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Parameters Identification in Population Dynamics Problem

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Abstract

We are interested in the identification of parameters in a problem of pollution modeled by a population dynamics problem. We use the notion of sentinel introduced by O.Nakoulima in [13]. We prove the existence of such sentinels by solving a problem of null-controllability with constraint on the control. The key of our results is an observability inequality of Carleman type adapted to the constraint.

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1 Introduction

In the modeling of the problems of pollution in population dynamics problem, the source terms as well as the initial or boundary conditions may be unknown. More precisely, the unknown real function y depends on variables t,a,x, where $t \in (0,T)$ stands for the running time, $a \in (0,A)$ for the age of individuals and $x \in \Omega \subset \mathbb{R}^N$ for space variable. The number $y(t,a,x)$ is the distribution of $a-year$ old individuals at time t at the point x. We set $U = (0,T)\times(0,A)$; $Q = U\times\Omega$; $Q_A = (0,A)\times\Omega$; $Q_T = (0,T)\times\Omega$; $\Sigma = U\times\Gamma$. The function y has satisfy the following two time scale varying equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y + \mu y = f + \sum\limits_{i=1}^{M} \lambda_i \widehat{f_i} & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0,a,x) = y^0(a,x) + \tau \widehat{y}^0(a,x), & \text{in } Q_A, \\ y(t,0,x) = \int_0^A \beta(t,a,x) y(t,a,x)\,da, & \text{in } Q_T. \end{cases} \tag{1.1}$$

It is assumed that Ω is open and bounded with C^2 boundary $\Gamma = \partial\Omega$ and $\mu(t,a,x) \geq 0; \beta(t,a,x) \geq 0$. The parameters of the problem have the following sense: the bound $T > 0$ is the horizon of the problem, the bound A is the expectation of life, the weight β is the natural fertility rate, the function $\mu = \mu(t,a,x)$ is the natural death rate of $a-year$ old individuals at

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time $t>0$ and in the position x, the function f corresponds to external flow and $y^0 = y^0(a,x)$ is the initial distribution of individuals.

Convenient assumptions for mesurability and integrability of functions are made. In particular $f \in L^2(Q)$.In the equation (1.1),we have:

- The source term is unknown and represents pollution source of the form $f + \sum_{i=1}^{M} \lambda_i \widehat{f_i}$.
  The functions f and $\{\widehat{f_i}\}_{1\leq i\leq M}$ are known whereas the real coefficients $\{\lambda_i\}_{1\leq i\leq M}$ are unknown.
  - The initial condition is of the form $y^0 + \tau\widehat{y}^0$ where the function y^0 is known while τ, real, is unknown.

We assume that

- y^0 and $\widehat{y}^0$ belong to $L^2(Q_A)$, f and $\widehat{f_i}$ belong to $L^2(Q)$,
  - the functions $\widehat{f_i}$, $1 \leq i \leq M$ are linearly independent,
  - the real τ is sufficiently small.

In the model (1.1), we are interested in identifying the parameters λ_i without any attempt of computing the missing term $\tau\widehat{y}^0$.To identify these parameters, we use the method of sentinels. In this paper we construct sentinels when the supports of the observation function and of the control function are included in two different open subsets of $\mathbb{R}^N$. This point of view has already been proposed by Nakoulima [13] for the parabolics equations. In [12] the authors use the previous point of view and build the sentinels with given sensitivity in order to identify parameters in a problem of pollution modeled by a semilinear parabolic equation.

The sentinels theory relies on three features:

- **A state equation** represented here by (1.1) whose solution
  $y = y(t,a,x,\lambda,\tau) = y(\lambda,\tau)$ depends on two families of parameter $\lambda = \{\lambda_1,...,\lambda_M\}$ and τ. We assume the following [1]:

$$(H1)\begin{cases} \beta \in L^\infty_+((0,A)\times(0,T)\times\Omega), \\ \exists\delta \in (0,A) \text{ s.t. } \beta(a,.,.) = 0 \text{ for } a \in (\delta,A); \end{cases}$$

$$(H2)\ \mu \in L^\infty_{loc}([0,A);L^\infty((0,T)\times\Omega)),\ \mu \geq 0 \text{ a.e. in } Q_A;$$

$$(H3)\begin{cases} 0<t<A,\ x\in\Omega \lim\limits_{a\longrightarrow A}\int_0^t \mu(\iota,a-t+\iota,x)d\iota = +\infty, \\ A<t<T,\ x\in\Omega \lim\limits_{a\longrightarrow A}\int_0^a \mu(t-a+\alpha,\alpha,x)d\alpha = +\infty. \end{cases}$$

