one can see that it is the Codazzi equation.

5 Spin equivalence

Two conformal immersions $f, \tilde{f} : M \to \mathbb{R}$ are called *spin-equivalent* if there exists $\lambda : M \to \mathbb{H}'_*$ such that

$$d\tilde{f} = \bar{\lambda} df \lambda \tag{5.1}$$

Here $\mathbb{H}'_*$ is the set of split quaternions with non-vanishing norm.

Note that for a simply connected domain any two conformal immersions are spin-equivalent: since $d\tilde{f}$ and df are conformal 1-forms pointwise they can be mapped into each other by a rotation and scaling. The description of rotations in Section 2 automatically makes them a special case of spin transform. In fact the spin transform consists of a rotation and a scaling, or simply a real multiple of one of the above rotations. So that for a general spin transform about a time-like or space-like axis

$$\lambda\bar{\lambda} = |\lambda|^2 > 0$$

and λ is constant if and only if $\tilde{f}$ can be obtained from f by a Euclidean motion and scaling. If in addition $|\lambda| = 1$, then f and $\tilde{f}$ are congruent immersions.

Locally, one can start with a given *reference* immersion $f: M \to \mathbb{R}$ and obtain new spin equivalent immersions by solving

$$dd\tilde{f} = 0 = d(\bar{\lambda} df \lambda) = d\bar{\lambda} \wedge df\lambda - \bar{\lambda} df \wedge d\lambda$$

by first property of the wedge product

$$= d\bar{\lambda} \wedge df\lambda + \overline{\overline{d\bar{\lambda}} \wedge \overline{df}\lambda}$$

since $\overline{d\lambda} = d\bar{\lambda}$ and because df is $Im\mathbb{H}'$-valued we have

$$0 = d\bar{\lambda} \wedge df\lambda - \overline{d\bar{\lambda} \wedge df\lambda}$$

so $d\bar{\lambda} \wedge df\lambda$ is real valued and $d\bar{\lambda} \wedge df\lambda \in \bigwedge^2(M,\mathbb{R})$. On the other hand

$$df \wedge *df(x, Jx) = df * *df - *df * df$$

$$= -edf^2 - (*df)^2 = -edf^2 - (NdfNdf)$$
$$= -edf^2 + NNdfdf$$
$$= -edf^2 - edf^2 = -2e|df|^2$$

is also in $\bigwedge^2(M,\mathbb{R})$. Because dim(M)=2, $dim(\bigwedge^2(M,\mathbb{R})) = 1$, so for any $\alpha, \beta \in \bigwedge^2(M,\mathbb{R})$ there exists $\rho' : M \to \mathbb{R}$ such that $\alpha = \rho'\beta$. This provides us with the equation

$$d\bar{\lambda} \wedge df\lambda = \rho'(-2e|df|^2)$$

or

$$d\bar{\lambda} \wedge df\lambda(\frac{\bar{\lambda}}{|\bar{\lambda}|^2}) = -2\rho' e|df|^2 \frac{\bar{\lambda}}{|\bar{\lambda}|^2}$$

and

$$d\bar{\lambda} \wedge df = -2\rho' 2e|df|^2 \frac{\bar{\lambda}}{|\bar{\lambda}|^2}$$

If we let $\rho = -2\rho'/|\bar{\lambda}|^2$.

$$d\bar{\lambda} \wedge df = e\rho|df|^2\bar{\lambda}$$

and if we conjugate

$$df \wedge d\lambda = e\rho df^2\lambda \tag{5.2}$$

Identifying the right hand side of (5.2) with its quadratic form via (3.1) we have

$$e\rho df^2\lambda = df * d\lambda - Ndfd\lambda =$$

$$= df * d\lambda + dfNd\lambda = df(*d\lambda + Nd\lambda) = e\rho df^2\lambda.$$

For $|df|^2 \neq 0$ we may divide by df point-wise as a quaternion, and we have

$$(*d\lambda + Nd\lambda) = e\rho df \tag{5.3}$$

This can be stated as

Lemma 5.1 *If $f, \tilde{f} : M \to \mathbb{R}$ are spin-equivalent via $d\tilde{f} = \bar{\lambda} df \lambda$ then $\lambda : M \to \mathbb{H}_*$ satisfies (5.3).*

Next we see how the Gauss map N, the induced metric, and the mean curvature H are related through spin transformations of conformal maps.

