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Symmetry Analysis, Integrability and Completeness of Geodesics of a Double of the Affine Lie Group of the Real Line

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Abstract

We investigate Lie point symmetries of a system of four nonlinear second-order ordinary differential equations (ODEs), appropriated to the geodesics of a Drinfel'd double Lie group of the affine Lie group of $\mathbb{R}$. The first integrals associated with Lie point symmetries are obtained by utilizing the constructive method due to Wafo Soh and Mahomed ([16]). This method deals with integrability when the symmetry vector fields and the operator associated to the system are unconnected. In certain cases we obtain the explicit expressions of the geodesics. We also show that the geodesics are not complete.

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1 Introduction

Let $G = \text{Aff}(\mathbb{R})$ denote the Lie group of rigid motions of the real line $\mathbb{R}$. It possesses numerous of interesting structures such as symplectic structure ([1], [2]), complex structure, Kählerian structure ([10]), affine structure ([4]), etc. From the works of Diatta and Medina ([7]), we know that G admits a left invariant symplectic structure which corresponds to a

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solution r of the classical Yang-Baxter equation. Using the solution r, one can construct a Poisson tensor π_r on G and then a Drinfel'd double Lie group $\mathcal{D}(G,r) = \mathcal{D}(G,\pi_r)$ of the Poisson Lie group (G,π_r). The double Lie group $\mathcal{D}(G,r)$ encompasses the most interesting informations about the Poisson Lie group (G,π_r). As a Drinfel'd double Lie group, $\mathcal{D}(G,r)$ is rich in structures. For example, it has an affine structure; that is a torsion free connection without curvature which is constructed in [7]. The study of the geodesics of this connection (see Eq. (2.5) below) gives rise to the following system of four second-order ordinary differential equations (ODEs):

$$\begin{cases} u_1'' = (u_1')^2 + 2u_1'u_4', \\ u_2'' = u_1'u_2' - 2u_1'u_3', \\ u_3'' = u_3'u_4' - 2u_2'u_4', \\ u_4'' = (u_4')^2 + 2u_1'u_4', \end{cases} \tag{1.1}$$

where the prime denotes differentiation with respect to the independent variable t.

In general, explicit solutions of a system of nonlinear ODEs are difficult to obtain. Arguably, the most powerful tool to study such systems, and in general differential equations (DEs) is due to Sophus Lie. Indeed, Lie noticed that invertible transformations leaving a given DE invariant and forming a local one-parameter group can be described equivalently through the vector tangent to the orbits of the group. He called this vector the symbol of the local one parameter group and the symmetry of the underlying equation. He showed that the set of symmetries of a DE forms a Lie algebra (an infinitesimal group in his terminology) and that the integrability of the equation depends upon the properties of this Lie algebra. Namely, he showed that a scalar n^{th}-order ODE admitting an n-dimensional solvable Lie algebra of symmetries is integrable by quadratures. Concretely, one arrives at the general solution of the ODE by successive reduction of order (which may be view as a quadrature). This is also equivalent to the construction of n first integrals of the underlying equation.

Symmetries are so useful that any known integration technique can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the DE, *i.e.* the Lie symmetry algebra. More generally, a symmetry is a mapping of one mathematical object into itself or into another mathematical object that preserves some property of the object. One of the most important applications of symmetries of DEs is to perform symmetry reduction. For partial differential equations (PDEs) this means a reduction of the number of independent variables.

However, the Lie's method has a major drawback. It is useless when applied to systems of n first-order ODEs. Such systems admit an infinite number of infinitesimal symmetries, and there is no systematic way to find these symmetries, except perhaps trivial ones such as translations and scalings when they are admitted by the underlying system. In fact, finding the symmetries of a system of first-order ODEs is equivalent to solving it.

First integrals are very important in the study of DEs and systems of DEs. The reason is the following. A DE or a system of DEs can be seen under the angle of its solutions. One can also see it under the first integrals it admits. Let $\Delta(x,u^{(n)}) = 0$ denotes a general n^{th}-order system of ODEs in p independent variables $x = (x^1,\dots,x^p)$ and q dependent variables $u = (u^1,\dots,u^q)$ with $u^{(n)}$ denoting the derivatives of u's with respect to the x's up to order n. For a system of ODEs $\Delta(x,u^{(n)}) = 0$, a first integral is a function $I(x,u^{(m)})$ $(m < n)$ which is constant on solutions. The fact that the function I is constant on solutions is equivalent to the statement that its total derivative vanishes on solutions, so $D_x I = 0$ whenever $u = f(x)$ solves $\Delta(x, u^{(n)}) = 0$ (see [14, page 242]). It is well known that for a Lagrangian system, first integrals are derived from Noether symmetries via Noether Theorem. Unfortunately, a lot of equations of mathematical and physical interest (*e.g.* the system of two second-order equations $y'' = y^2 + z^2, z'' = y$, see [12]) do not come from a variational problem. Fortunately, there are many studies which provide techniques to obtain first integrals of (systems of) DEs. In this work, we use the approach developed by Wafo Soh and Mahomed in [16] to determine the first integrals of the system of second-order ODEs (1.1).

The outline of this paper is the following. The second section is devoted to explaining the origin of the system of four nonlinear second-order ODEs (1.1). In it, we prove that the affine manifold $(\mathcal{D}(G,r), \nabla)$ is not geodesically complete (Theorem 2.2). In Section 3, we study the integrability of the system (1.1) using Lie point symmetries. Precisely, we provide sufficient conditions for the system to be integrable by quadratures (Proposition 3.5). In Section 4, we deal with the case where the system is solvable by quadratures and we compute the first integrals of the system (1.1) (Theorem 4.1). Section 5 is devoted to the study of cases where the sufficient conditions of Section 3 fail. In one part of this cases we exhibit the explicit expressions of the geodesics (Propositions 5.1, 5.2, 5.3 and 5.10). In the other part, we give some first integrals of the system (1.1) (see Theorems 5.6 and 5.9). We end this paper with discussions contained in Section 6.

2 Origin Of The System Of Second-order ODEs (1.1)

Although not central to the main purpose of this paper, this section allows to understand the context in which is obtained the system of ODEs (1.1).

2.1 Symplectic Form And Left Symmetric Algebra Structure On $\mathrm{aff}(\mathbb{R})$

A Left Symmetric Algebra (LSA) structure on a vector space V is a product $V \times V \to V$, $(x,y) \mapsto xy$ such that $(xy)z - x(yz) = (yx)z - y(xz)$, for all x, y, z in V. An LSA gives rise to a Lie bracket $[,]$ on V by $[x,y] := xy - yx$. Any connected Lie group with Lie algebra $(V, [,])$ possesses a left invariant affine structure with associated connection $\nabla_{x^+} y^+ := (xy)^+$. Here, if $z \in V$, the element z^+ stands for the left invariant vector field associated to z. Conversely, if ∇ is a left invariant affine structure on a Lie group G with unit ϵ, the formula $xy := (\nabla_{x^+} y^+)_{|\epsilon}$ gives an LSA structure on the Lie algebra $\mathcal{G}$ of G. In the case of a symplectic Lie group (G, ω), the connection $\bar{\nabla}$ defined by (see [6])

$$\omega(\bar{\nabla}_{x^+} y^+, z^+) = -\omega(y^+, [x^+, z^+]) \tag{2.1}$$

induces an LSA structure $(x,y) \to xy$ in $\mathcal{G}$, which is compatible with the Lie bracket of $\mathcal{G}$, *i.e.* $xy - yx = [x,y]$, for all x, y in $\mathcal{G}$. The formula (2.1) equips G with a left invariant affine structure.

The affine Lie group of the real line is $\mathrm{Aff}(\mathbb{R}) = \mathbb{R}^* \ltimes \mathbb{R}$, where $\mathbb{R}^*$ acts on $\mathbb{R}$ by multiplication of real numbers. The Lie algebra of $\mathrm{Aff}(\mathbb{R})$, said the affine Lie algebra, is $\mathrm{aff}(\mathbb{R}) = \mathbb{R}^2 = \mathrm{span}(\mathrm{e}_1, \mathrm{e}_2)$ with $[e_1, e_2] = e_2$. Note by e_2^* the form defined by $e_2^*(e_k) = \delta_{2k}$, for

$k=1,2$. The form ω_0, defined by $\omega_0(x,y)=e_2^*([x,y])$ for all x,y in aff($\mathbb{R}$), is nondegenerate and induces a left invariant symplectic form ω on Aff($\mathbb{R}$) (see [1]). As described above, the symplectic structure ω induces an affine structure $\bar{\nabla}$ through the formula (2.1).

From now on, G stands for the affine Lie group of $\mathbb{R}$ while $\mathcal{G}$ represents its Lie algebra.