For the biological comments about the model and for the basic existence of the solution to (1.1) we refer to [2, 8, 15]. For the sake of simplicity, we use indifferently y, $y(t,a,x;\lambda,\tau)$ or $y(\lambda,\tau)$ to denote the unique solution of (1.1).It is relevant since λ and τ are fixed parameters.

- **An observation** y_{obs} which is a measurement of the concentration of the pollutant taken on a non-empty open subset O of Ω, called observatory.
- **A function** $S = S(\lambda,\tau)$ called "sentinel". Let

$$h_0 \in L^2(U\times O) \tag{1.2}$$

and let $\omega \subset \Omega$, open and nonempty, $\omega \neq O$. For any control function $w \in L^2(U\times\omega)$, set

$$S(\lambda,\tau) = \int_U\int_O h_0 y(t,a,x;\lambda,\tau)dtdadx + \int_U\int_\omega wy(t,a,x;\lambda,\tau)dtdadx. \tag{1.3}$$

Choose now $w \in L^2(U\times\omega)$ such that the following holds:

- S is stationary to the first order with respect to the missing term $\tau\widehat{y}^0$:

$$\frac{\partial S}{\partial\tau}(0,0) = 0 \quad \forall\widehat{y}^0. \tag{1.4}$$

- S is sensitive to the first order with respect to the pollution terms $\lambda_i\widehat{f_i}$:

$$\frac{\partial S}{\partial\lambda_i}(0,0) = c_i,\ 1 \leq i \leq M, \tag{1.5}$$

where c_i, $(1 \leq i \leq M)$, are given constants not all identically zero.

- The control w is of minimal norm in $L^2(U\times\omega)$ among "the admissible controls" i.e

$$||w||_{L^2(U\times\omega)} = \min_{\overline{w}\in E}||\overline{w}||_{L^2(U\times\omega)}, \tag{1.6}$$

where $E = \{\overline{w} \in L^2(U\times\omega)$, such that $(\overline{w}, S(\overline{w}))$ satisfies (1.3)-(1.5)$\}$.

In the sequel, we assume without loss of generality that

$$f = 0 \text{ in } Q \text{ and } y^0 = 0 \text{ in } Q_A. \tag{1.7}$$

Remark 1.1. Consider the function $y_\tau = \dfrac{\partial y}{\partial\tau}$, where y corresponds to parameter values $\lambda = 0$, $\tau = 0$ and the function $y_{\lambda_i} = \dfrac{\partial y}{\partial\lambda_i}$, where y corresponds to parameter values $\lambda_i = 0$, $\tau = 0$. The functions y_τ and y_{λ_i} are respectively the solution of the problems

$$\begin{cases} \dfrac{\partial y_\tau}{\partial t} + \dfrac{\partial y_\tau}{\partial a} - \Delta y_\tau + \mu y_\tau = 0 \ \text{ in } Q, \\ y_\tau = 0 \ \text{ on } \Sigma, \\ y_\tau(0,a,x) = \widehat{y}^0 \ \text{ in } Q_A, \\ y_\tau(t,0,x) = \int_0^A \beta(t,a,x)y_\tau(t,a,x)da \ \text{ in } Q_T; \end{cases} \tag{1.8}$$

and

$$\begin{cases} \dfrac{\partial y_{\lambda_i}}{\partial t} + \dfrac{\partial y_{\lambda_i}}{\partial a} - \Delta y_{\lambda_i} + \mu y_{\lambda_i} = \widehat{f_i} \ \text{ in } Q, \\ y_{\lambda_i} = 0 \ \text{ on } \Sigma, \\ y_{\lambda_i}(0,a,x) = 0 \quad \text{in } Q_A, \\ y_{\lambda_i}(t,0,x) = \int_0^A \beta(t,a,x) y_{\lambda_i}(t,a,x)da \ \text{ in } Q_T. \end{cases} \tag{1.9}$$