Lemma 5.2 *Let $f, \tilde{f} : M \to \mathbb{R}^3$ be spin-equivalent via $d\tilde{f} = \bar{\lambda} df \lambda$. Then*

1)$\tilde{N} = \lambda^{-1} N \lambda$ where $\tilde{N}$ is the oriented normal to $\tilde{f}$

2)$|d\tilde{f}|^2 = |\lambda|^4 |df|^2$

3)$\tilde{H} = \frac{H+\rho}{|\lambda|^2}$ where $\rho : M \to \mathbb{R}$ is given by (5.3)

Proof: For 1) we have $*d\tilde{f} = *(\bar{\lambda} df \lambda)$ and

$$\tilde{N} d\tilde{f} = \bar{\lambda} * df \lambda = \bar{\lambda} N df \lambda$$

$$\tilde{N} \bar{\lambda} df \lambda = \bar{\lambda} N df \lambda$$

Since $\bar{\lambda}$ and λ are invertible and for $|df|^2 \neq 0$ we have

$$\tilde{N} \bar{\lambda} = \lambda N$$

From here by multiplication with $\bar{\lambda}^{-1} = \frac{\lambda}{|\lambda|^2}$ on the right

$$\tilde{N} = \tilde{N} \bar{\lambda} \frac{\lambda}{|\lambda|^2} = \frac{\bar{\lambda}}{|\lambda|^2} N \lambda = \lambda^{-1} N \lambda$$

For part 2) again a calculation gives us

$$|d\tilde{f}|^2 = |\bar{\lambda} df \lambda|^2$$

$$= (\bar{\lambda} df \lambda \bar{\lambda} df \lambda) = |\lambda|^4 |df|^2.$$

We can easily see that projection onto the conformal part is linear:

$$(a\alpha + b\beta)_+ = a\alpha_+ + b\beta_+$$

and that the spin transform preserves the conformal part

$$\bar{\lambda}(\alpha_+)\lambda = (\bar{\lambda}\alpha\lambda)_+.$$

Then

$$(\tilde{H} d\tilde{f} + \tilde{\omega})_+ = (d\tilde{N})_+ = (d(\lambda^{-1} N \lambda))_+$$

so that

$$\tilde{H}d\tilde{f} = (d\lambda^{-1}N\lambda + \lambda^{-1}dN\lambda + \lambda^{-1}Nd\lambda)_+.$$

If we note that since $d(\lambda^{-1}\lambda) = d(1) = 0$, then $d\lambda^{-1}\lambda = -\lambda^{-1}d\lambda$, we can simplify the above as

$$\tilde{H}d\tilde{f} = (-\lambda^{-1}d\lambda\lambda^{-1}N\lambda + \lambda^{-1}dN\lambda + \lambda^{-1}Nd\lambda)_+ =$$

$$= \lambda^{-1}(-d\lambda\lambda^{-1}N + Nd\lambda\lambda^{-1})_+\lambda + \frac{1}{|\lambda|^2}(\bar{\lambda}dN_+\lambda)$$

$$+\tilde{H}d\tilde{f} = \lambda^{-1}(-d\lambda\lambda^{-1}N + Nd\lambda\lambda^{-1})_+\lambda + \frac{1}{|\lambda|^2}(H\bar{\lambda}df\lambda)$$

By (4.4) on the right we must find the conformal part of $(Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N)$, so

$$2(Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N)_+ = Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N - eN^2 * d\lambda\lambda^{-1} + eN * d\lambda\lambda^{-1}N$$

$$= Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N + *d\lambda\lambda^{-1} + eN * d\lambda\lambda^{-1}N$$

employing equation (5.3)

$$= Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N + (e\rho df - Nd\lambda\lambda^{-1}) + eN(e\rho df - Nd\lambda\lambda^{-1})N$$

$$= Nd\lambda\lambda^{-1} - d\lambda\lambda^{-1}N + (e\rho df - Nd\lambda\lambda^{-1}) + eN\rho df + d\lambda\lambda^{-1})N$$

$$= 2e\rho df$$

So we finally arrive at

$$\tilde{H}d\tilde{f} = \lambda^{-1}(e\rho df)\lambda + \frac{1}{|\lambda|^2}(Hd\tilde{f}) = \frac{e\rho d\tilde{f}}{|\lambda|^2} + \frac{1}{|\lambda|^2}(Hd\tilde{f})$$

hence

$$\tilde{H} = \frac{H + e\rho}{|\lambda|^2}$$

From this last theorem we can derive the following corollary:

Corollary 5.1 *Let $f, \tilde{f} : M \to \mathbb{R}^3$ be spin-equivalent via $d\tilde{f} = \bar{\lambda}df\lambda$, Then the following are equivalent:*

1)$\tilde{H}|d\tilde{f}| = H|df|$

2)$df \wedge d\lambda = 0$*, equivalent to,* $*d\lambda + Nd\lambda = 0$

Proof: We check that $\tilde{H}|d\tilde{f}| = H|df|$ iff $\tilde{H}(|d\tilde{f}|^2)^{1/2} = H|df|$ iff $\tilde{H}(|\lambda|^4|df|^2)^{1/2} = H|df|$ iff

$$H + ep = H.$$

Hence $p = 0$, which is equivalent to

$$df \wedge d\lambda = epdf^2\lambda = 0.$$

6 Isothermic Surfaces and Bonnet Pairs

Definition 1: A conformal immersion is isothermic if there exists a non-zero $Im\mathbb{H}'$ valued anti-conformal 1-form, τ, such that $d\tau = 0$ but $t\tau \neq 0$ and $*\tau + N\tau = 0$ which is equivalent to $df \wedge \tau = 0$. Here actually $\tau = df^*$, where f^* is the dual surface to f.

Definition 2: Two conformal immersions form a Bonnet pair if $|d\tilde{f}|^2 = |df|^2$ and $\tilde{H} = H$ but are not congruent.

Now we state a theorem classifying such Bonnet pairs on a simply connected domain.

Theorem 6.1 *Let M be simply connected and $f : M \to \mathbb{R}^3$ be isothermic with dual surface $f^* : M \to \mathbb{R}^3$. Choose $r \in \mathbb{R}_*$, $a \in Im\mathbb{H}'$, with $r^2 > (f^* + a)^2$, and let $\lambda_\pm = \pm r + f^* + a$. Then the spin transforms $f_\pm : M \to \mathbb{R}^3$ given by $df_\pm = \bar{\lambda}_\pm df \lambda_\pm$ form a Bonnet pair. Conversely, every Bonnet pair arises from a 3-parameter family (determined up to scalings) of isothermic surfaces where the three parameters account for Euclidean rotations of the partners in the Bonnet pair.*

Proof: Given that f is isothermic implies

$$df \wedge d\lambda_\pm = df \wedge d(\pm r + f^* + a)$$

$$= df \wedge df^*$$

$$= df \wedge \tau = 0$$

which by Corollary 5.1 implies

$$H_\pm |df_\pm| = H|df|$$

and

$$|\lambda_\pm|^2 = (\pm r + f^* + a)\overline{(\pm r + f^* + a)}$$

$$= (\pm r + (f^* + a))(\pm r - (f^* + a))$$

$$= r^2 - (f^* + a)^2.$$

Since r is chosen so that $r^2 > (f^* + a)^2$, we have

$$|\lambda_+| = |\lambda_-|$$

so that

$$|df_\pm|^2 = (\bar{\lambda}_\pm df \lambda_\pm)^2 = |\lambda_\pm|^4 |df|^2 = |\lambda_+|^4 |df|^2$$

hence

$$|df_+|^2 = |df_-|^2.$$

Since we just saw that $H_\pm |df_\pm| = H|df|$, we have that also

$$H_+ = H_-.$$

If f_+, f_- were congruent then there would be a constant $q \in Im\mathbb{H}'$, such that

$$df_- = \bar{q} df_+ q$$

By $df_\pm = \bar{\lambda}_\pm df \lambda_\pm$ we have , $df_- = \overline{\lambda_+^{-1}\lambda_-} df_+ \lambda_+^{-1}\lambda_-$. This would in turn imply that f^* itself is constant, and so $df^* = 0 = \tau$, which contradicts the fact that f is isothermic. So indeed f_+, f_- form a Bonnet pair.