2.2 Drinfel'd Double Of $(\mathrm{Aff}(\mathbb{R}),\omega)=(\mathrm{G},\omega)$

Let $\mathcal{G}^*$ stand for the dual space of $\mathcal{G}$ and set $r:=q^{-1}:\mathcal{G}^*\mapsto\mathcal{G}$, with $\langle q(x),y\rangle=\omega_0(x,y)$. Here, $\langle,\rangle$ is the duality pairing between $\mathcal{G}$ and $\mathcal{G}^*$ given by $\langle\alpha,x\rangle=\alpha(x)$ for any α in $\mathcal{G}^*$ and x in $\mathcal{G}$. Since ω_0 is nondegenerate, r is invertible. Then one can define on $\mathcal{G}^*$ a Lie algebra structure $[,]_r$, dual to the one on $\mathcal{G}$ with respect to r as follows ([7]): for any α and β in $\mathcal{G}^*$,

$$[\alpha,\beta]_r:=ad^*_{r(\alpha)}\beta-ad^*_{r(\beta)}\alpha. \tag{2.2}$$

Here, ad stands for the adjoint action of $\mathcal{G}$ on itself while ad^* denotes the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$. Now, we define, on $\mathcal{G}\oplus\mathcal{G}^*$, a Lie bracket $[,]_{\mathcal{D}}$: for all (x,α) and (y,β) in $\mathcal{G}\oplus\mathcal{G}^*$,

$$[(x,\alpha),(y,\beta)]_{\mathcal{D}}=([x,y]+ad^*_\alpha y-ad^*_\beta x,[\alpha,\beta]_r+ad^*_x\beta-ad^*_y\alpha). \tag{2.3}$$

The Lie algebra $\mathcal{D}(\mathcal{G},r):=(\mathcal{G}\oplus\mathcal{G}^*,[,]_{\mathcal{D}})$ is called the double Lie algebra of G associated to r. Any Lie group with Lie algebra $\mathcal{D}(\mathcal{G},r)$ is called a Drinfel'd double Lie group of G associated to r.

In [11], the second author shows that $\mathbb{R}\ltimes\mathbb{H}_3$ is a Drinfel'd double Lie group of G when it is endowed with the following law: for all $x=(t;x_2,x_3,x_4)$, $y=(s;y_2,y_3,y_4)$ in $\mathbb{R}\ltimes\mathbb{H}_3$,

$$x\cdot y=\Big(t+s\ ;\ x_2+y_2e^t\ ,\ x_3+y_3+x_2y_4e^{-t}\ ,\ x_4+y_4e^{-t}\Big). \tag{2.4}$$

2.3 Left Invariant Connection And Geodesics On $\mathcal{D}(G,r)$

On $\mathcal{D}(G,r)$, consider the left invariant affine connection ∇ defined on any double Lie group of a symplectic Lie group by Diatta and Medina in [7]: for all (x,α) and (y,β) in $\mathcal{D}(\mathcal{G},r)$,

$$\nabla_{(x,\alpha)^+}(y,\beta)^+=(\bar{\nabla}_xy+ad^*_\alpha y,ad^*_{r(\alpha)}\beta+ad^*_x\beta)^+. \tag{2.5}$$

On some basis (e_1,e_2,e_3,e_4) of $\mathcal{D}(\mathcal{G},r)$, the connection is given by (see [11])

$$\begin{array}{lllll}\nabla_{e_1}e_1=-e_1; & \nabla_{e_1}e_3=e_2; & \nabla_{e_1}e_4=-e_1-e_4; & \nabla_{e_2}e_1=-e_2; & \nabla_{e_2}e_4=e_3\\ \nabla_{e_3}e_1=e_2; & \nabla_{e_3}e_4=-e_3; & \nabla_{e_4}e_1=-e_1-e_4; & \nabla_{e_4}e_2=e_3; & \nabla_{e_4}e_4=-e_4\end{array}$$

Now let $\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t),\gamma_4(t))$ be a geodesic of the affine manifold $(\mathcal{D}(G,r),\nabla)$; that is $\nabla_{\dot\gamma(t)}\dot\gamma(t)=0$, where $\dot\gamma(t)=\frac{d\gamma}{dt}(t)$ (differential of γ with respect to t at the point t). We have the following nonlinear second-order ODEs which are geodesic equations of $(\mathcal{D}(G,r),\nabla)$:

$$\left\{\begin{array}{lcl}\ddot\gamma_1-\dot\gamma_1^2-2\dot\gamma_1\dot\gamma_4 & = & 0,\\ \ddot\gamma_2-\dot\gamma_1\dot\gamma_2+2\dot\gamma_1\dot\gamma_3 & = & 0,\\ \ddot\gamma_3-\dot\gamma_3\dot\gamma_4+2\dot\gamma_2\dot\gamma_4 & = & 0,\\ \ddot\gamma_4-\dot\gamma_4^2-2\dot\gamma_1\dot\gamma_4 & = & 0.\end{array}\right. \tag{2.6}$$

The above system is just the system (1.1) where we have set $u_i=\gamma_i$, $\dot u_i'=\dot\gamma_i$ and $u_i''=\ddot\gamma_i$, for any $i=1,2,3,4$.

About the geodesic completeness of $(\mathcal{D}(G,r),\nabla)$, we first recall the following.

Theorem 2.1 ([7]). *Let H be a connected Lie group with Lie algebra $\mathfrak{h}$ and $r \in \wedge^2\mathfrak{h}$ an invertible solution of the Classical Yang-Baxter Equation. Consider the LSA structure in the double Lie algebra $\mathcal{D}(\mathfrak{h}, r)$. The following are equivalent:*
1. *on every connected Lie group whose Lie algebra is $\mathcal{D}(\mathfrak{h}, r)$, the left invariant affine structure, corresponding to the LSA, is geodesically complete;*
2. *the Lie group H is unimodular (and solvable).*

As a consequence, we have the

Theorem 2.2. *The affine manifold $(\mathcal{D}(G, r), \nabla)$ is not geodesically complete.*

Proof. From Theorem 2.1 and the fact that G is not unimodular, it follows that the manifold $(\mathcal{D}(G, r), \nabla)$ is not geodesically complete. □

We are now going to deal with the study of the system (1.1) via the Lie point symmetry approach developed by Wafo Soh and Mahomed in [16].

3 Integrability Of System (1.1)

The idea of S. Lie is to use the so-called Lie point symmetry vector fields to study the integrability of (systems of) ODEs. The fundamental question one can ask is: when can we reduce the equations to a simple quadrature? In other words, when can we find a factor Λ that allows, by simple multiplication, to rewrite the equations as the total differential of an expression I necessarily constant?

$$y^{(n)} - \omega(x, y, y', \dots, y^{(n-1)}) = 0 \Rightarrow \Big(y^{(n)} - \omega(x, y, y', \dots, y^{(n-1)})\Big)\Lambda = D_x I = 0.$$

The factor Λ, when it exists, is called an integrating factor and the function I is called a first integral, an exact invariant or a conserved quantity. The knowledge of symmetries of the underlying equation facilitates the search for integrating factors and first integrals.

3.1 A Basic Result

Consider a Lie group of transformations G acting on the open subset M of $\mathbb{R}^m \times \mathbb{R}$, associated to a given first-order linear homogeneous PDE involving m independent variables $y = (y_1, y_2, \dots, y_m)$ in $\mathbb{R}^m$ and one dependent variable t in $\mathbb{R}$. The solutions of the equation will be of the form $f = h(y)$. Now let us recall the following notion which is crucial for integrability in the case where $n \geq 2$ (see [8]): operators in n variables, $X_k = \phi^i_k(x)\frac{\partial}{\partial x^i}$, $k = 1, \dots, p \leqslant n$ are said to be unconnected if the rank of the matrix $(\phi^i_k(x))$ is p on M.

Lemma 3.1 ([16]). *A system of n p^{th}-order ODEs $x_i^{(p)} = f_i(t, x, \dots, x^{(p-1)})$, $i = 1, \dots, n$, which admits a pn-dimensional solvable symmetry algebra L_{pn} for which the $(p-1)^{th}$ prolongation of its symbols as well as $A = \frac{\partial}{\partial t} + x'_i\frac{\partial}{\partial x_i} + \cdots + x_i^{(p-1)}\frac{\partial}{\partial x_i^{(p-2)}} + f_i(t, x, ..., x^{(p-1)})\frac{\partial}{\partial x_i^{(p-1)}}$ are unconnected, is solvable by quadratures.*

3.2 Symmetries Of The PDE Associated To System (1.1)

From Lemma 3.1, we have the following result.

Lemma 3.2. *The operator A associated to the system of ODEs (1.1) is given by*

$$\begin{aligned} A = {} & \frac{\partial}{\partial t} + u'_1 \frac{\partial}{\partial u_1} + u'_2 \frac{\partial}{\partial u_2} + u'_3 \frac{\partial}{\partial u_3} + u'_4 \frac{\partial}{\partial u_4} + \Big((u'_1)^2 + 2u'_1 u'_4\Big)\frac{\partial}{\partial u'_1} + \\ & + (u'_1 u'_2 - 2u'_1 u'_3)\frac{\partial}{\partial u'_2} + (u'_3 u'_4 - 2u'_2 u'_4)\frac{\partial}{\partial u'_3} + \Big((u'_4)^2 + 2u'_1 u'_4\Big)\frac{\partial}{\partial u'_4}. \end{aligned} \tag{3.1}$$

A vector field $X = \xi(t, x_1, x_2, \ldots, x_n)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(t, x_1, x_2, \ldots, x_n)\frac{\partial}{\partial x_i}$ is a Lie point symmetry of the PDE $Af = 0$ if its first prolongation $X^{(1)} = X + \sum_{i=1} \dot{\eta}_i(t, x_i, \dot{x}_i)\frac{\partial}{\partial \dot{x}_i}$ satisfies $[X^{(1)}, A] = -\frac{d\xi}{dt}A$, where $\dot{\eta}_i = -\frac{d\xi}{dt}\dot{x}_i + \frac{d\eta_i}{dt}$. Let us now determine the symmetries of the first-order and homogeneous PDE $Af = 0$, where A is the operator associated to the system of second-order ODEs (1.1) and given by (3.1).

