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Screen Conformal Invariant Lightlike Hypersurfaces of Indefinite Sasakian Space Forms

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To Professor Augustin Banyaga on the occassion of his 65th Birthday

Abstract

In this paper, we investigate a class of screen conformal invariant lightlike hypersurfaces of an indefinite Sasakian manifold. The geometric configuration of such hypersurfaces is established. We prove that its geometry is closely related to the one of leaves of its conformal screen distributions. We also prove that, in any leaf of a conformal screen distribution of an invariant lightlike hypersurface of an indefinite Sasakian space form, the parallelism and semi-parallelism notions are equivalent.

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1 Introduction

A hypersurface M of a semi-Riemannian manifold $\overline{M}$ is called lightlike (degenerate) hypersurface if the induced metric on M is degenerate. In general, lightlike submanifolds have been studied widely in mathematical physics. They appear in general relativity as some smooth parts of event horizons of the Kruskal and Kerr black holes [8]. Lightlike submanifolds of semi-Riemannian manifold have been studied by Duggal-Bejancu and Kupeli in [2] and [7], respectively. Kupeli's approach is intrinsic while Duggal-Bejancus approach is extrinsic. Lightlike hypersurfaces of indefinite Sasakian manifolds are defined according to the behaviour of the almost contact structure of indefinite Sasakian manifolds and such submanifolds were studied by the author in [10], [11], [12], [13] and [14], and Duggal and

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Sahin whose details are given in the book [6] and references therein. They defined and studied invariant, contact CR-lightlike and screen CR- lightlike.

We know that the shape operator plays a key role in studying the geometry of submanifolds [2]. Motivated by above line of direction, the aim of this paper is to introduce the concept screen conformal distributions of invariant lightlike hypersufaces of Sasakian space forms. That is, we study invariant lightlike hypersufaces of Sasakian space forms whose shape operators are conformal to shape operators of their corresponding screen distributions. We also investigate the effect of conformal and invariance conditions on the geometry of leaves of some integrable distributions.

The paper is organized as follows. In section 2, we recall some basic definitions for indefinite Sasakian manifolds and lightlike hypersurfaces of semi-Riemannian manifolds. In section 3, we introduce a class of screen conformal invariant lightlike hypersurface M of an indefinite Sasakian space form $\overline{M}(c)$ supported by an example. By Theorem 3.6, we establish the geometric configuration of such hypersurfaces in indefinite Sasakian. We prove that the geometry of any leaf of the screen distribution is close to that of the screen conformal invariant lightlike hypersurface. We also prove that, in any leaf M' of a conformal screen distribution $S(TM)$ of an invariant lightlike hypersurface M of an indefinite Sasakian space form $\overline{M}(c)$, the parallelism and semi-parallelism notions are equivalent (Theorem 3.10).

2 Lightlike hypersurfaces

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \;\; \eta(\xi) = \varepsilon, \;\; \eta \circ \overline{\phi} = 0 \text{ and } \overline{\phi}\xi = 0, \tag{2.1}$$

where $\varepsilon = \pm 1$. Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an indefinite almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that [3], for any vector field $\overline{X}$, $\overline{Y}$ on $\overline{M}$,

$$\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \varepsilon\eta(\overline{X})\eta(\overline{Y}), \tag{2.2}$$

It follows that, for any vector $\overline{X}$ on $\overline{M}$,

$$\eta(\overline{X}) = \varepsilon\overline{g}(\xi, \overline{X}). \tag{2.3}$$

If, moreover, $d\eta(\overline{X}, \overline{Y}) = -\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y})$ and $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \varepsilon\eta(\overline{Y})\overline{X}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an *indefinite Sasakian manifold*. From (2.3), ξ is never a lightlike vector field on $\overline{M}$.

Sasakian manifolds with indefinite metrics have been first considered by Takahashi [17]. Their importance for physics have been pointed out by Duggal [5]. We have two classes of indefinite Sasakian manifolds [5]: ξ is spacelike ($\varepsilon = 1$ and the index of $\overline{g}$ is an even number $\nu = 2r$) and ξ is timelike ($\varepsilon = -1$ and the index of $\overline{g}$ is an odd number $\nu = 2r+1$).

Takahashi [17] shows that it suffices to consider those indefinite almost contact manifolds with space-like ξ. Hence, from now on, we shall restrict ourselves to the case of ξ a space-like unit vector (that is $\overline{g}(\xi, \xi) = 1$).

In this case, the equality

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X},\overline{Y})\xi - \eta(\overline{Y})\overline{X},$$

implies $\overline{\nabla}_{\overline{X}}\xi = -\overline{\phi}(\overline{X})$, $(\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = -\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})$ and ξ is a Killing vector field.

The Sasakian structure defined in [1] differs from the indefinite Sasakian one only by the positiveness of the metric involved and so, the main results in [1] remain unchanged for the indefinite case. We denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ.

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of the $\overline{\phi}$-sectional σ is called a $\overline{\phi}$-sectional curvature. If an indefinite Sasakian manifold $\overline{M}$ has constant $\overline{\phi}$-sectional curvature c, then, by virtue of the Theorem 7.19 in [1], the curvature tensor $\overline{R}$ of $\overline{M}$ is given by, for any $\overline{X}$, $\overline{Y}$, $\overline{Z}\in\Gamma(T\overline{M})$,

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} &= \frac{c+3}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\} + \frac{c-1}{4}\{\eta(\overline{X})\eta(\overline{Z})\overline{Y} - \eta(\overline{Y})\eta(\overline{Z})\overline{X}\\ &+ \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\,\overline{Y},\overline{Z})\overline{\phi}\,\overline{X} - \overline{g}(\overline{\phi}\,\overline{X},\overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}\,\overline{Z}\}.\end{aligned} \tag{2.4}$$

An indefinite Sasakian manifold $\overline{M}$ of constant $\overline{\phi}$-sectional curvature c will be called *indefinite Sasakian space form* and denoted $\overline{M}(c)$.

