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Reflected Generalized BSDEs with Random Time and Applications*

Auguste Aman†
UFR de Mathématiques et Informatique
Université Félix H. Boigny de Cocody, Côte d'Ivoire
22 BP 582 Abidjan 22

Abouo Elouaflin‡
UFR de Mathématiques et Informatique
Université Félix H. Boigny de Cocody, Côte d'Ivoire
22 BP 582 Abidjan 22

Modeste N'Zi§
UFR de Mathématiques et Informatique
Université Félix H. Boigny de Cocody, Côte d'Ivoire
22 BP 582 Abidjan 22

Abstract

In this paper, we aim to study solutions of reflected generalized BSDEs, involving the integral with respect to a continuous process, which is the local time of the diffusion on the boundary. We consider both a finite random terminal and a infinite horizon. In both case, we establish an existence and uniqueness result. As application, we give a characterization of an American pricing option in infinite horizon; and we also give a probabilistic formula for the viscosity solution of an obstacle problem for elliptic PDEs with a nonlinear Neumann boundary condition.

Keywords: American option pricing, elliptic PDEs, generalized backward stochastic differential equations, Neumann boundary condition, viscosity solution.
MSC: 60H20, 60H30, 60H99

1 Introduction

The theory of backward stochastic differential equations (BSDEs in short) was developed by Pardoux and Peng [16]. Precisely, given a data (ξ, f) consisting of a progressively measurable process f, so-called the generator, and a square integrable random variable ξ, they

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†E-mail address: `augusteaman5@yahoo.fr, Corresponding author`
‡E-mail address: `elabouo@yahoo.fr`
§E-mail address: `modestenzi@yahoo.fr`

proved the existence and uniqueness of an adapted process (Y,Z) solution to the following BSDEs:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T. \tag{1.1}$$

These equations have attracted great interest due to their connections with mathematical finance [6, 2], stochastic control and stochastic games [8]. Furthermore, it was shown in various papers that BSDEs give the probabilistic representation for the solution (at least in the viscosity sense) of a large class of systems of semi-linear parabolic partial differential equations (PDEs, in short) (see [15] for example).

Next, generalized backward stochastic differential equations (for short GBSDEs) has been considered by Pardoux and Zhang [17] as an extension of nonlinear BSDEs which involves an integral with respect to a non-decreasing and continuous process. More precisely, we have

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,\mathrm{d}s + \int_t^T g(s,Y_s)\,\mathrm{d}G_s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T, \tag{1.2}$$

where G and g are respectively a non-decreasing and continuous process and a progressively measurable function. They proved that process (Y,Z) solution of (1.2) provides probabilistic representation of viscosity solutions of both parabolic and elliptic PDEs with Neumann boundary condition. On other hand, El Karoui et al. [7] have considered the reflected BSDEs: for all $0 \le t \le T$

$$\begin{cases} (i)\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,\mathrm{d}s + K_T - K_t - \int_t^T Z_s\,\mathrm{d}B_s, \\ (ii)\ Y_t \ge S_t, \\ (iii)\ K \text{ is a non-decreasing and continuous process such that } K_0 = 0 \text{ and} \\ \int_t^T (Y_t - S_t)dK_s = 0 \quad \text{a.s.} \end{cases} \tag{1.3}$$

The solution of reflected BSDEs is a triplet of processes (Y,Z,K) where the non-decreasing and continuous process K is introduced to pushes the component Y upwards so that it may remain above the obstacle process S. Intuitively, dK_t/dt represents the amount of "push upwards" that we add to $-(dY_t/dt)$, so that the constraint (ii) is satisfied. In particular, condition (iii) means that the push is minimal, in the sense that we push only when the constraint is saturated, that is, when $Y_t = S_t$. In this setting, many others results have been established in the literature, among others, we note the work of Hamadène et al [9, 10], Cvitanic and Ma [3], Hamadène and Ouknine [11].

Recently, Ren and Xia [18] gave a probabilistic formula for the viscosity solution of an obstacle problem for parabolic PDEs with a nonlinear Neumann boundary condition. They show the connection with such PDEs and the reflected GBSDEs:

$$\begin{cases} (i)\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,\mathrm{d}s + \int_t^T g(s,Y_s)\,\mathrm{d}G_s + K_T - K_t - \int_t^T Z_s\,\mathrm{d}B_s, \\ (ii)\ Y_t \ge S_t, \\ (iii)\ K \text{ is a non-decreasing and continuous process such that } K_0 = 0 \text{ and} \\ \int_t^T (Y_t - S_t)dK_s = 0 \quad \text{a.s.} \end{cases} \tag{1.4}$$

In all above results of reflected BSDEs, the terminal horizon is supposed deterministic and the coefficients may be Lipschitz. These restrictive conditions limit the scope of several applications (finance, stochastic control, stochastic games, PDEs, etc,··). Let us consider, for example, the American option pricing framework. Here, the investor, in order to protect his advantages, can stop his controlling at any time before the maturity time. This is the first time when the financial asset price process, which is forced to live in a bounded domain, is on its boundary.

To correct this shortcoming, this paper is devoted to derive existence and uniqueness result to reflected GBSDEs with random terminal time and non Lipchitz coefficients. As application, we give an optimal stopping time problem related to American pricing option, using a infinite horizon reflected GBSDEs. With a finite random time, we derive a probabilistic formula for the viscosity solution of an obstacle problem for elliptic PDEs with a nonlinear Neumann boundary condition. The rest of this paper is organized as follows. We precise our problem in section 2. Section 3 and Section 4 are devoted to the main results. In section 5, we give as an application, the connection with American option pricing and an obstacle problem for elliptic PDEs with nonlinear Neumann boundary condition.

2 Formulation of the problem

Let $(W_t)_{t\geq 0}$ denote a d-dimensional Brownian Motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and for $t \geq 0, \{\mathcal{F}_t\}$ is the σ-algebra $\sigma(B_s,\ 0 \leq s \leq t)$, augmented with the $\mathbb{P}$-null set of $\mathcal{F}$ and $\mathcal{F}_\infty = \bigcup_{t\geq 0} \mathcal{F}_t$.
For any $d \geq 1$, we consider the following spaces of processes:

1. $M^2([0,T];\mathbb{R}^d)$ denotes the Banach space of all equivalence classes (with respect to the measure $d\mathbb{P}\otimes dt$) where each equivalence class contains an d-dimensional $\mathcal{F}_t$-progressively measurable stochastic process $\varphi_t; t \in [0,T]$, such that: for all $\lambda, \mu > 0$ $\|\varphi\|^2_{M^2} = \mathbb{E}\int_0^T e^{\lambda t+\mu G_t}|\varphi_t|^2 dt < +\infty;\ \ t \in [0,T]$.

2. $S^2([0,T];\mathbb{R})$ is the set of one dimensional, $\mathcal{F}_t$-measurable and continuous stochastic processes which satisfy: for all λ, μ, $\|\varphi\|^2_{S^2} = \mathbb{E}\Big(\sup_{0\leq t\leq T} e^{\lambda t+\mu G_t}|\varphi_t|^2\Big) < +\infty$.

In addition, we give the following assumptions:

$$
\mathbf{(A1)}\begin{cases} (i)\ \tau \text{ is a } \mathcal{F}_t\text{-stopping time.} \\[1ex] (ii)\ (G_t)_{t\geq 0} \text{ is a continuous real valued non-decreasing and continuous } \mathcal{F}_t\text{-progressively measurable} \\ \text{process verifying } G_0 = 0 \end{cases}
$$

(A2) f and g are $\mathbb{R}$-valued measurable functions defined respectively on $\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^d$ and $\Omega\times\mathbb{R}_+\times\mathbb{R}$ such that there are constants $\alpha \in \mathbb{R}$, $\beta < 0$, $K > 0$, $\lambda > 2|\alpha| + K^2$ and $\mu > 2|\beta|$ and $[1,+\infty)$-valued process $\{\varphi_t,\ \psi_t\}_{t\geq 0}$ satisfying

(i) $\forall t, \forall z, y \longmapsto (f(t,y,z), g(t,y))$ is continuous;

(ii) $(\omega, t) \longmapsto (f(\omega,t,y,z), g(\omega,t,y))$ is $\mathcal{F}_t$-progressively measurable;

(iii) $\forall t, \forall y, \forall (z,z')$, $|f(t,y,z) - f(t,y,z')| \leq K|z - z'|$;

(iv) $\forall t, \forall z, \forall (y,y')$, $(y - y')(f(t,y,z) - f(t,y',z)) \leq \alpha |y - y'|^2$;

(v) $\forall t$, $\forall (y,y')$, $(y - y')(g(t,y) - g(t,y')) \leq \beta |y - y'|^2$;

(vi) $\forall t, \forall y, \forall z$, $|f(t,y,z)| \leq \varphi_t + K(|y| + |z|)$, $|g(t,y)| \leq \psi_t + K|y|$;

(vii) $\mathbb{E}\Big[\int_0^\tau e^{\lambda s + \mu G(s)}[\varphi(s)^2 ds + \psi(s)^2]dG_s\Big] < \infty$.

(**A3**) ξ is a $\mathcal{F}_\tau$-measurable variable such that $\mathbb{E}(e^{\lambda\tau + \mu G(\tau)}|\xi|^2) < +\infty$.