Proposition 3.3. *For the operator A given by (3.1), the first-order linear homogeneous PDE $Af = 0$ admits the following eight symmetries:*

$$\begin{aligned} & X_1 = \tfrac{\partial}{\partial t},\ X_2 = \tfrac{\partial}{\partial u_1},\ X_3 = \tfrac{\partial}{\partial u_2},\ X_4 = \tfrac{\partial}{\partial u_3},\ X_6 = t\tfrac{\partial}{\partial t} - u'_1\tfrac{\partial}{\partial u'_1} - u'_2\tfrac{\partial}{\partial u'_2} - u'_3\tfrac{\partial}{\partial u'_3} - u'_4\tfrac{\partial}{\partial u'_4}, \\ & X_5 = \tfrac{\partial}{\partial u_4},\ X_7 = u_1\tfrac{\partial}{\partial u_2} - u_4\tfrac{\partial}{\partial u_3} + u'_1\tfrac{\partial}{\partial u'_2} - u'_4\tfrac{\partial}{\partial u'_3},\ X_8 = u_2\tfrac{\partial}{\partial u_2} + u_3\tfrac{\partial}{\partial u_3} + u'_2\tfrac{\partial}{\partial u'_2} + u'_3\tfrac{\partial}{\partial u'_3}. \end{aligned}$$

The non-vanishing brackets are:

$$[X_1, X_6] = X_1,\ [X_2, X_7] = X_3,\ [X_3, X_8] = X_3,\ [X_4, X_8] = X_4,\ [X_5, X_7] = -X_4,\ [X_7, X_8] = X_7.$$

Furthermore, the Lie algebra generated by these eight symmetries is solvable.

Proof. The proof is straightforward. The infinitesimal generators X_k, $k = 1, 2, \ldots, 8$ are obtained by applying the definition of a symmetry for the first-order PDE $Af = 0$. The brackets are obtained by computing the commutators $[X_i, X_j] = X_i X_j - X_j X_i$, $i, j = 1, 2, \ldots, 8$. Now since the basis $\{X_k, k = 1, 2, \ldots, 8\}$ satisfies the condition $[X_i, X_j] = \sum_{k=1}^{j-1} c_{ij}^k X_k$ whenever $i < j$, the Lie algebra, generated by the vector fields X_i, is solvable. □

Proposition 3.4. *The operators $\{X_i, i = 1, \ldots, 8\}$ of Proposition 3.3 and the operator A given by (3.1) are unconnected if and only if $u'_1 u'_4 \neq 0$, $u'_1 \neq u'_4$ and $u'_1 u'_3 + u'_2 u'_4 \neq 0$.*

Proof. Set $X = \{X_i, i = 1, \ldots, 8\}$ and let M_A be the matrix of the system $\{X, A\}$. We have

$$M_A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & u'_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & u_1 & u_2 & u'_2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -u_4 & u_3 & u'_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & u'_4 \\ 0 & 0 & 0 & 0 & 0 & -u'_1 & 0 & 0 & (u'_1)^2 + 2u'_1 u'_4 \\ 0 & 0 & 0 & 0 & 0 & -u'_2 & u'_1 & u'_2 & u'_1 u'_2 - 2u'_1 u'_3 \\ 0 & 0 & 0 & 0 & 0 & -u'_3 & -u'_4 & u'_3 & u'_3 u'_4 - 2u'_2 u'_4 \\ 0 & 0 & 0 & 0 & 0 & -u'_4 & 0 & 0 & (u'_4)^2 + 2u'_1 u'_4 \end{pmatrix}. \tag{3.2}$$

The determinant of M_A is $\det(M_A) = u'_1 u'_4 (u'_4 - u'_1)(u'_1 u'_3 + u'_2 u'_4)$. This determinant is nonzero if and only if $u'_1 u'_4 \neq 0$, $u'_1 \neq u'_4$ and $u'_1 u'_3 + u'_2 u'_4 \neq 0$. □

Hence, we have the conditions in which we can apply Lemma 3.1.

Proposition 3.5. *If $u'_1u'_4 \neq 0$, $u'_1 \neq u'_4$ and $u'_1u'_3 + u'_2u'_4 \neq 0$, then the system of four second-order ODEs (1.1) is solvable by quadratures.*

Proof. The Proposition 3.5 is a direct consequence of Proposition 3.4 and Lemma 3.1. □

4 First Integrals Of The System Of ODEs (1.1)

First integrals play an important role in the search of solutions to systems of DEs. As mentioned, a system of DEs can be either seen through its solutions or through the first integrals it possesses. In this section, we suppose that we are within the hypothesis of Lemma 3.1, that is $u'_1u'_4 \neq 0$, $u'_1 \neq u'_4$ and $u'_1u'_3 + u'_2u'_4 \neq 0$. We consider the following sequence of ideals: $\mathcal{L}_i = \mathrm{span}(\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{i})$, $i = 1, \ldots, 8$. One has $\mathcal{L}_i \subset \mathcal{L}_{i+1}$, for all $i = 1, \ldots, 7$.
Step 1. We solve the system

$$Af = 0, \quad X_i f = 0, i = 1, ..., 7. \tag{4.1}$$

From $X_i f = 0$, $i = 1, 2, 3, 4, 5$, we infer that $f = F(u'_1, u'_2, u'_3, u'_4)$. Now, $X_6 f = 0$ gives

$$-u'_1\partial_{u'_1}F - u'_2\partial_{u'_2}F - u'_3\partial_{u'_3}F - u'_4\partial_{u'_4}F = 0. \tag{4.2}$$

By solving (4.2) through its characteristic equations $\frac{du'_1}{u'_1} = \frac{du'_2}{u'_2} = \frac{du'_3}{u'_3} = \frac{du'_4}{u'_4}$, we obtain that $f = G(\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1 = \frac{u'_2}{u'_1}, \alpha_2 = \frac{u'_3}{u'_1}$ and $\alpha_3 = \frac{u'_4}{u'_1}$. Equation $X_7 f = 0$ implies that

$$\partial_{\alpha_1}G - \alpha_3\partial_{\alpha_2}G = 0. \tag{4.3}$$

Now (4.3) leads to $f = H(\beta, \alpha_3)$, where $\beta = \frac{u'_1u'_3 + u'_2u'_4}{(u'_1)^2}$ and $\alpha_3 = \frac{u'_4}{u'_1}$. Substituting the later expressions into $Af = 0$ yields to

$$\alpha_3(\alpha_3 - 1)\partial_{\alpha_3}H + (3\alpha_3 + 1)\beta\partial_\beta H = 0. \tag{4.4}$$

Solving of (4.4) gives $f = P(S_{01})$, where S_{01} is given by

$$S_{01} = \frac{\beta\alpha_3}{(\alpha_3 - 1)^4} = \frac{u'_1u'_4(u'_1u'_3 + u'_2u'_4)}{(u'_4 - u'_1)^4}. \tag{4.5}$$

One can readily check that $X_8 S_{01} = S_{01}$. If we put $S_1 = \int \frac{dS_{01}}{S_{01}}$, we obtain $X_8 S_1 = 1$, with

$$S_1 = \ln\left|\frac{\beta\alpha_3}{(\alpha_3 - 1)^4}\right| = \ln\left|\frac{u'_1u'_4(u'_1u'_3 + u'_2u'_4)}{(u'_4 - u'_1)^4}\right|. \tag{4.6}$$

Step 2. We use the following coordinates.

$$x^0 = t,\ x^1 = u_1,\ x^2 = u_2,\ x^3 = u_3,\ x^4 = u_4,\ x^5 = u'_1,\ x^6 = u'_2,\ x^7 = S_1,\ x^8 = u'_4.$$

In these coordinates the operators read

$$X_1 = \frac{\partial}{\partial x^0}, \quad X_2 = \frac{\partial}{\partial x^1}, \quad X_3 = \frac{\partial}{\partial x^2}, \quad X_4 = \frac{\partial}{\partial x^3}, \quad X_5 = \frac{\partial}{\partial x^4},$$
$$X_6 = x^0\frac{\partial}{\partial x^0} - x^5\frac{\partial}{\partial x^5} - x^6\frac{\partial}{\partial x^6} - x^8\frac{\partial}{\partial x^8}, \quad X_7 = x^1\frac{\partial}{\partial x^2} - x^4\frac{\partial}{\partial x^3} + x^5\frac{\partial}{\partial x^6},$$
$$A = \partial_{x^o} + x^5\partial_{x^1} + x^6\partial_{x^2} + \left[-\frac{x^6x^8}{x^5} \pm \frac{(x^8-x^5)^4}{(x^5)^2x^8}\exp(x^7)\right]\partial_{x^3} + x^8\partial_{x^4} + \left[(x^5)^2 + 2x^5x^8\right]\partial_{x^5}$$
$$+\left[x^5x^6 + 2x^6x^8 \mp \frac{2(x^8-x^5)^4\exp(x^7)}{x^5x^8}\right]\partial_{x^6} + \left[(x^8)^2 + 2x^5x^8\right]\partial_{x^8}.$$