Example 2.1. Let $\mathbb{R}^7$ be the 7-dimensional real number space. Let us consider $\{x_i\}_{1\leq i\leq 7}$ as Cartesian coordinates on $\mathbb{R}^7$ and define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x_i}\right\}$ a tensor field $\overline{\phi}$ of type $(1,1)$ by: $\overline{\phi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2}$, $\overline{\phi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_7}$, $\overline{\phi}(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_4}$, $\overline{\phi}(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_7}$, $\overline{\phi}(\frac{\partial}{\partial x_5}) = -\frac{\partial}{\partial x_6}$, $\overline{\phi}(\frac{\partial}{\partial x_6}) = \frac{\partial}{\partial x_5}$, $\overline{\phi}(\frac{\partial}{\partial x_7}) = 0$. The 1-form η is defined by $\eta = dx_7 - x_4dx_1 - x_6dx_3$. The vector field ξ is defined by $\xi = \frac{\partial}{\partial x_7}$. It is easy to check (2.1) and thus $(\overline{\phi},\xi,\eta)$ is an almost contact structure on $\mathbb{R}^7$. Finally, we define a metric $\overline{g}$ on $\mathbb{R}^7$ by

$$\begin{aligned}\overline{g} &= (x_4^2-1)dx_1^2 - dx_2^2 + (x_6^2+1)dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 + dx_7^2 - x_4dx_1\otimes dx_7\\ &- x_4dx_7\otimes dx_1 + x_4x_6dx_1\otimes dx_3 + x_4x_6dx_3\otimes dx_1 - x_6dx_3\otimes dx_7 - x_6dx_7\otimes dx_3,\end{aligned}$$

with respect to the natural field of frames. It is easy to check that $\overline{g}$ is a semi-Riemannian metric and $(\overline{\phi},\xi,\eta,\overline{g})$ given above is a Sasakian structure on $\mathbb{R}^7$. Therefore, $(\mathbb{R}^7,\overline{\phi},\xi,\eta,\overline{g})$ is an indefinite Sasakian space form of constant sectional curvature $c=-3$.

A hypersurface (M,g) with $g = \overline{g}_{|M}$ of an indefinite Sasakian manifold $\overline{M}$ is called a *lightlike hypersurface* if g is of constant rank $2n-1$ and the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1. Suppose that M is paracompact. Then, there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^\perp$ in TM, called a *screen distribution* on M, such that

$$TM = S(TM)\perp TM^\perp, \tag{2.5}$$

where $\perp$ denotes the orthogonal direct sum. In general, $S(TM)$ is not canonical (thus it is not unique) and the lightlike geometry depends on its choice but it is canonically isomorphic to the vector bundle $TM/\mathrm{Rad}\,TM$ [7]. We denote such a lightlike hypersurface by triple $(M,g,S(TM))$. It is well-known [2] that, for any null section E of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ of rank 1 in the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $\overline{M}$ satisfying

$$\overline{g}(N, E) = 1 \quad \text{and} \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.6}$$

In this case, the tangent bundle $T\overline{M}$ of $\overline{M}$ is decomposed as follow:

$$T\overline{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM), \tag{2.7}$$

where $\oplus$ denotes the nonorthogonal direct sum. We call $tr(TM)$ and N the transversal vector bundle and the null transversal vector field of M with respect to the screen $S(TM)$, respectively.

Let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.5). Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.9}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E, \tag{2.10}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{2.11}$$

for any $X,\ Y \in \Gamma(TM|_{\mathcal{U}})$, where ∇ and ∇^* are the liner connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively. A_N and A^*_E are the shape operators on TM and $S(TM)$, respectively, and τ is a differential 1-form on TM. Since $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric on TM. From the fact that $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$, for any $X,\ Y \in \Gamma(TM)$, we show that B is independent of the choice of a screen distribution and satisfies $B(\cdot, E) = 0$. The two local second fundamental forms B and C are related to their shape operators by

$$B(X, PY) = g(A^*_E X, PY),\ \ g(A^*_E X, N) = 0, \tag{2.12}$$

$$C(X, PY) = g(A_N X, PY),\ \ g(A_N X, N) = 0. \tag{2.13}$$

From (2.12), A^*_E is $S(TM)$-valued self-adjoint on TM such that

$$A^*_E E = 0. \tag{2.14}$$

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{M}$ and M, respectively. From Gauss-Codazzi equations [2], we have, for any $X,\ Y,\ Z \in \Gamma(TM_{|\mathcal{U}})$,

$$\begin{aligned} \overline{R}(X, Y)Z = R(X, Y)Z &+ B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ \overline{g}(\overline{R}(X, Y)Z, E)N, \end{aligned} \tag{2.15}$$

$$\overline{g}(\overline{R}(X, Y)Z, N) = \overline{g}(R(X, Y)Z, N), \tag{2.16}$$

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) &- (\nabla_Y C)(X, PZ) \\ &+ \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ), \end{aligned} \tag{2.17}$$

$$\overline{g}(\overline{R}(X, Y)E, N) = C(Y, A^*_E X) - C(X, A^*_E Y) - 2d\tau(X, Y). \tag{2.18}$$

Now, consider $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$ to be an indefinite Sasakian manifold and let (M,g) be a null hypersurface of $(\overline{M},\overline{g})$, *tangent to the structure vector field* ξ *($\xi\in TM$)*. If E is a local section of $TM^{\perp}$, it is easy to check that $\overline{\phi}E\neq 0$ and $\overline{g}(\overline{\phi}E,E)=0$, then $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp})\cap TM^{\perp}=\{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $tr(TM)$, we have $\overline{\phi}N\neq 0$. Since $\overline{g}(\overline{\phi}N,E)=-\overline{g}(N,\overline{\phi}E)=0$, we deduce that $\overline{\phi}E\in\Gamma(S(TM))$ and $\overline{\phi}N$ is also tangent to M. At the same time, $\overline{g}(\overline{\phi}N,N)=0$, i.e., $\overline{\phi}N$ has no component with respect to E. Thus $\overline{\phi}N\in\Gamma(S(TM))$, that is, $\overline{\phi}(tr(TM))$ is also a vector subbundle of $S(TM)$ of rank 1.