Conversely given a Bonnet pair $f_\pm$ by conformality in a simply connected domain, their exits $\lambda : M \to \mathbb{H}'_*$, such that

$$df_+ = \bar{\lambda} df_- \lambda$$

Since $|df_+|^2 = |df_-|^2$ and $|\lambda|^2 > 0$, by lemma 5.2 $|\lambda| = 1$. By Corollary 5.1 $df_\pm \wedge d\lambda = 0$. Now we seek an isothermic surface $f : M \to Im\mathbb{H}'$ for which $df_\pm = \bar{\lambda}_\pm df \lambda_\pm$ where $\lambda_\pm = \pm r + f^* + a$ as before. This would indicate that $\lambda = \lambda_-^{-1}\lambda_+$, since

$$-\bar{\lambda}_- = r + f^* + a = \lambda_+$$

then we have the equation, $\lambda = -\lambda_-^{-1}\bar{\lambda}_-$, which implies

$$-\lambda_-\lambda = \bar{\lambda}_-$$

$$-\lambda_-\lambda + \lambda_- = \bar{\lambda}_- + \lambda_- = 2Re(\lambda_-) = -2r$$

$$\lambda_-(1 - \lambda) = -2r$$

$$\lambda_- = 2r(\lambda - 1)^{-1}$$

so that $f^* = r - a + 2r(\lambda - 1)^{-1}$. We can guarantee that $(\lambda - 1)$ vanishes nowhere by multiplying λ by a unit quaternion, inducing a rotation of the Bonnet pair with respect to each other. Now

$$2Re(f^*) = f^* + \bar{f}^* = r - a + 2r(\lambda - 1)^{-1} + r + a + 2r\overline{(\lambda - 1)^{-1}}$$

$$= 2r(1 + (\lambda - 1)^{-1} + \overline{(\lambda - 1)^{-1}})$$

$$= \frac{2r}{|1 - \lambda|^2}((\lambda - 1)(\bar{\lambda} - 1) + \bar{\lambda} - 1 + \lambda - 1)$$

$$= \frac{2r}{|1 - \lambda|^2}(\lambda\bar{\lambda} - 1)$$

and $|\lambda|^2 = 1 = \lambda\bar{\lambda}$ implies $Re(f^*) = 0$, so f^* is purely imaginary. Setting

$$df = \overline{(\lambda - 1)} df_-(\lambda - 1)$$

we have

$$df \wedge df^* = 2r df \wedge d(\lambda - 1)^{-1}$$

and again using the fact that $(\lambda - 1)^{-1}(\lambda - 1) = 1$ we may write $d(\lambda - 1)^{-1} = -(\lambda - 1)^{-1} d\lambda(\lambda - 1)^{-1}$. So that

$$df \wedge df^* = \overline{(\lambda - 1)} df_-(\lambda - 1) \wedge (\lambda - 1)^{-1} d\lambda(\lambda - 1)^{-1}$$

since norms of quaternions commute freely in this expression we may write

$$(\lambda-1)^{-1}df_{-}(\lambda-1)\wedge\overline{(\lambda-1)}d\lambda(\lambda-1)^{-1}$$

or

$$(\lambda-1)^{-1}[df_{-}(\lambda-1)\wedge(\bar{\lambda}-1)d\lambda](\lambda-1)^{-1}.$$

So lets compute now $df_{-}(\lambda-1)\wedge(\bar{\lambda}-1)d\lambda$. Distribution and bilinearity yield

$$df_{-}(\lambda-1)\wedge(\bar{\lambda}-1)d\lambda = df_{-}\lambda\wedge\bar{\lambda}d\lambda - df_{-}\wedge\bar{\lambda}d\lambda - df_{-}\lambda\wedge d\lambda + df_{-}\wedge d\lambda.$$

Using spin equivalence, i.e $df_{-}\lambda = \bar{\lambda}^{-1}df_{+}$, and $d(\lambda\bar{\lambda}) = 0$, together with $df_{\pm}\wedge d\lambda = 0$ yield

$$df\wedge df^{*} = 0$$

as required. So $f : M \to \mathbb{R}^3$ is isothermic with dual surface f^*, and $f_{\pm}$ are spin transforms via $\lambda_{\pm} = \pm r + f^* + a$.