We deal with the system

$$Af = 0, \quad X_i f = 0, i = 1,\dots,6. \tag{4.7}$$

From $X_i f = 0$, $i = 1,2,3,4,5$, it comes that $f = F(x^5, x^6, x^8)$. Now $X_6 f = 0$ gives

$$-x^5\partial_{x^5}F - x^6\partial_{x^6}F - x^8\partial_{x^8}F = 0. \tag{4.8}$$

The solution of (4.8) through its characteristic equations $\frac{dx^5}{x^5} = \frac{dx^6}{x^6} = \frac{dx^8}{x^8}$ is $f = G(\beta_1,\beta_2)$, where $\beta_1 = \frac{x^6}{x^5}$ and $\beta_2 = \frac{x^8}{x^5}$. If we substitute the later expression of f in $Af = 0$, we obtain

$$\mp 2(\beta_2-1)^4\exp(x^7)\partial_{\beta_1}G + (\beta_2)^2(1-\beta_2)\partial_{\beta_2}G = 0. \tag{4.9}$$

The solution of (4.9) is $f = H(S_2)$, where S_2 satisfies $X_7S_2 = 1$ and is given by

$$S_2 = \frac{x^6}{x^5} \mp \exp(x^7)\left[-\left(\frac{x^8}{x^5}\right)^2 + \frac{6x^8}{x^5} - 6\ln\left|\frac{x^8}{x^5}\right| - \frac{2x^5}{x^8}\right]. \tag{4.10}$$

Step 3. We consider the following coordinates:

$$y^0 = t,\ y^1 = u_1,\ y^2 = u_2,\ y^3 = u_3,\ y^4 = u_4,\ y^5 = u_1',\ y^6 = S_2,\ y^7 = S_1,\ y^8 = u_4'.$$

The operators become

$$X_1 = \frac{\partial}{\partial y^0},\ X_2 = \frac{\partial}{\partial y^1},\ X_3 = \frac{\partial}{\partial y^2},\ X_4 = \frac{\partial}{\partial y^3},\ X_5 = \frac{\partial}{\partial y^4},\ X_6 = y^0\frac{\partial}{\partial y^0} - y^5\frac{\partial}{\partial y^5} - y^8\frac{\partial}{\partial y^8},$$
$$A = \partial_{y^o} + y^5\partial_{y^1} + ()\partial_{y^2} + ()\partial_{y^3} + y^8\partial_{y^4} + \left[(y^5)^2 + 2y^5y^8\right]\partial_{y^5} + \left[(y^8)^2 + 2y^5y^8\right]\partial_{y^8},$$

where the explicit form of the terms omitted in the parentheses are not needed in subsequent computations. We solve the system

$$Af = 0, \quad X_i f = 0, i = 1,2,3,4,5. \tag{4.11}$$

From $X_i f = 0$, $i = 1,2,3,4,5$, it comes that $f = F(y^5, y^8)$. Substituting the later expression in $Af = 0$, we obtain

$$\left[(y^5)^2 + 2y^5y^8\right]\partial_{y^5}F + \left[(y^8)^2 + 2y^5y^8\right]\partial_{y^8}F = 0. \tag{4.12}$$

The solution of (4.12) is $f = G(S_{03})$, where $S_{03} = \frac{y^5 y^8}{(y^8 - y^5)^3}$. We have $X_6 S_{03} = S_{03}$. We set $S_3 = \int \frac{dS_{03}}{S_{03}}$. Then,

$$S_3 = \ln\left|\frac{y^5 y^8}{(y^8 - y^5)^3}\right|. \tag{4.13}$$

Step 4. The new variables are

$$z^0 = t,\ z^1 = u_1,\ z^2 = u_2,\ z^3 = u_3,\ z^4 = u_4,\ z^5 = u_1',\ z^6 = S_2,\ z^7 = S_1,\ z^8 = S_3.$$

The operators are

$$X_1 = \frac{\partial}{\partial z^0}, \quad X_2 = \frac{\partial}{\partial z^1}, \quad X_3 = \frac{\partial}{\partial z^2}, \quad X_4 = \frac{\partial}{\partial z^3}, \quad X_5 = \frac{\partial}{\partial z^4}.$$

Using MATHEMATICA, we obtain

$$y^8 = y^5 + y^5 B(y^5, S_3) \pm \frac{1}{3\exp(S_3)}\left(B(y^5, S_3)\right)^{-1}, \tag{4.14}$$

where

$$B(y^5, S_3) = \left[\frac{2}{3\left(9(y^5)^2\exp(2S_3) + \sqrt{(y^5)^3\exp(3S_3)(\mp 4 + 27y^5\exp(S_3))}\right)}\right]^{\frac{1}{3}}. \tag{4.15}$$

Then it follows that

$$A = \partial_{z^o} + z^5\partial_{z^1} + ()\partial_{z^2} + ()\partial_{z^3} + P(z^5, z^8)\partial_{z^4} + Q(z^5, z^8)\partial_{y^5},$$

where the terms omitted in the parentheses are not needed and where $P(z^5, z^8)$ and $Q(z^5, z^8)$ are given by

$$P(z^5, z^8) = z^5 B(z^5, z^8) \pm \frac{\exp(-z^8)}{3}\left(B(z^5, z^8)\right)^{-1} + z^5, \tag{4.16}$$

$$Q(z^5, z^8) = 3(z^5)^2 + 2(z^5)^2 B(z^5, z^8) \pm \frac{2z^5\exp(-z^8)}{3}\left(B(z^5, z^8)\right)^{-1}. \tag{4.17}$$

Let us show that the expressions $P(z^5, z^8)$ and $Q(z^5, z^8)$ do not vanish. $P(z^5, z^8)$ is equal to u_4' which is different from zero by the hypothesis. Now, when expressed in the original variables u_1' and u_4', $Q(z^5, z^8)$ is equivalent to $(u_1')^2 + 2u_1'u_4'$. So, $Q(z^5, z^8) = 0$ if and only if $(u_1')^2 + 2u_1'u_4' = 0$, that is $u_1' = -2u_4'$ as $u_1' \neq 0$ by the hypothesis. In this case, we have $u_1'' = -2u_4''$ and the first and fourth equations of the system (1.1) yield respectively to $u_4'' = 0$ and $u_4'' = -3(u_4')^2 \neq 0$. This is not possible and then $Q(z^5, z^8) \neq 0$.

Now, we deal with the system

$$Af = 0, \quad X_i f = 0,\ i = 1, 2, 3, 4. \tag{4.18}$$

From $X_i f = 0$, $i = 1, 2, 3, 4$, it comes that $f = F(z^4, z^8)$. Then, $Af = 0$ gives

$$P(z^5, z^8)\partial_{z^4} F + Q(z^5, z^8)\partial_{z^5} F = 0. \tag{4.19}$$

Equation (4.19) leads to the already normalized solution

$$S_4 = z^4 - \int \frac{P(z^5, z^8)}{Q(z^5, z^8)} dz^5. \tag{4.20}$$

Step 5. We now have the coordinates

$$v^0 = t,\ v^1 = u_1,\ v^2 = u_2,\ v^3 = u_3,\ v^4 = S_4,\ v^5 = u_1',\ v^6 = S_2,\ v^7 = S_1,\ v^8 = S_3.$$

In these coordinates we have

$$X_1 = \frac{\partial}{\partial v^0},\ X_2 = \frac{\partial}{\partial v^1},\ X_3 = \frac{\partial}{\partial v^2},\ X_4 = \frac{\partial}{\partial v^3},\ A = \partial_{v^o} + v^5\partial_{v^1} + ()\partial_{v^2} + D(v^5, v^8)\partial_{v^3} + Q(v^5, v^8)\partial_{v^5},$$

where, with the change of variables, $Q(v^5, v^8)$ is given by (4.17); and

$$\begin{aligned} D(v^5, v^8) &= \pm 2v^5 \exp(v^7) - v^6 P(v^5, v^8) \pm \exp(v^7)\frac{\left[P(v^5, v^8)\right]^3}{(v^5)^2} \mp 6\exp(v^7)\frac{\left[P(v^5, v^8)\right]^2}{v^5} \\ &\quad \pm 6\exp(v^7)P(v^5, v^8)\times \ln\left|\frac{P(v^5, v^8)}{v^5}\right| \pm \frac{\left[P(v^5, v^8) - v^5\right]^4}{(v^5)^2 P(v^5, v^8)}\exp(v^7), \end{aligned} \tag{4.21}$$

where $P(v^5, v^8)$ is given by (4.16). The system we deal with is

$$Af = 0, \quad X_i f = 0,\ i = 1,2,3. \tag{4.22}$$

The equations $X_i f = 0$, $i = 1,2,3$ imply that $f = F(v^3, v^5)$. Then $Af = 0$ reads

$$D(v^5, v^8)\partial_{v^3} F + Q(v^5, v^8)\partial_{v^5} F = 0. \tag{4.23}$$

Equation (4.23) admits the normalized solution

$$S_5 = v^3 - \int \frac{D(v^5, v^8)}{Q(v^5, v^8)} dv^5. \tag{4.24}$$

Step 6. We use the following coordinates:

$$w^0 = t,\ w^1 = u_1,\ w^2 = u_2,\ w^3 = S_5,\ w^4 = S_4,\ w^5 = u_1',\ w^6 = S_2,\ w^7 = S_1,\ w^8 = S_3.$$