From (2.1), we have $\overline{g}(\overline{\phi}N,\overline{\phi}E)=1$. Therefore, $\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(tr(TM))$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2. If $\xi\in TM$, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E,\xi)=\overline{g}(\overline{\phi}N,\xi)=0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM)=\left\{\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(tr(TM))\right\}\perp D_0\perp<\xi>, \tag{2.19}$$

where $\langle\xi\rangle$ is the distribution spanned by ξ. The distribution D_0 is invariant under $\overline{\phi}$, i.e. $\overline{\phi}(D_0)=D_0$. Moreover, from (2.5) and (2.19) we obtain the decompositions

$$TM=\left\{\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(tr(TM))\right\}\perp D_0\perp<\xi>\perp TM^{\perp}, \tag{2.20}$$

$$T\overline{M}=\left\{\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(tr(TM))\right\}\perp D_0\perp<\xi>\perp(TM^{\perp}\oplus tr(TM)). \tag{2.21}$$

Now, we consider the distributions on M,

$$D:=TM^{\perp}\perp\overline{\phi}(TM^{\perp})\perp D_0 \quad\text{and}\quad D':=\overline{\phi}(tr(TM)). \tag{2.22}$$

Then D is invariant under $\overline{\phi}$ and

$$TM=(D\oplus D')\perp<\xi>. \tag{2.23}$$

Let us consider the local lightlike vector fields $U:=-\overline{\phi}N$, $V:=-\overline{\phi}E$. Then, from (2.23), any $X\in\Gamma(TM)$ is written as

$$X=RX+QX+\eta(X)\xi,\ \ QX=u(X)U, \tag{2.24}$$

where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot)=g(\cdot,V)$. Applying $\overline{\phi}$ to (2.24), using (2.1) and noting that $\overline{\phi}^2N=-N$, we obtain

$$\overline{\phi}X=\phi X+u(X)N,\ \ \forall X\in\Gamma(TM), \tag{2.25}$$

where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi X:=\overline{\phi}RX$, for any $X\in\Gamma(TM)$. Again, applying $\overline{\phi}$ to (2.25) and using (2.1), we also have

$$\phi^2X=-X+\eta(X)\xi+u(X)U,\ \ \forall X\in\Gamma(TM). \tag{2.26}$$

We have the following useful identities, for any $X\in\Gamma(TM)$,

$$\nabla_X\xi=-\phi X, \tag{2.27}$$

$$B(X,\xi)=-u(X), \tag{2.28}$$

$$C(X,\xi)=-v(X), \tag{2.29}$$

where v is a 1-form locally defined on M by $v(\cdot)=g(\cdot,U)$.

Example 2.2. Let M be a hypersurface of $(\mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$ in Example 2.1 defined as $M = \{(x_1, ..., x_7) \in \mathbb{R}^7 : x_5 = x_4\}$. Thus, the tangent space TM is spanned by $\{U_i\}_{1\leq i\leq 6}$, where $U_1 = \frac{\partial}{\partial x_1}$, $U_2 = \frac{\partial}{\partial x_2}$, $U_3 = \frac{\partial}{\partial x_3}$, $U_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$, $U_5 = \frac{\partial}{\partial x_6}$, $U_6 = \xi$ and the 1-dimensional distribution $TM^{\perp}$ of rank 1 is spanned by E, where $E = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$. It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\mathbb{R}^7$. Also, the transversal bundle $tr(TM)$ is spanned by $N = \frac{1}{2}(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5})$. On the other hand, by using the almost contact structure of $\mathbb{R}^7$ and also by taking into account of the decomposition (2.19), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = U_2$, $\overline{\phi}F = U_1 + x_4\xi$ and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(tr(TM))$ are spanned, respectively, by ξ, $\overline{\phi}E = U_3 - U_5 + x_6\xi$ and $\overline{\phi}N = \frac{1}{2}(U_3 + U_5 + x_6\xi)$. Hence M is a lightlike hypersurface of $\mathbb{R}^7$.

3 Screen conformal invariant lightlike hypersurfaces

Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$.

We say that M is invariant in $\overline{M}$ [18, p. 312] if M *is tangent to the structure vector field* ξ and

$$\overline{\phi}X \in \Gamma(TM), \ \ \forall X \in \Gamma(TM), \tag{3.1}$$

that is, using (2.26),

$$\overline{\phi}X = \phi X, \ \ \forall X \in \Gamma(TM). \tag{3.2}$$

It is easy to see that any invariant submanifold M with induced structure tensors, which will be denoted (ϕ, ξ, η, g), is also a Sasakian manifold.

From (2.8), (2.27) and (3.2), one obtains

$$h(X, \xi) = 0, \tag{3.3}$$

$$h(X, \overline{\phi}Y) = h(\overline{\phi}X, Y) = \overline{\phi}h(X, Y), \ \ \forall X, Y \in \Gamma(TM). \tag{3.4}$$

Therefore, an invariant lightlike hypersurface is not totally geodesic, in general.

A submanifod M is said to be parallel [10] if its second fundamental form h satisfies,

$$(\nabla_X h)(Y, Z) = 0, \ \ \forall X, \ Y, \ Z \in \Gamma(TM). \tag{3.5}$$

If M is an invariant lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$. Then,

$$(\nabla_X h)(Y, \xi) = h(Y, \overline{\phi}X), \ \ \forall X, \ Y \in \Gamma(TM). \tag{3.6}$$

This relation leads, using (3.4), to $h(X, Y) = -h(\overline{\phi}X, \overline{\phi}Y) = -(\nabla_X h)(\overline{\phi}Y, \xi)$. This means that, if the second fundamental form $h = B \otimes E$ of an invariant lightlike hypersurface M is parallel, then M is totally geodesic.