(**A4**) $(S_t)_{t\geq 0}$ is a continuous progressively measurable real-valued process satisfying:

(i) $\mathbb{E}\Big(\sup_{0\leq t\leq \tau} e^{\lambda t + \mu G_t}(S_t^+)^2\Big) < +\infty$;

(ii) $S_\tau \leq \xi$ $\mathbb{P}$ a.s.

Remark 2.1. Let us note that our monotonicity condition (**A2**)-(v) is not restrictive. Indeed if we assume $\beta > 0$, (Y_t, Z_t, K_t) solves the reflected GBSDE in (2.1) or (2.2) if and only if for every (some) $\eta > 0$ the pair $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t) = (e^{\eta G_t} Y_t, e^{\eta G_t} Z_t, e^{\eta G_t} K_t)$ solves an analogous reflected GBSDE, with f and g replaced respectively by

$$\begin{aligned} \bar{f}(t,y,z) &= e^{\eta G_t} f(t, e^{-\eta G_t} y, e^{-\eta G_t} z), \\ \bar{g}(t,y,z) &= e^{\eta G_t} g(t, e^{-\eta G_t} y). \end{aligned}$$

Then we can always choose η such that the function $\bar{g}$ satisfies (**A2**)-(v).

Remark 2.2.

1. Let (τ, G) satisfy the following conditions

   (i) $\mathbb{E}(e^{\lambda\tau}) < +\infty$,

   (ii) $\mathbb{E}(e^{\mu G_\tau}) < +\infty$,

   where the process G forces some diffusion to stay on some bounded domaine such that the above stopping time τ is its first time on this domaine boundary's (see Section 4, for more detail).

2. The condition (**A4**)-(i) implies the following:

   (**A4**)-(i') $\mathbb{E}\Big(\sup_{0\leq t\leq\tau}(S_t^+)^2\Big) < +\infty$.

Let (τ, ξ, f, g, S) be the data satisfying the previous conditions. We want to construct an adapted processes $(Y_t, Z_t, K_t)_{t\geq 0}$ solution of the reflected GBSDE

$$-dY_t = \mathbf{1}_{t\leq\tau} f(t, Y_t, Z_t)dt + \mathbf{1}_{t\leq\tau} g(t, Y_t)dG_t + dK_t - Z_t dW_t, \quad Y_\tau = \xi \tag{2.1}$$

or equivalently

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(t,Y_t,Z_t)dt + \int_{t\wedge\tau}^{\tau} g(t,Y_t)dG_t - \int_{t\wedge\tau}^{\tau} Z_t dW_t + K_\tau - K_{t\wedge\tau}. \tag{2.2}$$

Let us first recall the following

Definition 2.3. The solution to the equation (2.1) is a triplet of progressively measurable processes $(Y_t,Z_t,K_t)_{t\geq 0}$ with values in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ such that

1. $Y\in\mathcal{S}^2([0,\tau];\mathbb{R})$, $Z\in\mathcal{M}^2([0,\tau];\mathbb{R}^d)$;
2. for each nonnegative real T, $\forall t\in[0,\tau]$,
$$Y_t = Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s,Z_s)ds + \int_{t\wedge\tau}^{T\wedge\tau} g(s,Y_s)dG_s - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s + K_{T\wedge\tau} - K_{t\wedge\tau};$$
3. $Y_t \geq S_t, \quad t\leq\tau$;
4. For all $t\geq\tau$ *a.s.*, $Y_t=\xi,\ Z_t=0,\ K_t=K_\tau$;
5. K is a non-decreasing continuous process such that $K_0=0$ and $\int_0^\tau (Y_t - S_t)\,dK_t = 0$ a.s.

3 Reflected GBSDEs with finite random terminal time

The aim of this section is to prove the first main result of this paper, concerning the existence and uniqueness result for reflected GBSDEs (2.1) when the random time τ is suppose to be finite.

Theorem 3.1. *Assume that* (**A1**)-(**A4**) *hold. Moreover we suppose the obstacle process* $(S_t)_{t\geq 0}$ *to be an Itô process of the form* $dS_t = m_t\mathbf{1}_{[0,\tau]}dt + v_t\mathbf{1}_{[0,\tau]}dW_t$, *such that* $\mathbb{E}\left(\int_0^\tau e^{\lambda s+\mu G(s)}\left(|m_s|^2+|v_s|^2\right)ds\right) < +\infty$. *Then there exists a unique triple* (Y,Z,K) *solution of reflected GBSDE* (2.1).

Remark 3.2. If the random variable $\xi\equiv 0$ a.s., the condition (**A3**) remain true and Theorem 3.1 is available with assumptions (**A1**)-(**A4**), taking $\tau=+\infty$.

Before giving the proof of the Theorem 3.1, let us state this result (see [13], Lemma 8) which will be useful later.

Lemma 3.3. *Let* $(k^n : n\geq 1)$ *and* $(y^n : n\geq 1)$ *be two sequences in* $C([0,T],\mathbb{R}^d)$ *which converge uniformly to k and y respectively. Assume that* k^n *is of bounded variation and such that* $\sup_{n\in N^*}\|k^n\|_T < +\infty$, *where* $\|.\|_T$ *stands for the total variation on* $[0,T]$. *Then*

$$\int_0^T \langle y_s^n, dk_s^n\rangle \longrightarrow \int_0^T \langle y_s, dk_s\rangle.$$

Proof. We adopt this strategy for the proof.

Existence. For each integer n, let us denote $\xi_n = \mathbb{E}(\xi|\mathcal{F}_n)$ and consider the data $(\xi_n, \mathbf{1}_{[0,\tau]}f, \mathbf{1}_{[0,\tau]}g, S_{.\wedge\tau})$. Under (**A1**)-(**A4**), one can show, using the same argument as in [18] that there exists a unique process $(\overline{Y}^n, \overline{Z}^n, \overline{K}^n)$, solution of the classical (deterministic terminal time) reflected GBSDE

$$\begin{aligned}
\overline{Y}^n_t &= \xi_n + \int_t^n \mathbf{1}_{[0,\tau]}f(s, \overline{Y}^n_s, \overline{Z}^n_s)ds + \int_t^n \mathbf{1}_{[0,\tau]}g(s, \overline{Y}^n_s)dG_s \\
&\quad - \int_t^n \overline{Z}^n_s dW_s + \overline{K}^n_n - \overline{K}^n_t,\ 0 \le t \le n,
\end{aligned} \tag{3.1}$$

satisfying: $\overline{Y}^n_t \ge S_t$ such that

$$\int_0^{n\wedge\tau} (\overline{Y}^n_t - S_t)d\overline{K}^n_t = 0. \tag{3.2}$$

Since ξ belongs to $L^2(\mathcal{F}_\tau)$, there exists a process $(\eta_t)_{t\ge 0}$ in $M^2(0,\tau;\mathbb{R}^d)$ such that

$$\xi = \mathbb{E}[\xi] + \int_0^\tau \eta_s dW_s$$

and, we define $(\overline{Y}^n, \overline{Z}^n, \overline{K}^n)$ on the whole time axis by setting:

$$\forall\, t > n, \overline{Y}^n_t = \mathbb{E}(\xi|\mathcal{F}_t) = \xi_t \quad \overline{Z}^n_t = \eta_t \mathbf{1}_{[0,\tau]} \text{ and } \quad \overline{K}^n_t = \overline{K}^n_n.$$

In the sequel, we consider the process (Y^n, Z^n, K^n) defined by: $Y^n_t = \overline{Y}^n_{t\wedge\tau}$, $Z^n_t = \overline{Z}^n_{t\wedge\tau}$ and $K^n_t = \overline{K}^n_{t\wedge\tau}$.

The rest of the proof will be split in several steps and, C denotes a positive constant which may vary from one line to another.

Step 1: *A priori estimates uniform in n.*
First, there exists a constant $C > 0$ such that for all $s \ge 0$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le\tau} e^{\lambda t + \mu G_t} \left|Y^n_t\right|^2 + \int_0^\tau e^{\lambda s + \mu G_s} \left[(\left|Y^n_s\right|^2 + \left|Z^n_s\right|^2) ds + \left|Y^n_s\right|^2 dG_s \right] + |K^n_\tau|^2 \right) \\
&\le C\mathbb{E}\left(e^{\lambda\tau + \mu G_\tau} |\xi|^2 + \int_0^\tau e^{\lambda s + \mu G_s} \left[\varphi^2(s)ds + \psi^2(s)dG_s \right] + \sup_{0\le t\le\tau} e^{\lambda t + \mu G_t} \left|(S_t)^+\right|^2 \right).
\end{aligned} \tag{3.3}$$

Indeed, for any arbitrarily small $\varepsilon > 0$ and any $\rho < 1$ arbitrarily close to one, there exists a constant $C > 0$ such that for all $s > 0$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$\begin{aligned}
2\langle y, f(s,y,z)\rangle &\le (2\alpha + \rho^{-1}K^2 + \varepsilon)|y|^2 + \rho|z|^2 + c\varphi^2(s), \\
2\langle y, g(s,y)\rangle &\le (2\beta + \varepsilon)|y|^2 + c\psi^2(s).
\end{aligned}$$