Under the assumptions $(H_1)-(H_3)$, the linear problems (1.8), (1.9) gets respectively one only solution y_τ such that $y_\tau(t,A,x) = 0$ and y_{λ_i} such that $y_{\lambda_i}(t,A,x) = 0$. For the details of the proof we refer to [2, 8, 15].

Remark 1.2. If the function S defined by (1.3)-(1.5) exists, then it is unique since w verifies (1.6). In this case, to estimate the parameter λ_i one proceeds as follows: Assume that the solution of the state equation (1.1) when $\lambda = 0$ and $\tau = 0$ is known. Then one has the following information:

$$S(\lambda,\tau) - S(0,0) \approx \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0).$$

Therefore, fixing $i, j \in \{1,...,M\}$ and choosing i and j such that

$$\frac{\partial S}{\partial \lambda_j}(0,0) = 0 \text{ for } j \neq i \text{ and } \frac{\partial S}{\partial \lambda_i}(0,...,0) = 1,$$

one obtains the following estimate of the parameter λ_i :

$$\lambda_i \approx \frac{1}{c_i}(S(\lambda,\tau) - S(0,0)).$$

Definition 1.3. We will refer to the function S given by (1.3)-(1.5) as sentinel with given $\{c_i\}$ sensitivity.

Let χ_ω be the characteristic function of the set ω. We set

$$Y_\lambda = \text{Span}\{y_{\lambda_1}\chi_\omega, ..., y_{\lambda_M}\chi_\omega\}, \tag{1.10}$$

the vector subspace of $L^2(U \times \omega)$, generated by the M independent functions $y_{\lambda_i}\chi_\omega$, $1 \leq i \leq M$ and we denote by $Y_\lambda^\perp$ the orthogonal of Y_λ in $L^2(U \times \omega)$. Assume that

$$\begin{cases} \text{any function } k \in Y_\lambda \cap L^2(U, H^1(\omega)) \text{ such that} \\ \dfrac{\partial k}{\partial t} + \dfrac{\partial k}{\partial a} - \Delta k + \mu k = 0, \text{ in } U \times \omega, \text{ is identically zero in } U \times \omega. \end{cases} \tag{1.11}$$

Next, we consider the following general null-controllability problem: *Given $h \in L^2(Q)$, find $v \in L^2(U \times \omega)$ such that*

$$v \in Y_\lambda^\perp, \tag{1.12}$$

and such that $q = q(t,a,x,v) \in L^2(Q)$ which is the solution of

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x) + h + v\chi_\omega \text{ in } Q, \\ q = 0 \qquad \text{on } \Sigma, \\ q(T,a,x) = 0 \qquad \text{in } Q_A, \\ q(t,A,x) = 0 \qquad \text{in } Q_T; \end{cases} \tag{1.13}$$

satisfies

$$q(0,a,x,v) = 0 \text{ in } Q_A; \tag{1.14}$$

with v of minimal norm in $L^2(U\times\omega)$, that is

$$\|v\|_{L^2(U\times\omega)} = \min_{\overline{w}\in\mathcal{E}} \|\overline{w}\|_{L^2(U\times\omega)}; \tag{1.15}$$

where

$$\mathcal{E} = \left\{\overline{v} \in Y_\lambda^\perp \text{ such that } (\overline{v}, \overline{q} = q(t,a,x,\overline{v})) \text{ is subject to (1.13)-(1.14)}\right\}. \tag{1.16}$$

For the evolutions equations, others topics such as exact controllability and approximate controllability are considered. For example in [5], exact controllability of semilinear stochastic evolution equation is studied and, in [9], the interior approximate controllability of semilinear heat equation was proved.