An invariant lightlike hypersurface $(M, g, S(TM))$ of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ is screen locally conformal if the shape operators A_N and A^*_E are related by [6]

$$A_N = \varphi A^*_E, \tag{3.7}$$

or equivalently,

$$C(X, PY) = \varphi B(X, PY), \quad \forall X, Y \in \Gamma(TM), \tag{3.8}$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ in M. In case $\mathcal{U} = M$ the screen conformality is said to be global. Such a submanifold has some important and desirable properties, for instance, the integrability of its screen distribution [6].

Note that, for an invariant lightlike hypersurface (M, g) of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$, it is easy to check that the differential 1-forms u and v are vanishing identically on M. As an example, we have:

Example 3.1. Let M be the hypersurface of $\mathbb{R}^7$, of Example 2.1, given by $M = \{(x_1, ..., x_7) \in \mathbb{R}^7 : x_5 = x_4\}$, where $(x_1, ..., x_7)$ is a local coordinate system for $\mathbb{R}^7$. As explained in Example 2.1, M is a lightlike hypersurface of $\mathbb{R}^7$ having a local quasi-orthogonal field of frames $\{U_1, U_2, U_3, U_4 = E, U_5, U_6 = \xi, N\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^7$. Then, by straightforward calculations, the non-vanishing covariant derivative components of N and E along M are given by, $\nabla_{U_1} E = 2\overline{\nabla}_{U_1} N = -\frac{1}{2} x_4 U_1 - \frac{1}{2}(x_4^2 + 1)\xi$, $\nabla_{U_3} E = 2\overline{\nabla}_{U_3} N = -\frac{1}{2} x_6 U_1 - \frac{1}{2} x_4 x_6 \xi$ and $\nabla_\xi E = 2\overline{\nabla}_\xi N = \frac{1}{2} U_1 + \frac{1}{2} x_4 \xi$. Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae, the non-vanishing components of the shape operators A_N and A_E^* are given by, $A_E^* U_1 = 2A_N U_1 = \frac{1}{2} x_4 U_1 + \frac{1}{2}(x_4^2 + 1)\xi$, $A_E^* U_3 = 2A_N U_3 = \frac{1}{2} x_6 U_1 + \frac{1}{2} x_4 x_6 \xi$ and $A_E^* \xi = 2A_N \xi = -\frac{1}{2} U_1 - \frac{1}{2} x_4 \xi$. From these relations, we deduce that $A_N X = \frac{1}{2} A_E^* X$, for any $X \in \Gamma(TM)$ and $trA_N = 0$, i.e. the shape operator A_N is trace-free. Therefore, the hypersurface M of $\mathbb{R}^7$ is screen conformal and minimal. So, its screen distribution is integrable. The non-vanishing components of the local second fundamental form B are given by $B(U_1, U_1) = -x_4$, $B(U_1, U_3) = -\frac{1}{2} x_6$ and $B(U_1, U_6) = \frac{1}{2}$. It is easy to check that $B(X, \xi) = 0$, for any $X \in \Gamma(TM)$. Hence, M is a screen conformal invariant lightlike hypersurface.

Let M be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$. Using (2.4), (2.15) and (2.25), and comparing the tangential and transversal parts of both sides, we have, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \overline{g}(\overline{\phi}Y,Z)\phi X - \overline{g}(\overline{\phi}X,Z)\phi Y - 2\overline{g}(\overline{\phi}X,Y)\phi Z\} \\ & + \varphi\{B(Y,Z)A_E^* X - B(X,Z)A_E^* Y\}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = {} & \tau(Y)B(X,Z) - \tau(X)B(Y,Z) + \frac{c-1}{4}\{\overline{g}(\overline{\phi}Y,Z)u(X) \\ & - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\}. \end{aligned} \tag{3.10}$$

Also using (2.4) and (2.17), we obtain, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) = {} & \varphi\tau(X)B(Y,PZ) - \varphi\tau(Y)B(X,PZ) + \frac{c+3}{4}\{\overline{g}(Y,PZ)\theta(X) \\ & - \overline{g}(X,PZ)\theta(Y)\} + \frac{c-1}{4}\{\eta(X)\eta(PZ)\theta(Y) - \eta(Y)\eta(PZ)\theta(X) + \overline{g}(\overline{\phi}Y,PZ)v(X) \\ & - \overline{g}(\overline{\phi}X,PZ)v(Y) - 2\overline{g}(\overline{\phi}X,Y)v(Z)\}. \end{aligned} \tag{3.11}$$

Let us consider the following distribution

$$\widehat{D} = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))\right\} \perp D_0 \tag{3.12}$$

so that the tangent space of M is written

$$TM = \widehat{D} \perp \langle\xi\rangle \perp TM^{\perp}. \tag{3.13}$$

Let $\widehat{P}$ be the morphism of $S(TM)$ on $\widehat{D}$ with respect to the orthogonal decomposition of $S(TM)$ such that

$$\widehat{P}X = PX - \eta(X)\xi, \ \ \forall X \in \Gamma(TM), \tag{3.14}$$

and it is easy to see that the morphism $\widehat{P}$ is a projection.