From these and Itô's formula, we deduce that for any arbitrarily small $\delta > 0$

$$
\begin{aligned}
&\mathbb{E}\left(e^{\lambda t+\mu G_t}|Y_t^n|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G_s}[(\bar{\lambda}|Y_s^n|^2 + \bar{\rho}|Z_s^n|^2)ds + \bar{\mu}|Y_s^n|^2 dG_s]\right) \\
\le\;& \mathbb{E}\left(e^{\lambda\tau+\mu G_\tau}|\xi|^2 + 2c\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G_s}\left[\varphi^2(s)ds + \psi^2(s)dG_s\right] + 2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G_s}\langle S_s, dK_s^n\rangle\right) \\
\le\;& \mathbb{E}\left(e^{\lambda\tau+\mu G_\tau}|\xi|^2 + 2c\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G_s}\left[\varphi^2(s)ds + \psi^2(s)dG_s\right]\right. \\
&\left. +\delta^{-1}\sup_{0\le t\le\tau} e^{\lambda s+\mu G_s}(S_s^+)^2 + \delta(K_\tau^n - K_t^n)^2\right),
\end{aligned}
\tag{3.4}
$$

where $\bar{\lambda} = \lambda - 2\alpha - \rho^{-1}K^2 - \varepsilon$, $\bar{\rho} = 1-\rho$ and $\bar{\mu} = \mu - 2\beta - \varepsilon$. We may choose ε and ρ such that $\bar{\lambda} > 0, \bar{\rho} > 0$ and $\bar{\mu} > 0$. From the reflected GBSDE (3.1), estimate (3.4) and for every λ' such that $0 < \lambda' < \min(\lambda, \mu)$, we have

$$
\begin{aligned}
&\delta\mathbb{E}\left|K_\tau^n - K_t^n\right|^2 \\
\le\;& \delta\mathbb{E}\left(\left|Y_t^n\right|^2 + |\xi|^2 + (\lambda')^{-1}\int_{t\wedge\tau}^{\tau} e^{\lambda' s}\left(\varphi^2(s) + \left|Y_s^n\right|^2 + \left|Z_s^n\right|^2\right)ds\right. \\
&\left. +(\lambda')^{-1}\int_{t\wedge\tau}^{\tau} e^{\lambda' G_s}\left(\psi^2(s) + \left|Y_s^n\right|^2\right)dG_s\right) \\
\le\;& \delta\mathbb{E}\left(e^{\lambda t+\mu G_t}\left|Y_t^n\right|^2 + e^{\lambda\tau+\mu G_\tau}|\xi|^2\right) \\
&+\delta(\lambda')^{-1}\mathbb{E}\left(\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G(s)}\left[\left|Y_s^n\right|^2 + \varphi^2(s) + \left|Z_s^n\right|^2\right]ds\right) \\
&+\delta(\lambda')^{-1}\mathbb{E}\left(\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G(s)}(|Y_s^n|^2 + \psi^2(s))dG_s\right).
\end{aligned}
$$

Chosen δ small enough such that $1-\delta(\lambda')^{-1} > 0$, $\bar{\bar{\lambda}} = \bar{\lambda} - \delta(\lambda')^{-1} > 0$, $\bar{\bar{\rho}} = \bar{\rho} - \delta(\lambda')^{-1} > 0$ and $\bar{\bar{\mu}} = \bar{\mu} - \delta(\lambda')^{-1} > 0$, we get

$$
\begin{aligned}
&\mathbb{E}\left[(1-\delta(\lambda')^{-1})e^{\lambda t+\mu G_t}|Y_t^n|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G_s}\left([\bar{\bar{\lambda}}|Y_s^n|^2 + \bar{\bar{\rho}}|Z_s^n|^2]ds + \bar{\bar{\mu}}|Y_s^n|^2 dG_s\right)\right] \\
\le\;& C\mathbb{E}\left(e^{\lambda\tau+\mu G_\tau}|\xi|^2 + \int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu G_s}[\varphi^2(s)ds + \psi^2(s)dG_s] + \sup_{0\le t\le\tau} e^{\lambda t+\mu G(t)}(S_t^+)^2\right).
\end{aligned}
$$

Therefore, the result follows by using Burkhölder-Davis-Gundy inequality.

Step 2: *Convergence of the sequence* (Y^n, Z^n, K^n).

For $m > n$, let us set $\Delta Y_t = Y_t^m - Y_t^n$, $\Delta Z_t = Z_t^m - Z_t^n$, $\Delta K_t = K_t^m - K_t^n$. In view of (3.1), we get

$$
\begin{aligned}
-d(\Delta Y)_t &= (f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m))ds + (g(s, Y_s^n) - g(s, Y_s^m))dG_s \\
&\quad -\Delta Z_t dW_t + d(\Delta K)_s,
\end{aligned}
$$

from which, Itô's formula and above assumptions yield

$$
\begin{aligned}
&e^{\lambda t+\mu G_t}|\Delta Y_t|^2 + \int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s+\mu G_s}[(\bar{\lambda}|\Delta Y_s|^2 + \bar{\rho}|\Delta Z_s|)ds + \bar{\mu}|\Delta Y_s|^2 dG_s] \\
\le\;& e^{\lambda m+\mu G_m}|\Delta Y_m|^2 + \int_{t\wedge\tau}^{m\wedge\tau}\langle\Delta Y_s, d(\Delta K_s)\rangle - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s+\mu G_s}\langle\Delta Y_s, \Delta Z_s dW_s\rangle.
\end{aligned}
\tag{3.5}
$$

Furthermore, since one can show that

$$\int_{t\wedge\tau}^{m\wedge\tau}\langle\Delta Y_s, d(\Delta K_s)\rangle \leq 0,$$

by taking expectation in both side of (3.5) and using Burkhölder-Davis-Gundy inequality, we get

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{\lambda t+\mu G_t}|\Delta Y_t|^2 + \int_0^\tau e^{\lambda s+\mu G_s}[(\bar{\lambda}|\Delta Y_s|^2 + \bar{\rho}|\Delta Z_s|)ds + \bar{\mu}|\Delta Y_s|^2 dG_s]\right)\\ \leq\ &\mathbb{E}\left(e^{\lambda(m\wedge\tau)+\mu G_{m\wedge\tau}}|\Delta Y_m|^2\right).\end{aligned}$$

But, since $\Delta Y_m = \xi_{m\wedge\tau} - \xi_{n\wedge\tau}$,

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{\lambda t+\mu G_t}|\Delta Y_t|^2 + \int_0^\tau e^{\lambda s+\mu G_s}[(\bar{\lambda}|\Delta Y_s|^2 + \bar{\rho}|\Delta Z_s|)ds + \bar{\mu}|\Delta Y_s|^2 dG_s]\right)$$

tends to zero as n, m goes to infinity. Therefore, (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}([0,\tau])\times\mathcal{M}^2(0,\tau,\mathbb{R}^d)$ so that it converges in $\mathcal{S}([0,T])\times\mathcal{M}^2(0,T,\mathbb{R}^d)$ to (Y,Z). On the other hand, since $(Y^n,Z^n)\to(Y,Z)$ in $\mathcal{S}([0,T])\times\mathcal{M}^2(0,T,\mathbb{R}^d)$, then there exists $(Y',Z')\in\mathcal{S}([0,T])\times\mathcal{M}^2(0,T,\mathbb{R}^d)$ and a subsequence which we still denote (Y^n,Z^n) such that $\forall n, |Y^n|\leq Y'$, $\|Z^n\|\leq Z'$ and $(Y^n,Z^n)\to(Y,Z)$, $dt\otimes d\mathbb{P}$ a.e. Therefore, since in virtue of (3.1),

$$K_t^n = Y_0^n - Y_t^n - \int_0^t f(s,Y_s^n,Z_s^n)ds - \int_0^t g(s,Y_s^n)dG_s + \int_0^t Z_s^n\, dW_s; \tag{3.6}$$

and according to the fact that f, g are continuous and

- $\sup_{n\geq 0}|f(s,Y_s^n,Z_s)| \leq f_s + K\left\{\sup_{n\geq 0}|Y_s^n| + \|Z_s\|\right\}$,
- $\sup_{n\geq 0}|g(s,Y_s^n)| \leq \psi_s + K\sup_{n\geq 0}|Y_s^n|$,
- $\mathbb{E}\int_t^T|f(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s)|^2 ds \leq C\mathbb{E}\int_t^T\|Z_s^n - Z_s\|^2 ds$,

it follows by the dominated convergence theorem that there exist a non-decreasing and continuous process K verifying, for all $t\in[0,T]$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|K_t^n - K_t\right|^2\right)\longrightarrow 0$$

as n goes to infinity.