We have the following operators :

$$X_1 = \frac{\partial}{\partial w^0},\ X_2 = \frac{\partial}{\partial w^1},\ X_3 = \frac{\partial}{\partial w^2},\ A = \partial_{w^o} + w^5\partial_{w^1} + I(w^5, w^8)\partial_{w^2} + Q(w^5, w^8)\partial_{w^5},$$

where the expression $I(w^5, w^8)$ is given by

$$\begin{aligned} I(w^5, w^8) &= w^5 w^6 \mp \exp(w^7)\frac{\left[P(w^5, w^8)\right]^2}{w^5} \pm 6\exp(w^7)P(w^5, w^8) \\ &\quad \mp 6w^5 \exp(w^7)\ln\left|\frac{P(w^5, w^8)}{w^5}\right| \mp \frac{2(w^5)^2\exp(w^7)}{P(w^5, w^8)}, \end{aligned} \tag{4.25}$$

$P(w^5, w^8)$ and $Q(w^5, w^8)$ being defined as in (4.16) and (4.17). We deal with the system

$$Af = 0, \quad X_i f = 0, i = 1, 2. \tag{4.26}$$

From $X_i f = 0, i = 1, 2$ we have $f = F(v^3, v^5)$. Then $Af = 0$ implies that

$$I(w^3, w^5)\partial_{w^2} F + Q(w^5, w^8)\partial_{w^5} F = 0. \tag{4.27}$$

The solution of the later is the already normalized function

$$S_6 = w^2 - \int \frac{I(w^5, w^8)}{Q(w^5, w^8)} dw^5. \tag{4.28}$$

Step 7. The new variables are

$$q^0 = t,\ q^1 = u_1,\ q^2 = S_6,\ q^3 = S_5,\ q^4 = S_4,\ q^5 = u_1',\ q^6 = S_2,\ q^7 = S_1,\ q^8 = S_3.$$

The operators are

$$X_1 = \frac{\partial}{\partial q^0}, \quad X_2 = \frac{\partial}{\partial q^1}, \quad A = \partial_{q^o} + q^5 \partial_{q^1} + Q(q^5, q^8)\partial_{q^5}.$$

The system is

$$Af = 0, \quad X_1 f = 0. \tag{4.29}$$

From $X_1 f = 0$, we deduce that $f = F(q^1, q^5)$. Then equation $Af = 0$ becomes

$$q^5 \partial_{q^1} F + Q(q^5, q^8)\partial_{q^5} F = 0. \tag{4.30}$$

The solution of the later equation is

$$S_7 = q^1 - \int \frac{q^5 dq^5}{Q(q^5, q^8)}. \tag{4.31}$$

It is readily verified that $X_2 S_7 = 1$.

Step 8. In this final step we use the coordinates

$$r^0 = t,\ r^1 = S_7,\ r^2 = S_6,\ r^3 = S_5,\ r^4 = S_4,\ r^5 = u_1',\ r^6 = S_2,\ r^7 = S_1,\ r^8 = S_3.$$

In terms of these coordinates the operators are

$$X_1 = \partial_{r^o}, \quad A = \partial_{r^0} + Q(r^5, r^8)\partial_{r^5} \tag{4.32}$$

and the equation $Af = 0$ implies that

$$\partial_{r^0} F + Q(r^5, r^8)\partial_{r^5} F = 0, \tag{4.33}$$

for $f = F(r^0, r^5)$. The integration of (4.33) leads to

$$S_8 = r^0 - \int \frac{1}{Q(r^5, r^8)} dr^5, \tag{4.34}$$

where $Q(r^5, r^8)$ is defined as in (4.17). It is easy to see that $X_1 S_8 = 1$.

In summary, we have the

Theorem 4.1. *If $u'_1u'_4(u'_1-u'_4)(u'_1u'_3+u'_2u'_4)\neq 0$, then the system (1.1) is solvable by quadratures and admits the following height first integrals:*

$$
\begin{aligned}
&S_1=\ln\left|\frac{u'_1u'_4(u'_1u'_3+u'_2u'_4)}{(u'_4-u'_1)^4}\right|, \qquad S_2=\frac{u'_2}{u'_1}\mp\exp(S_1)\left[-\left(\frac{u'_4}{u'_1}\right)^2+\frac{6u'_4}{u'_1}-6\ln\left|\frac{u'_4}{u'_1}\right|-\frac{2u'_1}{u'_4}\right],\\
&S_3=\ln\left|\frac{u'_1u'_4}{(u'_4-u'_1)^3}\right|, \qquad S_4=u_4-\int\frac{P(u'_1,S_3)}{Q(u'_1,S_3)}du'_1, \qquad S_5=u_3-\int\frac{D(u'_1,S_3)}{Q(u'_1,S_3)}du'_1,\\
&S_6=u_2-\int\frac{I(u'_1,S_3)}{Q(u'_1,S_3)}du'_1, \qquad S_7=u_1-\int\frac{u'_1du'_1}{Q(u'_1,S_3)}, \qquad S_8=t-\int\frac{du'_1}{Q(u'_1,S_3)},
\end{aligned}
$$

where $P(u'_1,S_3)$, $Q(u'_1,S_3)$, $D(u'_1,S_3)$, $I(u'_1,S_3)$ are respectively given by (4.16), (4.17), (4.21), (4.25) and where we replace the variables z^5, v^5, w^5 by u'_1 and z^8, v^8, w^8 by S_3.

5 Cases where the symmetry vector fields and the operator A are connected

In this section, we study the cases where the symmetry vector fields X_i, $i=1,2,\dots,8$ and the operator A are connected; that is the cases where $\det(M_A)=u'_1u'_4(u'_4-u'_1)(u'_1u'_3+u'_2u'_4)=0$.

5.1 Case 1: $u'_1=0$

In this case the system (1.1) becomes:

$$u_1=c_1,\quad u''_2=0,\quad u''_3=u'_4(u'_3-2u'_2),\quad u''_4=(u'_4)^2, \tag{5.1}$$

where c_1 is an arbitrary constant. The resolution of this system is straightforward. We have:

Proposition 5.1. *The curve $\gamma:\mathbb{R}\to\mathcal{D}(G,r)$, $t\mapsto\gamma(t)=(u_1(t),u_2(t),u_3(t),u_4(t))$ defined by*

$$\gamma(t)=\Big(c_1\,,\,c_2t+c_3\,,\,-2c_2t+c_6\ln|t+c_4|+c_7\,,\,-\ln|t+c_4|+c_5\Big), \tag{5.2}$$

where $c_i\in\mathbb{R}, i=1,2,\dots,7$, is a geodesic of $(\mathcal{D}(G,r),\nabla)$. This geodesic is not complete since it is not defined for $t=-c_4$.

5.2 Case 2: $u'_4=0$

The system (1.1) becomes

$$u''_1=(u'_1)^2,\quad u''_2=u'_1(u'_2-2u'_3),\quad u''_3=0,\quad u_4=q_1, \tag{5.3}$$

where q_1 is an arbitrary constant. And we have the following result.

Proposition 5.2. *The curve $\gamma:\mathbb{R}\to\mathcal{D}(G,r)$, $t\mapsto\gamma(t)=(u_1(t),u_2(t),u_3(t),u_4(t))$ defined by*

$$\gamma(t)=\Big(-\ln|t+q_4|+q_5\,,-2q_2t+q_6\ln|t+q_4|+q_7\,,q_2t+q_3,\,q_1\Big), \tag{5.4}$$

where $q_i\in\mathbb{R}, i=1,2,\dots,7$ and $t\neq -q_4$, is a non complete geodesic of $(\mathcal{D}(G,r),\nabla)$.

5.3 Case 3: $u_1' = u_4' \neq 0$

In this case the system (1.1) becomes

$$u_1'' = 3(u_1')^2, \quad u_2'' = u_1'(u_2' - 2u_3'), \quad u_3'' = u_1'(u_3' - 2u_2'), \quad u_4' = u_1'. \tag{5.5}$$

This leads to the following result.

Proposition 5.3. *The curve $\gamma : \mathbb{R} \to \mathcal{D}(G,r)$, $t \mapsto \gamma(t) = (u_1(t), u_2(t), u_3(t), u_4(t))$ defined by*

$$\begin{cases} u_1(t) &=& -\frac{1}{3}\ln|3t+k_1| + k_2, \\ u_2(t) &=& \frac{1}{8}k_4(3t+k_1)^{\frac{4}{3}} + \frac{1}{6}k_6 \ln|3t+k_1| + \frac{1}{2}(k_5+k_7), \\ u_3(t) &=& \frac{1}{8}k_4(3t+k_1)^{\frac{4}{3}} - \frac{1}{6}k_6 \ln|3t+k_1| + \frac{1}{2}(k_5-k_7), \\ u_4(t) &=& -\frac{1}{3}\ln|3t+k_1| + k_3, \end{cases} \tag{5.6}$$

where $k_i \in \mathbb{R}, i = 1,2,...,7$ and $t \neq -\frac{1}{3}k_1$, is a non complete geodesic of $(\mathcal{D}(G,r), \nabla)$.