Using the projection morphism $\widehat{P}$ and the relation (3.3), we have the following identities, for any X, Y, $Z \in \Gamma(TM)$,

$$B(PX, PY) = B(\widehat{P}X, \widehat{P}Y), \tag{3.15}$$

$$\nabla_X PY = \nabla_X \widehat{P}Y + X(\eta(Y))\xi - \eta(Y)\phi X, \tag{3.16}$$

$$(\nabla_X B)(Y, PZ) = (\nabla_X B)(Y, \widehat{P}Z) + \eta(Z)\{B(\phi X, Y) + B(X, \phi Y)\}. \tag{3.17}$$

If M is a screen conformal invariant lightlike hypersurface, then, using (3.8), we have, for any X, Y, $Z \in \Gamma(TM)$,

$$(\nabla_X C)(Y, \widehat{P}Z) = X(\varphi)B(Y, \widehat{P}Z) + \varphi(\nabla_X B)(Y, \widehat{P}Z), \tag{3.18}$$

and using (3.3) and (3.18), the left hand side of (3.11) is given by

$$\begin{aligned}(\nabla_X C)(Y, \widehat{P}Z) - (\nabla_Y C)(X, \widehat{P}Z) &= X(\varphi)B(Y, \widehat{P}Z) - Y(\varphi)B(X, \widehat{P}Z)\\ &\quad + \varphi\{\tau(Y)B(X, \widehat{P}Z) - \tau(X)B(Y, \widehat{P}Z)\},\end{aligned} \tag{3.19}$$

On the other hand, using (3.14) and the fact that $v(X) = -\varphi B(X, \xi) = 0$, the relation (3.11) becomes

$$\begin{aligned}(\nabla_X C)(Y, \widehat{P}Z) - (\nabla_Y C)(X, \widehat{P}Z) &= \varphi\tau(X)B(Y, \widehat{P}Z) - \varphi\tau(Y)B(X, \widehat{P}Z)\\ &\quad + \frac{c+3}{4}\{\overline{g}(Y, \widehat{P}Z)\theta(X) - \overline{g}(X, \widehat{P}Z)\theta(Y)\}.\end{aligned} \tag{3.20}$$

Putting the pieces (3.19) and (3.20) together and using (3.8), we have

$$\begin{aligned}&\{X(\varphi) - 2\varphi\tau(X)\}B(Y, \widehat{P}Z) - \{Y(\varphi) - 2\varphi\tau(Y)\}B(X, \widehat{P}Z)\\ &\quad = \frac{c+3}{4}\{\overline{g}(Y, \widehat{P}Z)\theta(X) - \overline{g}(X, \widehat{P}Z)\theta(Y)\}.\end{aligned} \tag{3.21}$$

For $Y = E$, one obtains

$$\{E(\varphi) - 2\varphi\tau(E)\}B(X, \widehat{P}Z) = \frac{c+3}{4}\overline{g}(X, \widehat{P}Z). \tag{3.22}$$

Therefore,

Proposition 3.2. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$. Then,*

$$\{E(\varphi) - 2\varphi\tau(E)\}B(X, \widehat{P}Y) = \frac{c+3}{4} g(X, \widehat{P}Y), \tag{3.23}$$

for any X, $Y \in \Gamma(TM)$.

Let M' be a leaf of an integrable screen distribution $S(TM)$. Then, using the second equations of (2.8) and (2.10), we obtain, for any X, $Y \in \Gamma(TM')$,

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla^*_X Y + C(X, Y)E + B(X, Y)N \\ &= \nabla'_X Y + h'(X, Y),\end{aligned} \tag{3.24}$$

where ∇' and h' are the Levi-Civita connection and second fundamental form of M' in $\overline{M}$. Thus, for any X, $Y \in \Gamma(TM')$,

$$h'(X, Y) = C(X, Y)E + B(X, Y)N. \tag{3.25}$$

In the sequel, we need the following lemma.

Lemma 3.3. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with a leaf M' of $S(TM)$. Then, for any $X \in \Gamma(TM')$,*

$$\nabla'_X \xi = -\phi X, \tag{3.26}$$

$$\nabla'_X U = \varphi\overline{\phi}(A^*_E X) + \tau(X)U, \tag{3.27}$$

$$\nabla'_X V = \overline{\phi}(A^*_E X) - \tau(X)V. \tag{3.28}$$

Proof. The proof of this lemma follows the one of Lemma 4.2 in [12] using the fact that the differential 1-forms u and v vanish. □

Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$. Then, the local fundamental form C is symmetric on $S(TM)$. Thus, by Theorem 2.3.5 in [2, page 63], $S(TM)$ is integrable and M is locally a product manifold $L \times M'$, where L is an open subset of a lightlike geodesic ray in $\overline{M}$ and M' is a leaf of $S(TM)$.

Applying $\overline{\phi}$ to (3.27) and (3.28), and using (3.3), we have, for any $X \in \Gamma(TM')$,

$$\overline{\phi}\nabla'_X U = -\varphi A^*_E X + \tau(X)N, \tag{3.29}$$

$$\text{and } \overline{\phi}\nabla'_X V = -A^*_E X - \tau(X)E. \tag{3.30}$$

Putting the pieces (3.29) and (3.30) together into (3.25), we have,

$$h'(X, Y) = C(X, Y)E + B(X, Y)N = g(\nabla'_X U, \overline{\phi}Y)E + g(\nabla'_X V, \overline{\phi}Y)N. \tag{3.31}$$

From (3.31), it is easy to see that M' is totally geodesic if and only if the lightlike vector fields U and V are parallel with respect to the Levi-Civita connection ∇' on M'.

Using the relations (3.27) or (3.28), the parallelism of the lightlike vector fields U or V, with respect to ∇' on M', implies that the shape operator A^*_E and the differential 1-form τ vanish identically on M', and vice versa.

In the case of invariant lightlike hypersufaces, $B(X,V) = -B(A^*_E X,\xi) = 0$ and $B(X,U) = \varphi B(X,V) = 0$, and the shape operator A^*_E takes the form

$$A^*_E X = \sum_{i=1}^{2n-4} B(X,F_i)F_i, \quad \forall X \in \Gamma(S(TM)). \tag{3.32}$$

This means that $A^*_E X \in \Gamma(D_0)$ and since the distribution D_0 is non-degenerate, we have $g(A^*_E X, A^*_E X) = 0$ if and only if $A^*_E X = 0$, that is,

$$B(X,F_i) = 0, \quad \forall\, i = 1,2,...,2n-4. \tag{3.33}$$

Let $t(x)$ and $t'(x)$ be the type numbers of M and M', respectively, for any point $x \in M$, that is, the ranks of shape operators A_N and A^*_E, respectively, at x.