7 Time-like Bonnet pairs and split complex numbers

In the previous sections we showed that the local theory of Bonnet pairs in Minkowski space parallels the one in Euclidean space. Most of the theory of global space-like Bonnet pairs is the same as in the Euclidean case, so we consider here the time-like surfaces (called also Lorentzian). It is known that the formulation of the Gauss and Codazzi equations in the Euclidean space can be simplified if complex coordinate patch is used. In this section we provide a formulation of the time-like Bonnet pairs in terms of the so-called *split complex* numbers (called also para-complex and hyperbolic complex numbers). Split complex numbers have been used for describing Weierstrass representation and Björling problem for minimal time-like surfaces, [8], [9], [10], [3]. Our formulation is close to [2] for the Bonnet pairs in the Euclidean space. It allows one to construct a simple compact time-like Bonnet pairs of minimal tori in Minkowski space.

Consider an ordered pairs of real numbers with the usual addition but with multiplication $(a,b).(c,d) = (ac+bd, ad+bc)$. Denote by τ the pair $(0,1)$ and then the product is the same as the product of $(a+\tau b)(c+\tau d)$ extended by bilinearity and with the property $\tau^2 = 1$. One can see that this multiplication is commutative. The set $\mathbb{C}' = \{u+\tau v | u,v\in\mathbb{R}\}$ with this multiplication and the usual addition forms an algebra it elements are called split complex numbers. Split complex numbers can be regarded naturally as a commutative subalgebra of the split quaternions. Similarly the split complex number $\bar{z} = u-\tau v$ is called conjugate to $z = u+\tau v$ and $z\bar{z} = u^2 - v^2$. To use further this notations, we introduce

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - \tau\frac{\partial}{\partial v}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial u} + \tau\frac{\partial}{\partial v})$$

A function $f : \mathbb{C}' \to \mathbb{C}'$ with $\frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} = 0$ is called *split holomorphic*.

Let $F : M \to \mathbb{R}^{2,1}$ be an immersion and consider $\mathbb{R}^{2,1}$ as the real part of $(\mathbb{C}')^3$. Then the second partial derivatives of F satisfy $4F_{zz} = F_{uu} + F_{vv} - 2\tau F_{uv}$ and $4F_{z\bar{z}} = F_{uu} - F_{vv}$. It is straightforward to see that F is conformal and time-like if and only if

$$< F_z, F_z >=< F_{\bar{z}}, F_{\bar{z}} >= 0$$

and

$$< F_z, F_{\bar{z}} >= \frac{1}{2}e^w$$

for a real function w, so the induced metric is given by $g = e^w dz d\bar{z}$. The normal vector N satisfies

$$< F_z, N >=< F_{\bar{z}}, N >= 0, < N, N >= 1$$

In particular $F_z, F_{\bar{z}}, N$ are independent in $(\mathbb{C}')^3$ and are orthogonal with respect to the hermitian metric defined by $<,>$. Then the equations for the second derivatives become:

$$F_{zz} = w_z F_z + \Omega N$$

$$F_{z\bar{z}} = \frac{1}{2}e^w HN$$

$$N_z = -HF_z - 2e^{-w}\Omega F_{\bar{z}}$$

$$N_{\bar{z}} = -HF_{\bar{z}} - 2e^{-w}\overline{\Omega}F_z$$

and the Gauss-Codazzi equations, obtained from $F_{z\bar{z}z} = F_{zz\bar{z}}$, are given by:

$$w_{z\bar{z}} + \frac{1}{2}H^2 e^w - 2|\Omega|^2 e^{-w} = 0$$

$$\Omega_{\bar{z}} = \frac{1}{2}H_z e^w$$

Note that $\Omega =< F_{zz}, N > (dz)^2$ is quadratic differential, independent of the change of coordinates, called the Hopf differential.

Suppose now that we have a Bonnet pair F_1, F_2. Then $H_1 = H_2 = H$ and $w_1 = w_2 = w$, so for the Hopf diferentials we have

$$\frac{\partial}{\partial \bar{z}}(\Omega_1 - \Omega_2) = 0$$

and

$$|\Omega_1|^2 = |\Omega_2|^2$$

Then, in the same way as in the Euclidean case [2], there are split-holomorphic function h and real valued function α, such that $\Omega_1 = \frac{1}{2}h(\tau\alpha - 1), \Omega_2 = \frac{1}{2}h(\tau\alpha + 1)$. Note that $(\Omega_2 - \Omega_1)(dz)^2 = h(dz)^2$ is a split-holomorphic quadratic differential.