Step 4 *The limit process (Y,Z,K) solves our reflected GBSDE (τ,ξ,f,g,S).*

Taking the limit in BSDE (3.1), we get $\mathbb{P}$-a.s. for every $T>0$,

$$Y_t = \xi + \int_t^{\tau\wedge T} f(s,Y_s,Z_s)ds + \int_t^{\tau\wedge T} g(s,Y_s)dG_s + K_{\tau\wedge T} - K_t - \int_t^{\tau\wedge T} Z_s dW_s,\ \forall t\in[0,\tau]$$

and for all $t\geq\tau$, $Y_t=\xi$, $Z_t=0$, $K_t=K_\tau$. It remains to show that (Y,Z,K) solves (2.1) or (2.2). From the previous estimates, we deduce that $\sup_{n\in\mathbb{N}^*}\mathbb{E}\|K^n\|_{L^2} < +\infty$. Next, by taking a subsequence, we may assume that $(Y^n_t, K^n_t)_{0\le t\le T}$ converges uniformly on t to $(Y_t, K_t)_{0\le t\le T}$. Therefore since $Y^n \ge S$ for all $n \in \mathbb{N}$, we obtain passing to the limit that

$$Y_t \ge S_t, \forall t \in [0,\tau].$$

On the other hand, since the stopping time τ is finite, there exist $n_0 \ge 1$ such that for all $n \ge n_0$,

$$\int_0^{n\wedge\tau} (Y^n_s - S_s)dK^n_s = \int_0^{\tau} (Y^n_s - S_s)dK^n_s = 0.$$

Hence, by passing into the limit, it follows from Lemma 3.3 that

$$\int_0^{\tau} (Y_s - S_s)\, dK_s = 0.$$

Uniqueness

Let (Y_t, Z_t, K_t) and (Y'_t, Z'_t, K'_t) be two solutions of the reflected GBSDE (2.1), and $(\bar{Y}_t, \bar{Z}_t, \bar{K}_t) = (Y_t - Y'_t, Z_t - Z'_t, K_t - K'_t)$. It follows from Itô's formula, the assumptions (iii), (iv) and (v) of (**A2**) that

$$\begin{aligned}
& e^{\lambda(t\wedge\tau)+\mu G_{t\wedge\tau}}|\bar{Y}_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s+\mu G_s}[\lambda|\bar{Y}_s|^2 ds + \mu|\bar{Y}_s|^2 dG_s + |\bar{Z}_s|^2 ds] \\
\le\; & e^{\lambda(T\wedge\tau)+\mu G_{T\wedge\tau}}\left|\bar{Y}_{T\wedge\tau}\right|^2 + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s+\mu G(s)}[\alpha|\bar{Y}_s|^2 + K|\bar{Y}_s|\times|\bar{Z}_s|^2]ds \\
& 2\beta\int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s+\mu G(s)}|\bar{Y}_s|^2 dG_s - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s+\mu G(s)}\langle \bar{Y}_s, \bar{Z}_s dW_s\rangle.
\end{aligned}$$

Hence, with $\rho < 1, \bar{\lambda} = \lambda - 2\alpha - \rho^{-1}K^2 > 0, \bar{\mu} = \mu - 2\beta > 0$,

$$\begin{aligned}
& \mathbb{E}\left(e^{\lambda(t\wedge\tau)+\mu G_{t\wedge\tau}}|\bar{Y}_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s+\mu G_s}[\lambda|\bar{Y}_s|^2 ds + \mu|\bar{Y}_s|^2 dG_s + (1-\rho)|\bar{Z}_s|^2 ds]\right) \\
\le\; & \mathbb{E}\left(e^{\lambda(T\wedge\tau)+\mu G_{T\wedge\tau}}|\bar{Y}_{T\wedge\tau}|^2\right),
\end{aligned}$$

and consequently, letting $T \to \infty$, dominated convergence theorem yields

$$\mathbb{E}\left(e^{\lambda(t\wedge\tau)+\mu G(t\wedge\tau)}\left|\bar{Y}_{t\wedge\tau}\right|^2\right) = 0.$$

Then for all t, $\bar{Y}_{t\wedge\tau} = 0$ and $\bar{Z}_{t\wedge\tau} = 0$. Moreover, since

$$\begin{aligned}
\bar{K}_{t\wedge\tau} &= \bar{Y}_0 - \bar{Y}_{t\wedge\tau} - \int_0^{t\wedge\tau} f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)ds \\
&\quad - \int_0^{t\wedge\tau} g(s, Y_s) - g(s, Y'_s)dG_s + \int_0^{t\wedge\tau} \bar{Z}_s dW_s,
\end{aligned}$$

$\bar{K}_{t\wedge\tau} = 0$ for all t. □

4 Infinite horizon reflected GBSDEs

In this section, we study the following infinite horizon reflected GBSDE:

$$Y_t = \xi + \int_t^{\infty} f(s, Y_s, Z_s)ds + \int_t^{\infty} g(s, Y_s)ds - \int_t^{\infty} Z_s dW_s + K_\infty - K_t,\ 0 \le t \le \infty. \tag{4.1}$$

Let us introduce some spaces which our discussion will be carried on.

$$\mathcal{S}^2 = \left\{\varphi_t, 0 \le t \le \infty,\ \text{is an} \mathcal{F}_t\text{-adapted process such that,} \mathbb{E}\left(\sup_{0\le t\le\infty} |\varphi_t|^2\right) < \infty\right\},$$

$$\mathcal{H}^2 = \left\{\varphi_t, 0 \le t \le \infty,\ \text{is an} \mathcal{F}_t\text{-adapted process such that, } \mathbb{E}\left(\int_0^{\infty} |\varphi_t|^2 dt\right) < \infty\right\},$$

Throughout the remaining part of the paper, we propose the following assumptions:

(**A2′**) $f : \Omega\times[0,\infty)\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $g : \Omega\times[0,\infty)\times\mathbb{R} \to \mathbb{R}$ are two measurable mappings such that there exist three positives deterministic processes u, v and v' satisfying

$$\int_0^{\infty} [(v_t + v_t'^2)dt + u_t dG_t] < +\infty. \tag{4.2}$$

such that

(i) $|f(t,y,z) - f(t,y',z')| \le v_t|y-y'| + v'_t|z-z'|$,

(ii) $|g(t,y) - g(t,y')| \le u_t|y-y'|$

(iii) $\langle y-y', g(t,y) - g(t,y')\rangle \le \beta|y-y'|^2$

(iii) $|f(t,y,z)| \le \varphi_t + K(|y|+|z|),\ \ |g(t,y)| \le \psi_t + K|y|$

(iv) $\mathbb{E}\left(\int_0^{\infty} \varphi_t^2 ds + \psi_t^2 dG_t\right) < \infty.$

(**A3′**) a terminal value $\xi \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$

(**A4′**) The barrier $(S_t,\ t \ge 0)$ is a continuous progressively measurable real-valued process such that

(i) $\mathbb{E}[\sup_{t\ge0}(S_t^+)^2] < \infty$

(ii) $\limsup_{t\nearrow\infty} S_t \le \xi,\ a.s.$

With all the above preparations, we have

Definition 4.1. A solution to reflected GBSDE associated with the data (ξ, f, g, S) is a triple (Y_t, Z_t, K_t) of $\mathcal{F}_t$-progressively measurable processes such that (4.1) holds and

(i) $Y \in \mathcal{S}^2,\ \ Z \in \mathcal{H}^2,\ \ K_\infty \in L^2$;

(ii) $Y_t \geq S_t,\ \ t < \infty$;

(iii) K_t is a non-decreasing continuous process such that $K_0 = 0$ and $\int_0^\infty (Y_t - S_t)dK_t = 0$.

Our approach to solve above reflected GBSDEs with infinite horizon is to use the snell envelope theory connected to the contraction method. For this, let us establish the same result in case functions f and g do not depend on (Y,Z) and satisfy

$$\mathbb{E}\left(\int_0^\infty |f(t)|^2 dt + \int_0^\infty |g(t)|^2 dG_t\right) < \infty. \tag{4.3}$$

More precisely we have the following reflected GBSDE:

$$Y_t = \xi + \int_t^\infty f(s)ds + \int_t^\infty g(s)dG_s - \int_t^\infty Z_s dW_s + K_\infty - K_t,\ \ t \in [0,\infty]. \tag{4.4}$$

Proposition 4.2. *Assume that* $(\mathbf{A3'})$,$(\mathbf{A4'})$ *and* (4.3) *hold. Then reflected GBSDE* (4.4) *associated with* (ξ, f, g, S) *has a unique solution* (Y,Z,K).

Proof. Let $(F_t)_{0\leq t\leq\infty}$ be the process defined as follows:

$$F_t = \int_0^t f(s)ds + \int_0^t g(s)dG_s + S_t \mathbf{1}_{t<\infty} + \xi \mathbf{1}_{t=\infty}.$$

Then for $t < \infty$, F is continuous $\mathcal{F}_t$-adapted process and $\sup_{0\leq t\leq\infty} F_t \in L^2(\Omega, \mathcal{F}_\infty)$. So, the Snell envelope of F is the smallest continuous supermartingale which dominates the process F and it is given by:

$$\mathcal{S}_t(F) = ess \sup_{v\in\mathcal{K}_t} \mathbb{E}(F_v|\mathcal{F}_t),$$

where $\mathcal{K}_t$ is the set of all $\mathcal{F}_s$-stopping times taking values in $[t,+\infty]$. Then, we have

$$\mathbb{E}\left(\sup_{0\leq t\leq\infty} [\mathcal{S}_t(F)]^2\right) < \infty$$

hence $(\mathcal{S}_t(F))_{0\leq t\leq\infty}$ is of class [D] (A process X is said to belong to Class [D] on $[0,+\infty]$ if the family of random variables $\{X_\tau :\ \tau \in \mathcal{K}_0\}$ is uniformly integrable). Therefore, it has the following Doob-Meyer decomposition:

$$\mathcal{S}_t(F) = \mathbb{E}\left(\xi + \int_0^\infty f(t)ds + \int_0^\infty g(t)dG_t + K_\infty|\mathcal{F}_t\right) - K_t$$

where $(K_t)_{0\leq t\leq\infty}$ is an $\mathcal{F}_t$-adapted continuous non-decreasing process such that $K_0 = 0$. By the theory of Snell envelope (see Ren and Hu, [19]) we have $\mathbb{E}(K_\infty)^2 < \infty$. Therefore we derive

$$\mathbb{E}\left[\sup_{0\leq t\leq\infty}\left|\mathbb{E}\left(\xi + \int_0^\infty f(t)ds + \int_0^\infty g(t)dG_t + K_\infty|\mathcal{F}_t\right)\right|^2\right] < \infty$$

and then, through the martingale representation there exists a continuous uniformly integrable process $(Z_s)_{0\leq s\leq\infty}$such that

$$\begin{aligned} M_t &= \mathbb{E}\Big(\xi+\int_0^\infty f(t)ds+\int_0^\infty g(t)dG_t+K_\infty|\mathcal{F}_t\Big) \\ &= M_0+\int_0^t Z_s dW_s. \end{aligned}$$