The rank of a matrix is the maximum number of independent rows (or, the maximum number of independent columns) and if the shape operator A^*_E, locally, takes the form

$$A^*_E = \begin{bmatrix} \vdots & \vdots & \vdots & & \vdots \\ \vec{r}_1 & \vec{r}_2 & \vec{r}_3 & \cdots & \vec{r}_{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix}, \tag{3.34}$$

one has, $\operatorname{rank}(A^*_E) = \dim C(A^*_E)$, where $C(A^*_E) = \operatorname{Span}\{\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_{2n-1}\}$ which is the number of vectors in the basis for $C(A^*_E)$. Since φ is a non-vanishing smooth function on M, $\dim C(\varphi A^*_E) = \dim C(A^*_E)$, i.e., $\operatorname{rank}(A_N) = \operatorname{rank}(A^*_E)$, since $A_N = \varphi A^*_E$. Thus, if the rank of the shape operator A^*_E is vanishing identically on M, then A^*_E is a zero matrix which implies that $A_N = \varphi A^*_E$ is also a zero matrix. Consequently, by Theorem 2.2 and Proposition 2.7 in [2, pages 88, 89], the submanifolds M and M' are totally geodesic in $\overline{M}$. This means that if the rank of A^*_E is non-zero, then M and M' cannot be totally geodesic. Therefore, we have:

Proposition 3.4. *Let $(M,g,S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\overline{g})$ with a leaf M' of $S(TM)$. Then, $t(x) \equiv t'(x)$, for any $x \in M$. Moreover, $t(x) = 0$ if and only if M and M' are totally geodesic in $\overline{M}$.*

Next, we deal with the geometric configuration of the screen conformal invariant lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\overline{g})$.

By combining (2.8) and (2.10), we obtain, for any $X \in \Gamma(TM)$, $Y \in \Gamma(\widehat{M}')$,

$$\begin{aligned} \overline{\nabla}_X Y &= \widehat{\nabla}^*_X Y + g(\phi X, Y)\xi + B(X,Y)\{\varphi E + N\} \\ &= \widehat{\nabla}'_X Y + \widehat{h}'(X,Y), \end{aligned} \tag{3.35}$$

where $\widehat{\nabla}'$ and $\widehat{h}' = g(\phi\cdot,\cdot)\otimes\xi + B\otimes\{\varphi E + N\}$ are the Levi-Civita connection and the second fundamental form of $\widehat{M}'$ in $\overline{M}$, respectively. We have,

Lemma 3.5. *Let $(M,g,S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\overline{g})$. Then, the distribution $\widehat{D}$ in (3.12) is always integrable.*

Proof. The proof follows from $g(\phi X, Y) = 0$, for any X, $Y \in \Gamma(T\widehat{M}')$. □

Theorem 3.6. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$. Then, $c = -3$. Moreover, if the type number of M, $t(x) > 0$, for any $x \in M$, then M is not totally geodesic and the non-zero function φ satisfies the following partial differential equation,*

$$E(\varphi) - 2\varphi\tau(E) = 0.$$

Proof. Assume that $c \neq -3$. Then, by Proposition 3.2, we have, $E(\varphi) - 2\varphi\tau(E) \neq 0$ and $B \neq 0$, that is, M is not totally geodesic. From (3.8) and (3.23), we have

$$B(X, \widehat{P}Y) = \rho g(X, \widehat{P}Y), \;\; C(X, \widehat{P}Y) = \varphi\rho g(X, \widehat{P}Y), \tag{3.36}$$

for any X, $Y \in \Gamma(TM)$, where $\rho = \frac{c+3}{4}(E(\varphi) - 2\varphi\tau(E))^{-1} \neq 0$. The latter leads to

$$\rho\{E(\varphi) - 2\varphi\tau(E)\} = \frac{c+3}{4}. \tag{3.37}$$

The relations in (3.36) are equivalent to $A^*_E X = \rho\widehat{P}X$ and $A_N X = \varphi\rho\widehat{P}X$. Since $\varphi\rho \neq 0$, M and $\widehat{D}$ are not totally geodesic but proper totally umbilical. By Theorem 5.2 in [2], we have,

$$E(\rho) + \rho\tau(E) - \rho^2 = 0. \tag{3.38}$$

From (3.37) and (3.38), we have

$$E(\varphi\rho^2) = \rho(\frac{c+3}{4} + 2\varphi\rho^2). \tag{3.39}$$

By virtue of Lemma 3.5, let us consider a leaf $\widehat{M}'$ of $\widehat{D}$. Since $\{E(\varphi) - 2\varphi\tau(E)\} \neq 0$ and $B \neq 0$ along $\widehat{M}'$, then, by (3.36) $\widehat{M}'$ is not totally geodesic and its second fundamental form $\widehat{h}'$ is given by

$$\widehat{h}'(X, Y) = \widehat{H}' g(X, Y), \;\; \forall X, \; Y \in \Gamma(T\widehat{M}'),$$

where $\widehat{H}' = \rho(\varphi E + N)$ is the mean curvature vector of the leaf $\widehat{M}'$. The relation (3.35) becomes $\overline{\nabla}_X Y = \widehat{\nabla}'_X Y + \widehat{H}' g(X, Y)$, and the curvature tensors $\overline{R}$ and $\widehat{R}'$ of $\overline{M}$ and $\widehat{M}'$, respectively, are related as $\overline{R}(X, Y)Z = \widehat{R}'(X, Y)Z + g(Y, Z)\overline{\nabla}_X\widehat{H}' - g(X, Z)\overline{\nabla}_Y\widehat{H}'$, which leads to, for any X, Y, Z, $W \in \Gamma(T\widehat{M}')$,

$$\overline{R}(X, Y)Z = \widehat{R}'(X, Y)Z - 2\varphi\rho^2\{g(Y, Z)X - g(X, Z)Y\}, \tag{3.40}$$

$$X(\rho) + \rho\tau(X) = 0 \;\text{ and }\; X(\varphi\rho) - \varphi\rho\tau(X) = 0. \tag{3.41}$$