5.4 Case 4: $u_1'u_3' + u_2'u_4' = 0$, $u_1' \neq u_4'$, $u_1'u_4' \neq 0$.

5.4.1 $u_3' = u_2' = 0$

Then the system becomes

$$u_1'' = (u_1')^2 + 2u_1'u_4', \quad u_2 = p_1, \quad u_3 = p_2, \quad u_4'' = (u_4')^2 + 2u_1'u_4', \tag{5.7}$$

where $p_1, p_2 \in \mathbb{R}$. We now consider the following system

$$u_1'' = (u_1')^2 + 2u_1'u_4', \quad u_4'' = (u_4')^2 + 2u_1'u_4'. \tag{5.8}$$

Lemma 5.4. *If $u_1' \neq u_4'$ and $u_1'u_4' \neq 0$, then the system (5.8) is solvable by quadratures.*

Proof. The operator A_1 associated to this system is given by

$$A_1 = \frac{\partial}{\partial t} + u_1' \frac{\partial}{\partial u_1} + u_4' \frac{\partial}{\partial u_4} + \left((u_1')^2 + 2u_1'u_4'\right)\frac{\partial}{\partial u_1'} + \left((u_4')^2 + 2u_1'u_4'\right)\frac{\partial}{\partial u_4'}. \tag{5.9}$$

The first-order linear homogeneous PDE $A_1 f = 0$ admits the following four symmetries:

$$X_1 = \tfrac{\partial}{\partial t}, \quad X_2 = \tfrac{\partial}{\partial u_1}, \quad X_3 = \tfrac{\partial}{\partial u_4}, \quad X_4 = t\tfrac{\partial}{\partial t} - u_1' \tfrac{\partial}{\partial u_1'} - u_4' \tfrac{\partial}{\partial u_4'}.$$

The non-vanishing bracket is: $[X_1, X_4] = X_1$. Furthermore, the Lie algebra generated by these four symmetries is solvable, since the basis $\{X_k, k = 1, \ldots, 4\}$ satisfies the condition $[X_i, X_j] = \sum_{k=1}^{j-1} c_{ij}^k X_k$ whenever $i < j$. The matrix M_{A_1} of the system $\{X_i, A_1, i = 1, \ldots, 4\}$ is

$$M_{A_1} = \begin{pmatrix} 1 & 0 & 0 & t & 1 \\ 0 & 1 & 0 & 0 & u_1' \\ 0 & 0 & 1 & 0 & u_4' \\ 0 & 0 & 0 & -u_1' & (u_1')^2 + 2u_1'u_4' \\ 0 & 0 & 0 & -u_4' & (u_4')^2 + 2u_1'u_4' \end{pmatrix}. \tag{5.10}$$

The determinant of M_{A_1} is $\det(M_{A_1}) = u_1'u_4'(u_4' - u_1')$. By the hypothesis, $\det(M_{A_1}) \neq 0$ and then the system $\{X_i, A_1, i = 1, \ldots, 4\}$ is unconnected. We conclude with Lemma 3.1. □

Now let us determine the first integrals of the system (5.8). We have

Lemma 5.5. *If $u'_1 \neq u'_4$ and $u'_1 u'_4 \neq 0$, the system (5.8) admits the following first integrals:*

$$J_1 = \ln\left|\frac{u'_1 u'_4}{(u'_4 - u'_1)^3}\right|, J_2 = u_4 - \int \frac{P'(u'_1, J_1)}{Q'(u'_1, J_1)} du'_1, J_3 = u_1 - \int \frac{u'_1}{Q'(u'_1, J_1)} du'_1, J_4 = t - \int \frac{du'_1}{Q'(u'_1, J_1)},$$

where $P'(u'_1, J_1)$ and $Q'(u'_1, J_1)$ are non-vanishing expressions depending only on u'_1 and J_1.

Proof. Consider the ideals $\mathcal{L}_i = \text{span}(\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{i})$, $i = 1,2,3,4$. We have $\mathcal{L}_i \subset \mathcal{L}_{i+1}$, $i = 1,2,3$.
Step 1. We solve the system

$$A_1 f = 0,\ X_i f = 0,\ i = 1,2,3. \tag{5.11}$$

Equations $X_i f = 0$, $i = 1,2,3$ imply that $f = F(u'_1, u'_3)$. Substituting the later expression of f in $A_1 f = 0$ yields to

$$\left[(u'_1)^2 + 2u'_1 u'_4\right]\partial_{u'_1} F + \left[(u'_4)^2 + 2u'_1 u'_4\right]\partial_{u'_4} F = 0. \tag{5.12}$$

The solution of (5.12) is $f = G(J_{01})$, where $J_{01} = \frac{u'_1 u'_4}{(u'_4 - u'_1)^3}$. We have $X_4 J_{01} = J_{01}$. We set $J_1 = \int \frac{dJ_{01}}{J_{01}}$. Then,

$$J_1 = \ln\left|\frac{u'_1 u'_4}{(u'_4 - u'_1)^3}\right|. \tag{5.13}$$

Using MATHEMATICA, we obtain

$$u'_4 = u'_1 + u'_1 B'(u'_1, J_1) \pm \frac{1}{3\exp(J_1)}[B'(u'_1, J_1)]^{-1}, \tag{5.14}$$

where

$$B'(u'_1, J_1) = \left[\frac{2}{3\left(9(u'_1)^2\exp(2J_1) + \sqrt{(u'_1)^3\exp(3J_1)\left(\mp 4 + 27u'_1\exp(J_1)\right)}\right)}\right]^{\frac{1}{3}}. \tag{5.15}$$

Step 2. The new variables are

$$x^0 = t,\ x^1 = u_1,\ x^2 = u_3,\ x^3 = u'_1,\ x^4 = J_1.$$

The operators are

$$X_1 = \frac{\partial}{\partial x^0}, X_2 = \frac{\partial}{\partial x^1}, X_3 = \frac{\partial}{\partial x^2}, A_1 = \partial_{x^0} + x^3\partial_{x^1} + P'(x^3, x^4)\partial_{z^4} + Q'(x^3, x^4)\partial_{y^5},$$

where $P'(x^3, x^4)$ and $Q'(x^3, x^4)$ are given by

$$\begin{aligned} P'(x^3, x^4) &= x^3 B(x^3, x^4) \pm \frac{\exp(-x^4)}{3}\left[B(x^3, x^4)\right]^{-1} + x^3, & (5.16)\\ Q'(x^3, x^4) &= 3(x^3)^2 + 2(x^3)^2 B(x^3, x^4) \pm \frac{2x^3\exp(-x^4)}{3}\left[B(x^3, x^4)\right]^{-1}. & (5.17) \end{aligned}$$

As shown above for Q, the expression $Q'(x^3,x^4) \neq 0$. We deal with the system

$$A_1 f = 0, \quad X_i f = 0, i = 1,2. \tag{5.18}$$

From $X_i f = 0$, $i = 1,2$ it comes that $f = F(x^2,x^4)$. Then $A_1 f = 0$ gives

$$P'(x^3,x^4)\partial_{x^2} F + Q'(x^3,x^4)\partial_{x^3} F = 0. \tag{5.19}$$

The solution of the equation (5.19) is the already normalized function

$$J_2 = x^2 - \int \frac{P'(x^3,x^4)}{Q'(x^3,x^4)} dx^3. \tag{5.20}$$

Step 3. We deal with the coordinates

$$y^0 = t, \quad y^1 = u_1, \quad y^2 = u_3, \quad y^3 = J_2, \quad y^4 = J_1.$$

In these coordinates, we have

$$X_1 = \frac{\partial}{\partial y^0}, \quad X_2 = \frac{\partial}{\partial y^1}, \quad A_1 = \partial_{y^o} + y^3\partial_{y^1} + Q'(y^3,y^4)\partial_{y^3}.$$

The system is

$$A_1 f = 0, \quad X_1 f = 0. \tag{5.21}$$

From $X_1 f = 0$, we deduce that $f = F(y^1,y^3)$. Then, equation $A_1 f = 0$ becomes

$$y^3\partial_{y^1} F + Q'(y^3,y^4)\partial_{y^3} F = 0. \tag{5.22}$$

The solution of the later equation is the normalized function

$$J_3 = y^1 - \int \frac{y^3 dy^3}{Q'(y^3,y^4)}.$$

Step 4. In this final step we use the coordinates

$$z^0 = t, \quad z^1 = u_1, \quad z^2 = J_3, \quad z^3 = J_2, \quad z^4 = J_1.$$

In terms of these coordinates the operators are

$$X_1 = \partial_{z^o}, \quad A_1 = \partial_{z^0} + Q'(z^3,z^4)\partial_{z^3} \tag{5.23}$$

and $A_1 f = 0$ implies that

$$\partial_{z^0} F + Q'(z^3,z^4)\partial_{z^3} F = 0, \tag{5.24}$$

for $f = F(z^0,z^3)$. We have the following solution of (5.24):

$$J_4 = z^0 - \int \frac{1}{Q'(z^3,z^4)} dz^3. \tag{5.25}$$

We have $X_1 J_4 = 1$. □

Combining Lemmas 5.4 and 5.5, we obtain the following result.

Theorem 5.6. *If $u_1'u_3' + u_2'u_4' = 0$, $u_1' \neq u_4'$, $u_1'u_4' \neq 0$ and $u_3' = u_2' = 0$, then*
1. *$u_3(t) = p_2$ and $u_2(t) = p_1$, for all $t \in \mathbb{R}$, where $p_1, p_2 \in \mathbb{R}$;*
2. *the system formed by the first and the fourth equations of (1.1) is solvable by quadratures and admits the following four first integrals:*

$$J_1 = \ln\left|\frac{u_1'u_4'}{(u_4'-u_1')^3}\right|, J_2 = u_4 - \int \frac{P'(u_1', J_1)}{Q'(u_1', J_1)} du_1', J_3 = u_1 - \int \frac{u_1'}{Q'(u_1', J_1)} du_1', J_4 = t - \int \frac{du_1'}{Q'(u_1', J_1)},$$

where $P'(u_1', J_1)$, $Q'(u_1', J_1)$ are expressions depending only on u_1' and J_1.