Now let us set

$$Y_t = ess\sup_{\nu\in\mathcal{K}_t}\mathbb{E}\Big[\int_t^\nu f(s)ds+\int_t^\nu g(s)dG_s+S_\nu\mathbf{1}_{\nu<\infty}+\xi\mathbf{1}_{\nu=\infty}\Big].$$

Then

$$\begin{aligned} Y_t+\int_0^t f(s)ds+\int_0^t g(s)dG_s &= \mathcal{S}_t(F) \\ &= M_t-K_t \end{aligned}$$

henceforth, we have

$$Y_t+\int_0^\infty f(s)ds+\int_0^\infty g(s)dG_s = \xi+\int_0^\infty f(s)ds+\int_0^\infty g(s)dG_s+\int_0^t Z_s dW_s-K_t.$$

So, we obtain

$$Y_t = \xi+\int_t^\infty f(s)ds+\int_t^\infty g(s)dG_s+K_\infty-K_t-\int_t^\infty Z_s dW_s,\ 0\leq t\leq\infty.$$

Since, $Y_t+\int_0^t f(s)ds+\int_0^t g(s)dG_s = \mathcal{S}_t(F)$ and $\mathcal{S}_t(F)\geq F_t = \int_0^t f(s)ds+\int_0^t g(s)dG_s+S_t\mathbf{1}_{t<\infty}+\xi\mathbf{1}_{t=\infty}$, then $Y_t\geq S_t$.

Finally, by using again the theory of Snell envelope, we know that $\int_0^\infty(\mathcal{S}_t(F)-F_t)dK_t = 0$ i.e.

$$\int_0^\infty (Y_t-S_t)dK_t = \int_0^\infty(\mathcal{S}_t(F)-F_t)dK_t = 0.$$

Therefore, the triple (Y,Z,K) satisfies the reflected GBSDE (4.4) and properties (i)-(iii) above.

Let us prove uniqueness. If (Y',Z',K') is another solution of the reflected generalized GBSDE (4.4) associated with (ξ,f,g,S) satisfying properties (i)-(iii) above, define $\bar{Y} = Y-Y'$, $\bar{Z} = Z-Z'$, and $K = K-K'$. Using Itô's formula to $|\bar{Y}_t|^2$,

$$|\bar{Y}_t|^2+\int_t^\infty|\bar{Z}_s|^2 ds = 2\int_t^\infty \bar{Y}_s d\bar{K}_s-2\int_t^\infty \bar{Y}_s\bar{Z}_s dW_s, \tag{4.5}$$

by the integrable conditions (i)-(iii) and Burkholder-Davis-Gundy's inequality, we have

$$\mathbb{E}\Big(|\bar{Y}_t|^2+\int_t^\infty|\bar{Z}_s|^2 ds\Big) = 2\mathbb{E}\Big(\int_t^\infty \bar{Y}_s d\bar{K}_s\Big)\leq 0.$$

So $\mathbb{E}(\bar{Y}_t) = 0$ a.s. for all $t\in[0,\infty]$ and $\mathbb{E}\Big(\int_t^\infty|\bar{Z}_s|^2ds\Big) = 0$. Then $|\bar{Y}_t|^2 = |\bar{Z}_t|^2 = 0$ a.s., so that $Y = Y'$ by the continuity of $\bar{Y}_t$ and $Z = Z'$. Finally, it is easy to get $K = K'$ a.s. □

We now establish the main result of this section.

Theorem 4.3. *Assume that* (**A2′**), (**A3′**) *and* (**A4′**) *hold. Then the reflected GBSDE* (4.1) *associated with* (ξ, f, g, S) *has a unique solution* (Y, Z, K).

Proof. We first prove the uniqueness. Let (Y, Z, K) and (Y', Z', K') be two solutions of the reflected GBSDE (4.1) associated with (ξ, f, g, S). By use the notation of the uniqueness proof of Proposition 4.2, and applying Itô's formula to $|\bar{Y}_t|^2$, we have

$$
\begin{aligned}
|\bar{Y}_t|^2 + \int_t^\infty |\bar{Z}_s|^2 ds &= 2\int_t^\infty \bar{Y}_s(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s))ds + 2\int_t^\infty \bar{Y}_s(g(s, Y_s) - g(s, Y'_s))dG_s \\
&\quad +2\int_t^\infty \bar{Y}_s d\bar{K}_s - 2\int_t^\infty \bar{Y}_s \bar{Z} dW_s.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}\left(|\bar{Y}_t|^2 + \int_t^\infty |\bar{Z}_s|^2 ds\right) &\leq 2\mathbb{E}\int_t^\infty |\bar{Y}_s|(v_s|\bar{Y}_s| + v'_s|\bar{Z}_s|)ds \\
&\quad +2\beta\mathbb{E}\int_t^\infty |\bar{Y}_s|^2 dG_s + 2\mathbb{E}\int_t^\infty \bar{Y}_s d\bar{K}_s \\
&\leq \frac{1}{2}\mathbb{E}\int_t^\infty |\bar{Z}_s|^2 ds + \mathbb{E}\int_t^\infty (2v_s + 2v'^2_s)|\bar{Y}_s|^2 ds. \qquad (4.6)
\end{aligned}
$$

From Gronwall's lemma we obtain $\mathbb{E}|\bar{Y}_t|^2 = 0$ for all $t \in [0, \infty]$. Then $|\bar{Y}_t|^2 = 0$ a.s., so $Y = Y'$ by the continuity of $\bar{Y}$. Now, going back to (4.6), we have

$$
\mathbb{E}\int_0^\infty |\bar{Z}_s|^2 ds \leq \mathbb{E}\sup_{0\leq t\leq\infty} |\bar{Y}_s|^2 \int_0^\infty (2v_s + 2v'^2_s)ds,
$$

so

$$
\mathbb{E}\int_0^\infty |\bar{Z}_s|^2 ds = 0.
$$

Finally, we show easily that $K = K'$.

At last, It remains to prove the existence of (4.1). It will be divided into two steps.

Step 1. Assume $\left(\int_0^\infty v_s ds + u_s dG_s\right)^2 + \int_0^\infty v'^2_s ds < \frac{1}{32}$.

Let us denote $\mathcal{D} = \mathcal{S}^2 \times \mathcal{H}^2$ and $\|(Y, Z)\|_{\mathcal{D}} = \|Y\|^2_{\mathcal{S}^2} + \|Z\|^2_{\mathcal{H}^2}$. We define a mapping $\Psi : \mathcal{D} \to \mathcal{D}$ as follows: for any $(U, V) \in \mathcal{D}$, $(Y, Z) = \Psi(U, V)$ is a element of $\mathcal{D}$ such that (Y, Z, K) is a unique solution to reflected GBSDE associated with $(\xi, f(s, U_s, V_s), g(s, U_s), S)$. Similarly we define $(Y', Z') = \Psi(U', V')$ for $(U', V') \in \mathcal{D}$ and set $\bar{U} = U - U'$, $\bar{V} = V - V'$, $\bar{Y} = Y - Y'$, $\bar{Z} = Z - Z'$, $\bar{K} = K - K'$, $\bar{f} = f(s, U_s, V_s) - f(s, U'_s, V'_s)$ and $\bar{g} = g(s, U_s) - g(s, U'_s)$. From above we have

$$
\begin{aligned}
Y_t &= ess\sup_{\nu\in\mathcal{K}_t} \mathbb{E}\left(\int_t^\nu f(s, U_s, V_s)ds + \int_t^\nu g(s, U_s)dG_s + S_\nu \mathbf{1}_{\nu<\infty} + \xi\mathbf{1}_{\nu=\infty}|\mathcal{F}_t\right), \\
Y'_t &= ess\sup_{\nu\in\mathcal{K}_t} \mathbb{E}\left(\int_t^\nu f(s, U'_s, V'_s)ds + \int_t^\nu g(s, U'_s)dG_s + S_\nu \mathbf{1}_{\nu<\infty} + \xi\mathbf{1}_{\nu=\infty}|\mathcal{F}_t\right).
\end{aligned}
$$