The latter leads to $X(\varphi) - 2\varphi\tau(X) = 0$. On the other hand, we have

$$\overline{R}(X, Y)Z = R(X, Y)Z + \varphi\rho^2\{g(X, Z)Y - g(Y, Z)X\}. \tag{3.42}$$

Putting (3.40) and (3.42) together, we obtain

$$R(X, Y)Z = \widehat{R}'(X, Y)Z - \varphi\rho^2\{g(Y, Z)X - g(X, Z)Y\}. \tag{3.43}$$

Also, using (3.9), the curvature R is expressed along the leaf $\widehat{M}'$ as

$$R(X,Y)Z = (\frac{c+3}{4} + \varphi\rho^2)\{g(Y,Z)X - g(X,Z)Y\}.$$

Using this and (3.43), the curvature tensor $\widehat{R}'$ of $\widehat{M}'$ is given by

$$\widehat{R}'(X,Y)Z = (\frac{c+3}{4} + 2\varphi\rho^2)\{g(Y,Z)X - g(X,Z)Y\}. \tag{3.44}$$

Thus, $\widehat{M}'$ is a semi-Riemannian manifold of constant curvature $(\frac{c+3}{4} + 2\varphi\rho^2)$. Let $\widehat{Ric}'$ be the induced symmetric Ricci tensor of $\widehat{M}'$. From (3.44), we have,

$$\widehat{Ric}'(X,Y) = (2n-3)(\frac{c+3}{4} + 2\varphi\rho^2)g(X,Y),\ \forall X,Y \in \Gamma(\widehat{D}). \tag{3.45}$$

Thus, $\widehat{M}'$ is an Einstein manifold. Since $\dim \widehat{M}' > 2$, $(\frac{c+3}{4} + 2\varphi\rho^2)$ is a constant and $\widehat{M}'$ has a constant curvature $\frac{c+3}{4} + 2\varphi\rho^2$. Thus, $0 = E(\varphi\rho^2) = \rho(\frac{c+3}{4} + 2\varphi\rho^2)$. Since $(\frac{c+3}{4} + 2\varphi\rho^2)$ is a constant and $\rho \neq 0$, we have, $\frac{c+3}{4} + 2\varphi\rho^2 = 0$. $\widehat{M}'$ is a semi-Euclidean space. Using the second relation of (3.41), the covariant derivative of the second fundamental form $h^* = \varphi\rho(g \otimes E)$ of $S(TM)$ with respect to the induced connection ∇', along $\widehat{M}'$, gives,

$$(\nabla'_X h^*)(Y,Z) = \{X(\varphi\rho) - \varphi\rho\tau(X)\}g(Y,Z)E = 0, \quad \forall X,\, Y,\, Z \in \Gamma(T\widehat{M}'),$$

which means that $S(TM)$ is parallel along $\widehat{M}'$. By Proposition 2.7 [2, page 89], $h^* = 0$ along $\widehat{M}'$. Since $h^* = \varphi\rho(g \otimes E)$, we have $\varphi\rho = 0$. This implies that $c = -3$, which is a contradiction to $c \neq -3$. This contradicts the assumption. The last assertion follows from the relation (3.23) and this completes the proof. □

From the Example 3.1, the non-vanishing components of the shape operators A^*_E and A_N are $A^*_E U_1 = 2A_N U_1 = \frac{1}{2}x_4 U_1 + \frac{1}{2}(x_4^2 + 1)\xi$, $A^*_E U_3 = 2A_N U_3 = \frac{1}{2}x_6 U_1 + \frac{1}{2}x_4 x_6 \xi$ and $A^*_E \xi = 2A_N \xi = -\frac{1}{2}U_1 - \frac{1}{2}x_4\xi$. Locally, the matrix form A^*_E, in the local orthogonal field of frames $\{U_1,\, U_2,\, U_3,\, U_4 = E,\, U_5,\, U_6 = \xi\}$ on M is

$$A^*_E = \begin{bmatrix} \frac{1}{2}x_4 & 0 & \frac{1}{2}x_6 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(x_4^2+1) & 0 & \frac{1}{2}x_4x_6 & 0 & 0 & -\frac{1}{2}x_4 \end{bmatrix}.$$

It is easy to see that the rank of A^*_E is 2. Therefore, for any $x \in \mathbb{R}^7$, $t(x) = 2 > 0$. One of the non-vanishing components of the local second fundamental form B is $B(U_1, \xi) = \frac{1}{2} \neq 0$ and this means that M cannot be totally geodesic. Since $\varphi = \frac{1}{2}$ and $\tau(E) = 0$ (Example 3.1), we have $E(\varphi) - 2\varphi\tau(E) = 0$.

Therefore, there exist screen conformal invariant lightlike hypersurfaces in indefinite Sasakian space forms $(\overline{M}(c), \overline{g})$ with $\xi \in TM$ which satisfy the results in Theorem 3.6.