5.4.2 $u_3' \neq 0$, $u_2' \neq 0$

In this case we have $u_1'u_3' = -u_2'u_4'$ and the system (1.1) becomes

$$u_1'' = (u_1')^2 + 2u_1'u_4', \quad u_2'' = u_1'u_2' - 2u_1'u_3', \quad u_3'' = u_3'u_4' + 2u_1'u_3', \quad u_4'' = (u_4')^2 + 2u_1'u_4'. \tag{5.26}$$

By the change of variables $u = u_1', v = u_2', w = u_3'$ and $z = u_4'$, the system (5.26) becomes

$$\frac{u'}{u} = u + 2z, \quad \frac{v'}{v} = u + 2z, \quad \frac{w'}{w} = z + 2u, \quad \frac{z'}{z} = z + 2u. \tag{5.27}$$

Equations (5.27) lead to $v = \alpha_1 u$, $z = \alpha_2 w$ which can be writing as

$$u_2' = \alpha_1 u_1', \quad u_4' = \alpha_2 u_3'. \tag{5.28}$$

When coupled with equation $u_1'u_3' = -u_2u_4$, equation (5.28) gives $u_1'u_3'(1+\alpha_1\alpha_2) = 0$, that is $1+\alpha_1\alpha_2 = 0$, since $u_1'u_3' \neq 0$. Now the first and the third equations of (5.26) become:

$$u_1'' = (u_1')^2 + 2\alpha_2 u_1'u_3', \quad u_3'' = \alpha_2(u_3')^2 + 2u_1'u_3'. \tag{5.29}$$

Lemma 5.7. *If $u_1' \neq 0$, $u_3' \neq 0$ and $u_1' \neq \alpha_2 u_3'$, the system (5.29) is solvable by quadratures.*

Proof. The operator A_2 associated to the system of ODEs (5.29) is given by

$$A_2 = \frac{\partial}{\partial t} + u_1'\frac{\partial}{\partial u_1} + u_3'\frac{\partial}{\partial u_3} + \left((u_1')^2 + 2\alpha_2 u_1'u_3'\right)\frac{\partial}{\partial u_1'} + \left(\alpha_2(u_3')^2 + 2u_1'u_3'\right)\frac{\partial}{\partial u_3'}. \tag{5.30}$$

For the operator A given by (5.30), the first-order linear homogeneous PDE $A_2 f = 0$ admits the following four symmetries:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial u_1},\ X_3 = \frac{\partial}{\partial u_3},\ X_4 = t\frac{\partial}{\partial t} - u_1'\frac{\partial}{\partial u_1'} - u_3'\frac{\partial}{\partial u_3'}.$$

The non-vanishing bracket is: $[X_1, X_4] = X_1$. Furthermore, the Lie algebra generated by these four symmetries is solvable, since the basis $\{X_k, k = 1,2,\dots,4\}$ satisfies the condition $[X_i, X_j] = \sum_{k=1}^{j-1} c_{ij}^k X_k$ whenever $i < j$. The matrix of the system $\{X_i, A_2, i = 1,2,3,4\}$ is

$$M_{A_2} = \begin{pmatrix} 1 & 0 & 0 & t & 1 \\ 0 & 1 & 0 & 0 & u_1' \\ 0 & 0 & 1 & 0 & u_3' \\ 0 & 0 & 0 & -u_1' & (u_1')^2 + 2\alpha_2 u_1'u_3' \\ 0 & 0 & 0 & -u_3' & \alpha_2(u_3')^2 + 2u_1'u_3' \end{pmatrix}. \tag{5.31}$$

The determinant of M_{A_2} is $\det(M_{A_2}) = u_1'u_3'(\alpha_2 u_3' - u_1')$. □

Lemma 5.8. *If $u_1' \neq 0$, $u_3' \neq 0$ and $u_1' \neq \alpha_2 u_3'$, then the system (5.29) has the following four first integrals:*

$$I_1 = \ln\left|\frac{u_1' u_3'}{(u_3' - u_1')^3}\right|,\ I_2 = u_4 - \int \frac{P''(u_1', I_1)du_1'}{Q''(u_1', I_1)},\ I_3 = u_1 - \int \frac{u_1'}{Q''(u_1', I_1)}du_1',\ I_4 = t - \int \frac{du_1'}{Q''(u_1', I_1)},$$

where $P''(u_1', I_1)$ and $Q''(u_1', I_1)$ are expressions depending only on u_1' and I_1.

Proof. In the conditions listed in Lemma 5.8, $\det(M_{A_2}) \neq 0$ and the symmetry vector fields X_i, $i = 1,2,3,4$ and A_2 are unconnected. We consider the ideals $\mathcal{L}_i = \mathrm{span}(\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{i})$, $i = 1,2,3,4$. One has $\mathcal{L}_i \subset \mathcal{L}_{i+1}$, for all $i = 1,2,3$.
Step 1. We solve the system

$$A_2 f = 0, \quad X_i f = 0, i = 1,2,3. \tag{5.32}$$

From $X_i f = 0$, $i = 1,2,3$, we infer that $f = F(u_1', u_3')$. Now, if we substitute the later expression of f in $A_2 f = 0$, we have

$$\left[(u_1')^2 + 2\alpha_2 u_1' u_3'\right]\partial_{u_1'} F + \left[\alpha_2 (u_3')^2 + 2u_1' u_3'\right]\partial_{u_3'} F = 0. \tag{5.33}$$

The solution of (5.33) is $f = G(I_{01})$, where $I_{01} = \frac{u_1' u_3'}{(\alpha_2 u_3' - u_1')^3}$. We have $X_4 I_{01} = I_{01}$. So, we set $I_1 = \int \frac{dI_{01}}{I_{01}}$ which leads to the following expression:

$$I_1 = \ln\left|\frac{u_1' u_3'}{(\alpha_2 u_3' - u_1')^3}\right|. \tag{5.34}$$

Using MATHEMATICA, one obtains

$$u_3' = \frac{u_1'}{\alpha_2} + u_1' B''(u_1', I_1) \pm \frac{1}{3(\alpha_2)^3 \exp(I_1)}\left[B''(u_1', I_1)\right]^{-1}, \tag{5.35}$$

where

$$B''(u_1', I_1) = \left[\frac{2}{3\left(9(u_1')^2\exp(2I_1) + \sqrt{(u_1')^3\exp(3I_1)\left(\mp 4 + 27u_1'\exp(I_1)\right)}\right)}\right]^{\frac{1}{3}}. \tag{5.36}$$

Step 2. The new variables are

$$x^0 = t, \quad x^1 = u_1, \quad x^2 = u_3, \quad x^3 = u_1', x^4 = I_1.$$

The operators become

$$X_1 = \frac{\partial}{\partial x^0}, \quad X_2 = \frac{\partial}{\partial x^1}, \quad X_3 = \frac{\partial}{\partial x^2}, \quad A_2 = \partial_{x^o} + x^3\partial_{x^1} + P(x^3, x^4)\partial_{x^2} + Q(x^3, x^4)\partial_{x^3},$$

where $P''(x^3, x^4)$ and $Q''(x^3, x^4)$ are given by

$$P''(x^3, x^4) = x^3 B''(x^3, x^4) \pm \frac{\exp(-x^4)}{3(\alpha_2)^3}\left(B''(x^3, x^4)\right)^{-1} + \frac{x^3}{\alpha_2}, \tag{5.37}$$

$$Q''(x^3, x^4) = 3(x^3)^2 + 2\alpha_2 (x^3)^2 B''(x^3, x^4) \pm \frac{2x^3\exp(-x^4)}{3(\alpha_2)^2}\left(B''(x^3, x^4)\right)^{-1}. \tag{5.38}$$

One can verify that $Q''(x^3,x^4) \neq 0$. We deal with the system

$$A_2 f = 0, \quad X_i f = 0, i = 1,2. \tag{5.39}$$

From $X_i f = 0$, $i = 1,2$, it comes that $f = F(x^2,x^3)$. Then $A_2 f = 0$ gives

$$P''(x^3,x^4)\partial_{x^2}F + Q''(x^3,x^4)\partial_{x^3}F = 0. \tag{5.40}$$

The solution of the equation (5.40) is the normalized solution

$$I_2 = x^2 - \int \frac{P''(x^3,x^4)}{Q''(x^3,x^4)}dx^3. \tag{5.41}$$

Step 3. We consider the coordinates

$$y^0 = t, \quad y^1 = u_1,\ y^2 = u_3,\ y^3 = I_2,\ y^4 = I_1.$$

In these coordinates we have

$$X_1 = \frac{\partial}{\partial y^0}, \quad X_2 = \frac{\partial}{\partial y^1},\ A_2 = \partial_{y^o} + y^3\partial_{y^1} + Q''(y^3,y^4)\partial_{y^3}.$$

The system is

$$A_2 f = 0,\ X_1 f = 0. \tag{5.42}$$

From $X_1 f = 0$, we deduce that $f = F(y^1,y^3)$. Then, equation $A_2 f = 0$ becomes

$$y^3\partial_{y^1}F + Q''(y^3,y^4)\partial_{y^3}F = 0. \tag{5.43}$$

The solution of the later is already normalized and is given by

$$I_3 = y^1 - \int \frac{y^3 dy^3}{Q''(y^3,y^4)}. \tag{5.44}$$

Step 4. Now we use the coordinates:

$$z^0 = t, \quad z^1 = u_1, \quad z^2 = I_3, \quad z^3 = I_2, \quad z^4 = I_1.$$

In terms of these coordinates the operators become

$$X_1 = \partial_{z^o}, \quad A_2 = \partial_{z^0} + Q''(z^3,z^4)\partial_{z^3}. \tag{5.45}$$

Now equation $A_2 f = 0$ implies that

$$\partial_{z^0}F + Q''(z^3,z^4)\partial_{z^3}F = 0, \tag{5.46}$$

for $f = F(z^0,z^3)$. Integrating (5.46), we obtain the following solution normalized

$$I_4 = z^0 - \int \frac{1}{Q''(z^3,z^4)}dz^3. \tag{5.47}$$

$\square$

Finally, we have the following result.