Then

$$\begin{aligned}|\bar{Y}_t| &\leq ess\sup_{\nu\in\mathcal{K}_t}\mathbb{E}\left(\int_t^{\nu}|\bar{f}(s)|ds+\int_t^{\nu}|\bar{g}(s)|dG_s|\mathcal{F}_t\right)\\ &\leq \mathbb{E}\left(\int_0^{\infty}|\bar{f}(s)|ds+\int_0^{\infty}|\bar{g}(s)|dG_s|\mathcal{F}_t\right)\end{aligned}$$

which provides

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\infty}|\bar{Y}_t|^2\right) &\leq \mathbb{E}\left[\sup_{0\leq t\leq\infty}\mathbb{E}\left(\int_0^{\infty}|\bar{f}(s)|ds+\int_0^{\infty}|\bar{g}(s)|dG_s|\mathcal{F}_t\right)^2\right]\\ &\leq 4\mathbb{E}\left(\int_0^{\infty}|\bar{f}(s)|ds+\int_0^{\infty}|\bar{g}(s)|dG_s\right)^2\end{aligned}$$

by Doob's inequality. Using Itô's formula to $|\bar{Y}_t|^2$, we get

$$\begin{aligned}|\bar{Y}_t|^2+\int_t^{\infty}|\bar{Z}_s|^2ds &= 2\int_t^{\infty}\bar{Y}_s\bar{f}(s)ds+2\int_t^{\infty}\bar{Y}_s\bar{g}(s)ds+2\int_t^{\infty}\bar{Y}_sd\bar{K}_s-2\int_t^{\infty}\bar{Y}_s\bar{Z}_sdW_s.\\ &\leq 2\int_t^{\infty}\bar{Y}_s\bar{f}(s)ds-2\int_t^{\infty}\bar{Y}_s\bar{Z}_sdW_s.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left(\int_t^{\infty}|\bar{Z}_s|^2ds\right) &\leq 2\int_0^{\infty}\bar{Y}_s\bar{f}(s)ds\\ &\leq \mathbb{E}\left(\sup_{0\leq t\leq\infty}|Y_t|^2\right)+\mathbb{E}\left(\int_0^{\infty}|\bar{f}(s)|ds\right)^2.\\ &\leq 4\mathbb{E}\left(\int_0^{\infty}[|\bar{f}(s)|ds+|\bar{g}(s)|dG_s]\right)^2+\mathbb{E}\left(\int_0^{\infty}|\bar{f}(s)|ds\right)^2.\end{aligned}$$

From (**A2′**) we get

$$\begin{aligned}&\mathbb{E}\left(\int_0^{\infty}[|\bar{f}(s)|ds+|\bar{g}(s)|dG_s]\right)^2+\mathbb{E}\left(\int_0^{\infty}|\bar{f}(s)|ds\right)^2\\ \leq\ &\mathbb{E}\left(\int_0^{\infty}(v_s|\bar{U}_s|+v'_s|\bar{V}_s|)ds+u_s|\bar{U}_s|dG_s\right)^2\\ \leq\ &4\left[\left(\int_0^{\infty}v_sds+u_sdG_s\right)^2+\int_0^{\infty}v'^2ds\right]\|(\bar{U},\bar{V})\|_{\mathcal{D}}.\end{aligned}$$

Finally, we have

$$\|(\bar{Y},\bar{Z})\|_{\mathcal{D}}\leq 32\left[\left(\int_0^{\infty}v_sds+u_sdG_s\right)^2+\int_0^{\infty}v'^2ds\right]\|(\bar{U},\bar{V})\|_{\mathcal{D}}. \tag{4.7}$$

From the inequality $\left(\int_0^{\infty}v_sds+u_sdG_s\right)^2+\int_0^{\infty}v'^2_sds<\frac{1}{32}$ we infer that Ψ is a strict contraction and has a unique fixed point, which is a unique solution of the reflected GBSDE (4.1).

Step 2. For the general case i.e (4.2), there exists $T_0 > 0$ such that

$$\left(\int_{T_0}^{\infty} v_s ds + u_s dG_s\right)^2 + \int_{T_0}^{\infty} v_s'^2 ds < \frac{1}{32}.$$

From Step 1 we know that the reflected GBSDE

$$\begin{aligned}\widehat{Y}_t &= \xi + \int_t^{\infty} \mathbf{1}_{\{s\geq T_0\}} f(s,\widehat{Y}_s,\widehat{Z}_s)ds + \int_t^{\infty} \mathbf{1}_{\{s\geq T_0\}} g(s,\widehat{Y}_s)ds \\ &\quad - \int_t^{\infty} \widehat{Z}_s dW_s + \widehat{K}_\infty - \widehat{K}_t,\ 0 \leq t \leq \infty,\end{aligned} \tag{4.8}$$

has a unique solution $(\widehat{Y},\widehat{Z},\widehat{K})$. Then we consider the reflected GBSDE

$$\begin{aligned}\widetilde{Y}_t &= \xi + \int_t^{T_0} f(s,\widetilde{Y}_s,\widetilde{Z}_s)ds + \int_t^{T_0} g(s,\widetilde{Y}_s)ds \\ &\quad - \int_t^{T_0} \widetilde{Z}_s dW_s + \widetilde{K}_{T_0} - \widetilde{K}_t,\ 0 \leq t \leq T_0.\end{aligned} \tag{4.9}$$

It follows from [18], the existence of a unique solution $(\widetilde{Y},\widetilde{Z},\widetilde{K})$ of reflected GBSDE (4.9).

Let us set

$$Y_t = \begin{cases} \widetilde{Y}_t, & t\in[0,T_0], \\ \widehat{Y}_t, & t\in[T_0,\infty], \end{cases} \qquad Z_t = \begin{cases} \widetilde{Z}_t, & t\in[0,T_0], \\ \widehat{Z}_t, & t\in[T_0,\infty], \end{cases} \qquad K_t = \begin{cases} \widetilde{K}_t, & t\in[0,T_0] \\ \widetilde{K}_{T_0} + \widehat{K}_t - \widehat{K}_{T_0}, & t\in[T_0,\infty]. \end{cases}$$

If $t\in[T_0,\infty]$, $(\widehat{Y}_t,\widehat{Z}_t,\widehat{K}_t)$ is the solution of (4.8), and then $(\widehat{Y}_t,\widehat{Z}_t,\widetilde{K}_{T_0}+\widehat{K}_t-\widehat{K}_{T_0})$ also satisfies (4.8). Now, if $t\in[0,T_0]$, $(\widetilde{Y}_t,\widetilde{Z}_t,\widetilde{K}_t)$ is the solution of (4.9) and $\widetilde{Y}_{T_0} = \widehat{Y}_{T_0}$, $\widetilde{K}_{T_0} = \widetilde{K}_{T_0} + \widehat{K}_{T_0} - \widehat{K}_{T_0}$. So Y and K are continuous, and (Y,Z,K) is a unique solution of reflected GBSDE (4.1). □

5 Applications

In this section we will investigate the reflected generalized BSDEs, studied in the previous section, in Markovian framework, in order to give a interpretation of an American option pricing as well as a probabilistic representation of the viscosity solution of an elliptic obstacle problem.

5.1 A class of reflected diffusion process

Let $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ be functions such that

$$\left|b(x) - b(x')\right| + \left|\sigma(x) - \sigma(x')\right| \leq K\left|x - x'\right|.$$

Let Θ be an open connected bounded subset of $\mathbb{R}^d$, defined as follows: there exist some function $\phi \in C_b^2(\mathbb{R}^d)$ such that $\Theta = \{\phi > 0\}$, $\partial\Theta = \{\phi = 0\}$, and $|\nabla\phi(x)| = 1,\ x\in\partial\Theta$. Note that at any boundary point $x\in\partial\Theta$, $\nabla\phi(x)$ is a unit normal vector to the boundary, pointing

towards the interior of $\partial\Theta$.
By Lions and Szitman [12] (see also Saisho [20]) for each $x \in \overline{\Theta}$ there exists a unique pair of progressively measurable continuous processes $\{(X_s^x, G_s^x) : t \geq 0\}$, with values in $\overline{\Theta} \times \mathbb{R}_+$, such that

$$\begin{aligned}
& s \mapsto G_s^x \text{ is non-decreasing,} \\
& X_s^x = x + \int_0^s b(X_r^x)dr + \int_0^s \sigma(X_r^x)dW_r + \int_0^s \nabla\phi(X_r^x)dG_r^x, \quad s \geq 0, \\
& G_s^x = \int_0^s 1_{\{X_r^x \in \partial\Theta\}} dG_r^x.
\end{aligned} \tag{5.1}$$

Let state some properties of processes $\{(X_s^x, G_s^x), s \geq 0\}$. We refer the reader to Pardoux and Zhang, [17].

Proposition 5.1. *For each $T \geq 0$, there exits a constant C_T such that for all $x, x' \in \overline{\Theta}$*

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |X_s^x - X_s^{x'}|^4\right) \leq C_T |x - x'|^4$$

and

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |G_s^x - G_s^{x'}|^4\right) \leq C_T |x - x'|^4.$$

Moreover, there exists a constant C_p such that for all $(t, x) \in \mathbb{R}_+ \times \overline{\Theta}$,

$$\mathbb{E}(|G_t^x|^p) \leq C_p \left(1 + t^p\right),$$

and for each $\mu, t > 0$, there exists $C_{\mu,t}$ such that for all $x \in \overline{\Theta}$,

$$\mathbb{E}\left(e^{\mu G_t^x}\right) \leq C_{\mu,t}.$$

Since we state in Markovian framework, the (ξ, f, g, S) are defined as follows:

$$f(s,y,z) = f(s, X_s^x, y, z), \ g(s,y) = g(s, X_s^x, y), \ S_s = h(X_s^x),$$

where f, g satisfy the previous assumptions on random finite or infinite horizon and $h \in C(\mathbb{R}^d; \mathbb{R})$ with at most polynomial growth at infinity.