Let us now deal with the leaf M' of the conformal screen distribution $S(TM)$ of the lightlike hypersurface M of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$ with $\xi \in TM$. Using (3.7) and (3.25), the second fundamental form h' of M' becomes, for any X, $Y \in \Gamma(TM')$,

$$h'(X,Y) = B(X,Y)K, \tag{3.46}$$

where $K = \varphi E + N \in \Gamma(TM^{\perp} \otimes tr(TM))$ and the relation (3.24) reduces to

$$\overline{\nabla}_X Y = \nabla'_X Y + B(X,Y)K. \tag{3.47}$$

If the type number of M, $t(x) > 0$, by Theorem 3.6, the covariant derivative of W with respect to $\overline{\nabla}$ is given by

$$\begin{aligned} \overline{\nabla}_X K &= X(\varphi)E + \varphi\overline{\nabla}_X E + \overline{\nabla}_X N \\ &= -\{\varphi A^*_E X + A_N X\} + \{X(\varphi) - \varphi\tau(X)\}E + \tau(X)N \\ &= -2\varphi A^*_E X + \tau(X)K. \end{aligned} \tag{3.48}$$

We have developed the first set of basic formulas for submanifolds, namely,

$$\overline{\nabla}_X Y = \nabla'_X Y + B(X,Y)K, \tag{3.49}$$

$$\text{and } \overline{\nabla}_X K = -\mathcal{A}_K X + \nabla'^{\perp}_X K, \ \forall X, Y \in \Gamma(TM'), \tag{3.50}$$

where $\mathcal{A}_K X = 2\varphi A^*_E X$ and $\nabla'^{\perp}_X K = \tau(X)K$. Since $g(K,K) = 2\varphi \neq 0$, it is easy to check, for any X, $Y \in \Gamma(TM')$,

$$g(h'(X,Y), W) = g(\mathcal{A}_W X, Y). \tag{3.51}$$

Theorem 3.7. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$ with a leaf M' of $S(TM)$. Then,*

(i) M is totally geodesic,

(ii) M is totally contact umbilical,

(iii) M is minimal,

if and only if M' is so immersed as a submanifold of $\overline{M}$.

Using (3.49) and (3.50), one obtains

$$\overline{R}(X,Y)Z = R'(X,Y)Z + (\nabla'_X h')(Y,Z) - (\nabla'_Y h')(X,Z), \tag{3.52}$$

from which, using (2.4), we have

$$g((\nabla'_X h')(Y,Z) - (\nabla'_Y h')(X,Z), K) = 0, \tag{3.53}$$

which lead to

$$(\nabla'_X B)(Y,Z) - (\nabla'_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z), \tag{3.54}$$

since $(\nabla'_X h')(Y,Z) = \{(\nabla'_X B)(Y,Z) + \tau(X)B(Y,Z)\}K$. Taking $Z = \xi$ in this equation, one obtains $(\nabla'_X B)(Y,\xi) - (\nabla'_Y B)(X,\xi) = 0$. Since $(\nabla'_X B)(Y,\xi) = X(B(Y,\xi)) - B(\nabla'_X \xi, Y) - B(\xi, \nabla'_X Y) = B(\phi X, Y)$. We have,

$$B(\phi X, Y) = B(X, \phi Y).$$

Proposition 3.8. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$ with a leaf M' of $S(TM)$ immersed in $\overline{M}$ as non-degenerate submanifold. Then, $\phi X \in \Gamma(TM')$, for any $X \in \Gamma(TM')$, and the second fundamental form h' of M' satisfies*

$$h'(\phi X, Y) = h'(X, \phi Y), \quad \forall X, Y \in \Gamma(TM'). \tag{3.55}$$

Moreover, M' is parallel if and only if it is totally geodesic.

Proof. For any $X \in \Gamma(TM')$, $\phi X = P\phi X + \theta(\phi X)E = P\phi X$. If the leaf M' is parallel, then $(\nabla'_X h')(Y,Z) = \{(\nabla'_X B)(Y,Z) + \tau(X)B(Y,Z)\}K = 0$, that is, $(\nabla'_X B)(Y,Z) = -\tau(X)B(Y,Z)$. Taking $Z = \xi$ in these relation leads to $(\nabla'_X B)(Y,\xi) = 0$ which gives $B(\phi X, Y) = 0$, i.e $h'(X,Y) = 0$. Thus M' is totally geodesic. The converse is obvious. □

Putting the relation (3.54) into (3.52) leads to, for any X, Y, $Z \in \Gamma(TM')$,

$$\begin{aligned} R'(X,Y)Z = \overline{R}(X,Y)Z = &\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \{\eta(Y)X - \eta(X)Y\}\eta(Z). \end{aligned} \tag{3.56}$$

A submanifold M' is said to be semi-parallel [14] if its second fundamental form h' satisfies, for any X, Y, Z, $W \in \Gamma(TM')$,

$$(R'(X,Y) \cdot h')(Z,W) = -h'(R'(X,Y)Z, W) - h'(Z, R'(X,Y)W) = 0. \tag{3.57}$$

By direct calculation, the right hand-side of (3.57) reduces to

$$\begin{aligned} (R'(X,Y) \cdot h')(Z,W) = &-\{\eta(Y)h'(X,W) - \eta(X)h'(Y,W)\}\eta(Z) \\ &- \{\eta(Y)h'(X,Z) - \eta(X)h'(Y,Z)\}\eta(W). \end{aligned} \tag{3.58}$$

If M' is semi-parallel, then, putting $W = \xi$ into (3.58), one obtains,

$$h'(X,Z)Y = h'(Y,Z)X, \tag{3.59}$$

which implies that $h'(X,Z) = 0$, i.e., M' is totally geodesic. The converse is a straightforward calculation. We have:

Proposition 3.9. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$ with a leaf M' of $S(TM)$ immersed in $\overline{M}$ non-degenerate a submanifold. Then, M' is semi-parallel if and only if it is totally geodesic.*

From Proposition 3.8 and Proposition 3.9, we obtain

Theorem 3.10. *Let $(M, g, S(TM))$ be a screen conformal invariant lightlike hypersurface of an indefinite Sasakian space form $(\overline{M}(c), \overline{g})$. Let M' be a leaf of $S(TM)$ immersed in $\overline{M}$ as non-degenerate submanifold. Then, M' is parallel if and only it is semi-parallel.*

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