The Gauss-Codazzi equations become:

$$w_{z\bar{z}} + \frac{1}{2}H^2 e^w - \frac{1}{2}|h|^2(1 - \alpha^2)e^{-w} = 0$$

$$\frac{\tau}{2}h\alpha_{\bar{z}} = \frac{1}{2}H_z e^w$$

Notice that for any real-valued function f, the functions $f_\pm = (1 \pm \tau)f(u \mp v)$ are split-holomorphic and $|f_\pm|^2 = 0$. Also for any split-holomorphic function $h = h_1 + \tau h_2$ h_1 and h_2 satisfy the equation $(h_i)_{z\bar{z}} = 0, i = 1, 2$. Now we can see that quadruple of functions (H, α, h, w) where $H = 0$, $\alpha = const \neq 0$, h - any nonconstant split-holomorphic function with $|h|^2 = 0$, and w satisfying $w_{z\bar{z}} = 0$, satisfy the Gauss-Codazzi equations and give rise to a Bonnet pair. So based on this we consider an example of a Bonnet pair of time-like tori.

Example. An immersed torus M is given by a map F which is doubly-periodic. Consider the simplest case where both periods are 2π: $F(u+2\pi, v) = F(u, v+2\pi) = F(u, v)$. Then the functions H, α, h, w are also 2π-doubly periodic. The converse is also true - if H, α, h, w are periodic and satisfy the Gauss-Codazzi equations, then there exists a unique surface, up to a Euclidean motion and with given H, α, h, w which is given by periodic F. Take for example $H = 0$, $h = (1+\tau)\sin(u-v)$, $\alpha = 1+\tau$, and $w = \cos(u+v)$. These functions give rise to solution of the equations above. Notice that the corresponding Bonnet pair F_1, F_2 has $H_1 = H_2 = 0$ and is a pair of minimal immersed time-like tori in $\mathbb{R}^{2,1}$. Also the points in which $h = (1+\tau)\sin(u+v) = 0$ are umbilical, in contrast to the Euclidean space.

Aknowledgemnts: We thank M. Magid for informing us about his work.

References

[1] L. Bianchi, , *Sulla superficie e linee di curvature isotherme.* Rend. Accad. Naz. dei Lincei (5)**12** (1903), 511-520.

[2] A. Bobenko, *Exploring surfaces through methods from the theory of integrable systems: the Bonnet problem.* Surveys on geometry and integrable systems, 153, Adv. Stud. Pure Math., 51, Math. Soc. Japan, Tokyo, 2008.

[3] R. Chaves, M. Dussan, M. Magid, *Bjrling problem for timelike surfaces in the Lorentz-Minkowski space.* J. Math. Anal. Appl. 377 (2011), no. 2, 481494.

[4] M. Dussan, M. Magid, *Timelike Christoffel pairs in the split-quaternions.* Ann. Polon. Math. 99 (2010), no. 2, 201214.

[5] A. Fujioka, J. Inoguchi, *Timelike surfaces with harmonic inverse mean curvature.* Surveys on geometry and integrable systems, 113141, Adv. Stud. Pure Math., 51, Math. Soc. Japan, Tokyo, 2008.

[6] G. Kamberov, F. Pedit, U. Pinkall, *Bonnet Pairs and Isothermic Surfaces*, Duke Math.J. ,92 (3), (1998) 637-644.

[7] J. Konderak, *A Weierstrass representation theorem for Lorentz surfaces.* Complex Var. Theory Appl. 50 (2005), no. 5, 319332.

[8] W. Kühnel, *Differential geometry. Curves - surfaces - manifolds*, Translated from the 1999 German original by Bruce Hunt. Student Mathematical Library, 16. American Mathematical Society, Providence, RI, 2002.

[9] M.- A. Lawn, *Immersions of Lorentzian surfaces in $R^{2,1}$.* J. Geom. Phys. 58 (2008), no. 6, 683700.

[10] J. Lira, M. Melo, F. Mercuri,*A Weierstrass representation for minimal surfaces in 3-dimensional manifolds.* Results Math. 60 (2011), no. 1-4, 311323.

[11] M. Magid, *Lorentzian isothermic surfaces and Bonnet pairs.* Ann. Polon. Math. 83 (2004), no. 2, 129139.

[12] I. Sabitov *Isometric surfaces with common mean curvature and the problem of Bonnet pairs.* (Russian) Mat. Sb. 203 (2012), no. 1, 115158.