5.2 American option pricing revisited

In this section, we use the result on infinite horizon reflected GBSDEs with one barrier to deal with optimal stopping time problem. Roughly speaking, let us consider the following reflected GBSDE:

1.

$$
\begin{aligned}
Y_s^x &= \xi + \int_s^\infty f(r, X_r^x, Y_r^x, Z_r^x)dr + \int_s^\infty g(r, X_r^x, Y_r^x)dG_r^x \\
&\quad - \int_s^\infty Z_r^x dW_r + K_\infty^x - K_s^x,\ 0 \le s \le \infty,
\end{aligned} \tag{5.2}
$$

2. $Y_s^x \ge h(X_s^x)$,

3. $\mathbb{E}\Big(\sup_{0\le t\le\infty} |Y_t^x|^2 + \int_0^\infty \left|Z_r^x\right|^2 dr\Big) < +\infty$,

4. K_s^x is a non-decreasing process such that $K_0 = 0$ and $\int_0^\infty (Y_s^x - h(X_s^x))dK_s^x = 0$.

From Theorem 4.1, the previous reflected GBSDE has a unique solution (Y^x, Z^x, K^x). Unlike of the work of Cvitanic and Ma (see [2]), we interpret X^x in (5.1) as a price process of financial assets which might affect the wealth of a controller which is forced to live in a bounded domain; Y^x and Z^x are the wealth process and the trading strategy, respectively, of a "small" investor or a "small" shareholder in the market in the sense that both Y^x and Z^x might no affect the price X^x. The investor acts to protect his advantages so that he has possibility at any time $\theta \in \mathcal{K}$ (set of all $\mathcal{F}_s$-stopping time with values in $[0,\infty]$) to stop controlling. The control is not free. We define the pay off by

$$
\begin{aligned}
R(\theta) &= \mathbb{E}\Big\{\int_0^\theta f(r, X_r^x, Y_r^x, Z_r^x)dr + \int_0^\theta g(r, X_r^x, Y_r^x)dG_r^x \\
&\quad + h(X_\theta^x)\mathbf{1}_{\{\theta<\infty\}} + \xi \mathbf{1}_{\{\theta=\infty\}}\Big\}
\end{aligned}
$$

for all $\theta \in \mathcal{K}$. For the investor, $f(X^x, Y^x, Z^x)$, (resp. $f(X^x, Y^x, Z^x) + g(X^x, Y^x)\dot{G}^x$) is the instantaneous reward on Θ (resp. on $\partial\Theta$), and $h(X^x)$ and ξ are respectively the rewards if he decides to stop before or until infinite time. The problem is to look for an optimal strategy for the investor, i.e. a strategy $\widehat{\theta}$ such that

$$
R(\theta) \le R(\widehat{\theta}) \quad \text{for all } \theta \in \mathcal{K}.
$$

Now we give the main result of this section, an analogue of that in Cvitanic and Ma, [2].

Theorem 5.2. *Let $(Y_.^x, Z_.^x, K_.^x)$ be a unique solution of reflected GBSDE* (5.7)*. Then there exists an optimal stopping time given by*

$$
\widehat{\theta} = \begin{cases} inf\{t \in [0,\infty),\ \ Y_t^x \le h(X_t^x)\}, \\ \\ \infty \quad otherwise. \end{cases}
$$

Then $Y_0^x = R(\widehat{\theta})$, and $\widehat{\theta}$ is an optimal strategy for the investor.

Proof. Since (Y^x, Z^x, K^x) is a unique solution of reflected GBSDE (5.7), Y_0^x is deterministic and we have

$$\begin{aligned} Y_0^x = \mathbb{E}(Y_0^x) &= \mathbb{E}\bigg(\xi + \int_0^\infty f(X_r^x, Y_r^x, Z_r^x)dr + \int_0^\infty g(r, X_r^x, Y_r^x)dG_r^x \\ &\quad - \int_0^\infty Z_r^x dW_r + K_\infty^x\bigg) \\ &= \mathbb{E}\bigg(Y_{\widehat{\theta}}^x + \int_0^{\widehat{\theta}} f(X_r^x, Y_r^x, Z_r^x)dr + \int_0^{\widehat{\theta}} g(r, X_r^x, Y_r^x)dG_r^x \\ &\quad - \int_0^{\widehat{\theta}} Z_r^x dW_r + K_{\widehat{\theta}}^x\bigg) \end{aligned} \tag{5.3}$$

In view of $\widehat{\theta}$ and reflected GBSDE's properties one knows that the process K_t does not increase between 0 and $\widehat{\theta}$, hence then $K_{\widehat{\theta}} = 0$.

On the other hand, since $\int_0^{\widehat{\theta}} Z_r^x dW_r$ is a martingale, we get

$$Y_0^x = \mathbb{E}\bigg(Y_{\widehat{\theta}}^x + \int_0^{\widehat{\theta}} f(X_r^x, Y_r^x, Z_r^x)dr + \int_0^{\widehat{\theta}} g(r, X_r^x, Y_r^x)dG_r^x\bigg).$$

Next, $Y_{\widehat{\theta}}^x = h(X_{\widehat{\theta}_t}^x)\mathbf{1}_{\{\widehat{\theta}<\infty\}} + \xi\mathbf{1}_{\{\widehat{\theta}=\infty\}}$ a.s., implies $Y_0^x = R(\widehat{\theta})$.

Now from (5.3), we deduce that for every $\theta \in \mathcal{K}$,

$$\begin{aligned} Y_0^x &= \mathbb{E}\bigg\{Y_\theta^x + \int_0^\theta f(r, X_r^x, Y_r^x, Z_r^x)dr \\ &\quad + \int_0^\theta g(r, X_r^x, Y_r^x)dG_r^x + K_\theta^x\bigg\}. \end{aligned}$$

But $K_\theta^x \geq 0$ and $Y_\theta^x \geq h(X_\theta^x)\mathbf{1}_{\{\theta<\infty\}} + \xi\mathbf{1}_{\{\theta=\infty\}}$. Then,

$$\begin{aligned} R(\widehat{\theta}) = Y_0^x &\geq \mathbb{E}\bigg\{\int_0^\theta f(r, X_r^x, Y_r^x, Z_r^x)dr + \int_0^\theta g(r, X_r^x, Y_r^x)dG_r^x + h(X_\theta^x)\mathbf{1}_{\{\theta<\infty\}} + \xi\mathbf{1}_{\{\theta=\infty\}}\bigg\} \\ &\geq R(\theta). \end{aligned}$$

Hence the stopping time $\widehat{\theta}$ is optimal. □

5.3 An obstacle problem for elliptic PDEs with nonlinear Neumann boundary condition

In this subsection, we will show that in the Markovian case the solution of the reflected GBSDEs with random terminal time is a solution of an obstacle problem for elliptic PDEs with a nonlinear Neumann boundary condition. Let $\{X_t^x;\ t \geq 0\}$ is defined as above. For each $x \in \overline{\Theta}$, let consider the stopping time

$$\tau_x = \inf\{t > 0;\ X_t^x \in \partial\Theta\}.$$

We assume that

$$\mathbb{P}(\tau_x < +\infty) = 1 \quad \text{for all } \; x \in \overline{\Theta} \tag{5.4}$$

that the set of singular points

$$\Gamma = \{x \in \partial\Theta,\ \mathbb{P}(\tau_x > 0) > 0\} \quad \text{is empty,} \tag{5.5}$$

that for some λ, and all $x \in \overline{\Theta}$,

$$\mathbb{E}(e^{\lambda\tau_x}) < +\infty. \tag{5.6}$$

Let us recall the following result (see Proposition 5.2. in [15]):

Proposition 5.3. *Under the conditions* (5.5) *and* (5.6)*, the mapping $x \mapsto \tau_x$ is a.s. continuous on $\overline{\Theta}$.*

Let $l : \overline{\Theta} \to \mathbb{R}$ be a continuous function satisfies $l(x) \geq h(x)$. It follows from the results of the Section 3 that for all $x \in \overline{\Theta}$, there exists a unique triple (Y^x, Z^x, K^x) be the unique solution of the following reflected GBSDE:

1.
$$\begin{aligned} Y_s^x &= l(X_{\tau_x}^x) + \int_s^{\tau_x} f(r, X_r^x, Y_r^x, Z_r^x)dr + \int_s^{\tau_x} g(r, X_r^x, Y_r^x)dG_r^x \\ &\quad - \int_s^{\tau_x} Z_r^x dW_r + K_{\tau_x}^x - K_s^x,\ 0 \leq s \leq \tau_x, \end{aligned} \tag{5.7}$$

2. $Y_s^x \geq h(X_s^x)$,

3. $\mathbb{E}\Big(\sup_{0\leq t\leq\tau_x} |Y_t^x|^2 + \int_0^{\tau_x} \left|Z_r^x\right|^2 dr\Big) < +\infty$,

4. K_s^x is a non-decreasing process such that $K_0 = 0$ and $\int_0^{\tau_x}(Y_s^x - h(X_s^x))dK_s^x = 0$.

We now consider the related obstacle problem for elliptic PDEs with a nonlinear Neumann boundary condition. Roughly speaking, a solution of the obstacle problem is a function $u \in C(\overline{\Theta}; \mathbb{R})$ which satisfies:

$$\begin{aligned} \min\{u(x) - h(x), Lu(x) + f(x, u(x), (\nabla u)^* \sigma(x))\} &= 0, \;\; x \in \Theta, \\ \frac{\partial u}{\partial n}(x) + g(x, u(x)) &= 0, \;\; x \in \partial\Theta, \end{aligned} \tag{5.8}$$

where

$$L = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}$$

and at point $x \in \partial\Theta$

$$\frac{\partial}{\partial n} = \sum_{i=1}^{d} \frac{\partial \psi}{\partial x_i}(x) \frac{\partial}{\partial x_i}.$$

More precisely, solutions of Equation (5.8) is taken in viscosity sense.