Theorem 5.9. *If $u_1'u_3' + u_2'u_4' = 0$, $u_1' \neq u_4'$, $u_1'u_4' \neq 0$, $u_3' \neq 0$, $u_2' \neq 0$ and $u_1' \neq \alpha_2 u_3'$, then*
1. $u_2(t) = -\frac{1}{\alpha_2}u_1(t) + p_1$ *and* $u_4(t) = \alpha_2 u_3(t) + p_2$, *where* $p_1, p_2 \in \mathbb{R}$;
2. *the system* $u_1'' = (u_1')^2 + 2\alpha_2 u_1'u_3'$, $u_3'' = \alpha_2(u_3')^2 + 2u_1'u_3'$ *is solvable by quadratures and has the following four first integrals:*

$$I_1 = \ln\left|\frac{u_1'u_3'}{(u_3' - u_1')^3}\right|, I_2 = u_4 - \int \frac{P''(u_1', I_1)du_1'}{Q''(u_1', I_1)}, I_3 = u_1 - \int \frac{u_1'}{Q''(u_1', I_1)}du_1', I_4 = t - \int \frac{du_1'}{Q''(u_1', I_1)},$$

where $P''(u_1', I_1)$ *and* $Q''(u_1', I_1)$ *are expressions depending only on* u_1' *and* I_1.

Now we examine the case where $\det(M_{A_2}) = 0$, that is the case where $u_1' = \alpha_2 u_3'$. In this case, X_i, $i = 1,2,3,4$ and A_2 are connected. Then, the second equation of (5.29) becomes $u_3'' = 3\alpha_2(u_3')^2$. The later equation integrates to

$$u_3(t) = -\frac{1}{3\alpha_2}\ln|3\alpha_2 t + q| + r_1, \tag{5.48}$$

where $r_1 \in \mathbb{R}$. It comes from $u_1' = \alpha_2 u_3'$ that $u_1 = \alpha_2 u_3 + r_2'$, $(r_2' \in \mathbb{R})$. That is

$$u_1(t) = -\frac{1}{3}\ln|3\alpha_2 t + q| + r_2, \tag{5.49}$$

where $r_2 \in \mathbb{R}$. Now, from (5.28) and the equality $\alpha_1\alpha_2 = -1$, we deduce that there exists r_3, r_4 in $\mathbb{R}$ such that

$$u_2(t) = \frac{1}{3\alpha_2}\ln|3\alpha_2 t + q| + r_3, \quad u_4(t) = -\frac{1}{3\alpha_2}\ln|3\alpha_2 t + q| + r_4. \tag{5.50}$$

So we have proved the following.

Proposition 5.10. *If $u_1'u_3' + u_2'u_4' = 0$, $u_1' \neq u_4'$, $u_1'u_4' \neq 0$, $u_3' \neq 0$, $u_2' \neq 0$ and $u_1' = \alpha_2 u_3'$ for some $\alpha_2 \in \mathbb{R}^*$, then the curve $\gamma : t \mapsto (u_1(t), u_2(t), u_3(t), u_4(t))$ given by*

$$\gamma(t) = \left(-\frac{1}{3}\ln|3\alpha_2 t + q| + r_2; -\frac{1}{3\alpha_2}\ln|3\alpha_2 t + q| + r_1; \frac{1}{3\alpha_2}\ln|3\alpha_2 t + q| + r_3; -\frac{1}{3\alpha_2}\ln|3\alpha_2 t + q| + r_4\right),$$

where $r_1, r_2, r_3, r_4, q \in \mathbb{R}$, is a geodesic of $\mathcal{D}(G, \nabla)$. This geodesic is not complete as it is not defined for $t = -\frac{q}{3\alpha_2}$.

6 Discussions

The system of four second-order ODEs (1.1), which admits an 8-dimensional symmetry Lie algebra is solvable by quadratures. By studying the conditions of Lemma 3.1 due to Wafo Soh and Mahomed ([16]) and related to the integrability of the system (1.1), two cases arise. In the first case, the symmetry generators of the system and the equations vector field (the operator associated to the system) are unconnected. In this case, the system (1.1) is reducible to quadratures and the eight first integrals are found, although not completely explicit. The second situation corresponds to cases where the conditions of Lemma 3.1 are not satisfied. In these cases, the geodesics of the Drinfel'd double of the affine Lie group of $\mathbb{R}$ are sometimes completely determined and they are not complete as they are not defined for all t in $\mathbb{R}$. This confirms Theorem 2.2 which states that the affine manifold $(\mathcal{D}(G,r),\nabla)$ is not geodesically complete. In this second situation, we also meet the cases where we are just able to find the explicit expressions of only two components of the geodesics; the others two components being characterized by the first integrals of the system of ODEs they satisfy.

Ultimately, the integration of the system (1.1) is very delicate. The use of Lie theory can make steps towards obtaining solutions, or at least the first integrals, of the system. For this purpose, the method of Wafo Soh and Mahomed has been helpful.

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References

[1] **Agaoka, Y.**, Uniqueness of left invariant symplectic structures on the affine Lie group. *Proc. Amer. Math. Soc.* **129** (2001), no 9, 2753 – 2762.

[2] **Bajo, I. ; Benayadi, S. ; Medina, A.**, Symplectic structures on quadratic Lie algebras. *J. Algebra* **316**, (2007), no 1, 174 – 188.

[3] **Bluman, G. W. ; Kumei, S.**, Symmetries and Differential Equations. *Springer Verlag*, New York, 1989.

[4] **Bordemann, M. ; Medina, A. ; Ouadfel, A.**, Le groupe affine comme variété symplectique. *Tôhoku Math. J. (2)* **45**, (1993), no 3, 423 – 436.

[5] **Campoamor-Stursberg, R.**, Systems of Second-Order Linear ODEs with Constant Coefficients and their Symmetries II: the Case of Non-diagonal Coefficient Matrices. *Commun. Nonlinear Sci. Numer. Simul.* **17**, (2012), no 3, 1178 – 1193.

[6] **Chu B.-Y.**, Symplectic homogeneous spaces. *Trans. Amer. Math. Soc.* **197**, (1974), 145 – 159.

[7] **Diatta, A. ; Medina, A.**, Classical Yang-Baxter Equation and Left Invariant Affine Geometry on Lie Groups. *Manuscripta Math.* **114**, (2004), no. 4, 477 – 486.

[8] **Eisenhart, L. P.**, Continuous Groups of Transformations. *Princeton Univ. Press*, Princeton, 1933.

[9] **Gorringe, V. M. ; Leach, P. G. L.**, Lie point Symmetries for Systems of second-order Linear Odinary Differential Equations. *Quaestiones Math.* **111**, (1988), no 1, 95 – 117.

[10] **Lichnerowicz, A. ; Medina, A.**, On Lie groups with left invariant symplectic or Kählerian structures. *Lett. Math. Phys.* **16**, (1988), no 3, 225 – 235.

[11] **Manga, B.**, On the Geometry of Cotangent Bundles of Lie Groups. *PhD Thesis, Université d'Abomey-Calavi (UAC), Institut de Mathématiques et de Sciences Physiques (IMSP)*, Benin, 2010.

[12] **Naz, R.** ; **Naeem, I.** ; **Mahomed, F. M.**, First Integrals for Two Linearly Coupled Nonlinear Duffing Oscillators. *Math. Probl. Eng.*, **2011**, Art. ID 831647, 14 pp.

[13] **Olver, P. J.**, Applications of Lie Groups to Differential Equations. 2^{nd} Ed., GTM, **Vol. 107**, *Springer Verlag*, New York, 1993.

[14] **Olver, P. J.**, Equivalence, Invariants and Symmetry. *Cambridge University Press*, 1995.

[15] **Ovsiannikov, L. V.**, Group Analysis of Differential Equations. *Academic Press*, New York, 1982.

[16] **Wafo Soh, C. ; Mahomed F. M.**, Reduction of Order for Systems of Ordinary Differential Equations. *J. Nonlinear Math. Phys.* **11**, (2004),no 1, 13 – 20.

[17] **Wafo Soh, C. ; Mahomed F. M.**, Canonical Forms for systems of two second-order Ordinary Differential Equations. *J. Phys. A* **34**, (2001), no 13, 2883-2911.

[18] **Wafo Soh, C. ; Mahomed, F. M.**, Linearization criteria for a system of second-order ordinary differntial equations. *Internat. J. Non-Linear Mech.* **36**, (2001),no 4, 671-677.