Definition 5.4. (a) $u \in C\left(\overline{\Theta}, \mathbb{R}^d\right)$ is said to be a viscosity subsolution of (5.8) if for any point $x_0 \in \overline{\Theta}$, such that $u(x_0) > h(x_0)$ and for any $\varphi \in C^2(\overline{\Theta})$ such that $\varphi(x_0) = u(x_0)$ and $u - \varphi$ attains its minimum at x_0, then

$$
\begin{aligned}
&-L\varphi(x_0) - f(x_0, u(x_0), (\nabla\varphi\sigma)(x_0)) \leq 0, \text{ if } x_0 \in \Theta \\
&\min\left(-L\varphi(x_0) - f(x_0, u(x_0), (\nabla\varphi\sigma)(x_0)),\ -\tfrac{\partial\varphi}{\partial n}(x_0) - g(x_0, u(x_0))\right) \leq 0, \text{ if } x_0 \in \partial\Theta.
\end{aligned}
\tag{5.9}
$$

(b) $u \in C\left(\overline{\Theta}, \mathbb{R}^d\right)$ is said to be a viscosity supersolution of (5.8) if for any point $x_0 \in \overline{\Theta}$, such that $u(x_0) \geq h(x_0)$ and for any $\varphi \in C^2(\overline{\Theta})$ such that $\varphi(x_0) = u(x_0)$ and $u - \varphi$ attains its maximum at x_0, then

$$
\begin{aligned}
&-L\varphi(x_0) - f(x_0, u(x_0), (\nabla\varphi\sigma)(x_0)) \geq 0, \text{ if } x_0 \in \Theta \\
&\min\left(-L\varphi(x_0) - f(x_0, u(x_0), (\nabla\varphi\sigma)(x_0)),\ -\tfrac{\partial\varphi}{\partial n}(x_0) - g(x_0, u(x_0))\right) \geq 0, \text{ if } x_0 \in \partial\Theta.
\end{aligned}
\tag{5.10}
$$

(c) u is a viscosity solution of (5.8) if it is both a viscosity subsolution and supersolution.

We define

$$u(x) = Y_0^x, \quad x \in \overline{\Theta} \tag{5.11}$$

which is a deterministic quantity since Y_0^x is measurable with respect to the σ-algebra $\sigma(W_r : 0 \leq r \leq \tau_x)$. From Proposition 5.1 and standard estimates for reflected GBSDEs (see Proposition 5.1, [18]), one can show:

Proposition 5.5. *The function u is continuous and $u(x) \geq h(x)$ $\forall\, x \in \overline{\Theta}$.*

The main result in this subsection is the following.

Theorem 5.6. *The function defined by* (5.11) *is a viscosity solution of* (5.8).

Proof. First, let us show that u is a viscosity subsolution of (5.8). Let $x_0 \in \overline{\Theta}$ and $\varphi \in C^2(\overline{\Theta}; \mathbb{R}^d)$ be such that $\varphi(x_0) = u(x_0)$ and $\varphi(x_0) \geq u(x)$ for all $x \in \overline{\Theta}$.

Step 1: Suppose that $u(x_0) > h(x_0)$ and $x_0 \in \Theta$ and

$$-L\varphi(x_0) - f(x_0, \varphi(x_0), (\nabla\varphi\sigma)(x_0)) > 0,$$

and we will find a contradiction.

Indeed, by continuity, we can suppose that there exist $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for each $x \in \{y : |y - x_0| < \eta_\varepsilon \subset \Theta$, we have $u(x) \geq h(x) + \varepsilon$ and

$$-Lu(x) - f(x, \varphi(x), (\nabla\varphi\sigma)(x)) \geq \varepsilon. \tag{5.12}$$

Define

$$\overline{\tau} = \inf\{s \geq 0 : |X_s^{x_0} - x_0| > \eta_\varepsilon\} \wedge \tau_{x_0} \tag{5.13}$$

Note that, for all $s \in [0, \overline{\tau}]$

$$u(X_s^{x_0}) \geq h(X_s^{x_0}) + \varepsilon.$$

Consequently, the process $K_s^{x_0}$ is constant on $[0,\overline{\tau}]$ and, hence,

$$Y_s^x = Y_{\overline{\tau}}^{x_0} + \int_s^{\overline{\tau}} f(X_r^{x_0}, Y_r^{x_0}, Z_r^{x_0})dr - \int_s^{\overline{\tau}} Z_r^{x_0} dW_r,\ 0 \le s \le \overline{\tau}.$$

On the other hand, applying Itô's formula to $\varphi(X_s^{x_0})$ gives

$$\varphi(X_s^{x_0}) = \varphi(X_{\overline{\tau}}^{x_0}) - \int_s^{\overline{\tau}} L\varphi(X_r^{x_0})dr - \int_s^{\overline{\tau}} \nabla\varphi\sigma(X_r^{x_0})dW_r,\ 0 \le s \le \overline{\tau}.$$

Now, by inequality (5.12),

$$-L\varphi(X_s^{x_0}) - f(X_s^{x_0}, \varphi(X_s^{x_0}), (\nabla\varphi\sigma)(X_s^{x_0})) \ge \varepsilon.$$

Also,

$$\varphi(X_{\overline{\tau}}^{x_0}) \ge u(X_{\overline{\tau}}^{x_0}) = Y_{\overline{\tau}}^{x_0}.$$

Consequently, comparison theorem for GBSDEs (see [17]) implies

$$\varphi(x_0) > \varphi(X_{\overline{\tau}}^{x_0}) - \overline{\tau}\varepsilon \ge u(x_0),$$

which leads to a contradictions.

Step 2: If we further suppose that $u(x_0) > h(x_0)$ and $x_0 \in \partial\Theta$ and

$$\min\left(-L\varphi(x_0) - f(x_0, \varphi(x_0), (\nabla\varphi\sigma)(x_0)),\ -\frac{\partial\varphi}{\partial n} - g(x_0, \varphi(x_0))\right) > 0. \tag{5.14}$$

By continuity, we can suppose that there exist $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for each $x \in \{y : |y - x_0| < \eta_\varepsilon \subset \Theta$, we have $u(x) \ge h(x) + \varepsilon$ and

$$\min\left(-Lu(x) - f(x, \varphi(x), (\nabla\varphi\sigma)(x)),\ -\frac{\partial\varphi}{\partial n} - g(x, \varphi(x))\right) \ge \varepsilon. \tag{5.15}$$

Let $\overline{\tau}$ be the stopping time defined as above by (5.13) and note that, for all $s \in [0,\overline{\tau}]$

$$u(X_s^{x_0}) \ge h(X_s^{x_0}) + \varepsilon.$$

Consequently, the process $K_s^{x_0}$ is constant on $[0,\overline{\tau}]$ and, hence,

$$\begin{aligned} Y_s^x &= Y_{\overline{\tau}}^{x_0} + \int_s^{\overline{\tau}} f(X_r^{x_0}, Y_r^{x_0}, Z_r^{x_0})dr + \int_s^{\overline{\tau}} g(r, X_r^{x_0}, Y_r^{x_0})dG_r^{x_0} \\ &\quad - \int_s^{\overline{\tau}} Z_r^{x_0} dW_r,\ 0 \le s \le \overline{\tau}. \end{aligned}$$

On the other hand, applying Itô's formula to $\varphi(X_s^{x_0})$ gives

$$\varphi(X_s^{x_0}) = \varphi(X_{\overline{\tau}}^{x_0}) - \int_s^{\overline{\tau}} L\varphi(X_r^{x_0})dr - \int_s^{\overline{\tau}} \frac{\partial\varphi}{\partial n}(X_r^{x_0})dG_r^{x_0} - \int_s^{\overline{\tau}} \nabla\varphi\sigma(X_r^{x_0})dW_r,\ 0 \le s \le \overline{\tau}.$$

Now, by (5.15),

$$\min\left(-L\varphi(X_s^{x_0}) - f(X_s^{x_0}, \varphi(X_s^{x_0}), (\nabla\varphi\sigma)(X_s^{x_0})),\ -\frac{\partial\varphi}{\partial n}(X_s^{x_0}) - g(r, X_r^{x_0}, Y_r^{x_0})\right) \geq \varepsilon.$$

Also,

$$\varphi(X_{\overline{\tau}}^{x_0}) \geq u(X_{\overline{\tau}}^{x_0}) = Y_{\overline{\tau}}^{x_0}.$$

Consequently, comparison theorem for GBSDEs (see [17]) implies

$$\varphi(x_0) > \varphi(X_{\overline{\tau}}^{x_0}) - \overline{\tau}\varepsilon \geq u(x_0),$$

which leads to a contradiction.

By the same argument as above one can show that u given by (5.11) is also a viscosity supersolution of elliptic reflected PDEs (5.8) and ends the proof. □

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