For the problem (1.12)-(1.15), two matters are considered. The first one consists in solving the null-controllability problem, and the second one consists in characterizing the optimal solution (1.15) by some optimality system. The problem (1.12)-(1.15) is solved when Y_λ={0} (i.e. setting without constraints or free constraints) in several issues by various methods [1], [4]. In the present paper both points are considered in the general setting $Y_\lambda \neq \{0\}$.More precisely, we have the following results:

Theorem 1.4. *Assume that the above hypotheses on Ω, ω, O and the data of the equation (1.1) are satisfied. Then the existence of sentinel (1.3)-(1.6) holds if and only if, null controllability problem with constraint on the control (1.12)-(1.15) holds.*

The proof of the null controllability problem with constraint on the control (1.12)-(1.15) lies on the existence of a function θ and a Carleman inequality adapted to the constraint (cf Subsection 2.2), for which we have the following result:

Theorem 1.5. *Assume that the hypotheses of Theorem 1.4 and the condition (1.11) are satisfied. Then there exists a positive weight function θ such that, for any function $h \in L^2(Q)$ with $\theta h \in L^2(Q)$, null controllability problem with constraint on the control (1.12)-(1.15) holds. Moreover, the control is given by:*

$$\widehat{v_\theta} = -(\widehat{\rho_\theta} - P\widehat{\rho_\theta}\chi_\omega)\chi_\omega, \tag{1.17}$$

where $\widehat{\rho_\theta}$ is a solution of:

$$\begin{cases} \frac{\partial\widehat{\rho_\theta}}{\partial t} + \frac{\partial\widehat{\rho_\theta}}{\partial a} - \Delta\widehat{\rho_\theta} + \mu\widehat{\rho_\theta} = 0 \textit{ in } Q, \\ \widehat{\rho_\theta} = 0 \qquad\qquad \textit{on } \ \Sigma, \\ \widehat{\rho_\theta}(t,0,x) = \int_0^A \beta(t,a,x)\widehat{\rho_\theta}(t,a,x)da \textit{ in } Q_T; \end{cases} \tag{1.18}$$

and P is the orthogonal projection operator from $L^2(U\times\omega)$ into Y_λ.

The remaining of paper is organized as follows. Section 2 is devoted to some preliminary results. In this section, we prove Theorem 1.4 and establish the inequality adapted to the constraint (1.12). In Section 3, we prove the existence and the uniqueness of the solution for the controllability problem (1.12)-(1.15) of Theorem 1.4 and give the proof of Theorem 1.5. We finish with Section 4 where the expression of the sentinel S defined by (1.3)-(1.5) and the estimate of the parameters λ_i are given.

2 Preliminary results

2.1 Proof of Theorem 1.4

Since y_τ and y_{λ_i} are respectively solutions of (1.8) and (1.9), the stationary condition (1.4) and respectively the sensitivity conditions (1.5) hold if and only if:

$$\int_U\int_O h_0 y_\tau dtdadx + \int_U\int_\omega w y_\tau dtdadx = 0,\ \forall\, \widehat{y}^0, \|\widehat{y}^0\|_{L^2(Q_A)} \le 1, \tag{2.1}$$

and

$$\int_U\int_O h_0 y_{\lambda_i} dtdadx + \int_U\int_\omega w y_{\lambda_i} dtdadx = c_i\ 1 \le i \le M. \tag{2.2}$$

In order to transform equation (2.1), we introduce the classical adjoint state. More precisely, we consider the solution $q = q(t,a,x)$ of the linear problem

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q = \beta q(t,0,x) + h_0\chi_O + w\chi_\omega \text{ in } Q, \\ q = 0 \qquad\qquad \text{on } \Sigma, \\ q(T,a,x) = 0 \qquad \text{in } Q_A, \\ q(t,A,x) = 0 \qquad \text{in } Q_T; \end{cases} \tag{2.3}$$

where χ_O and χ_ω are indicator functions for the respective open sets O and ω. There is only one solution in $L^2(Q)$ as some consequence of the fixed point theorem for contracting mapping [2, 3]. The so called adjoint state q depends on the unknown function w and its utility comes from the following process.

First, multiplying both members of the differential equation in (2.3) by y_τ, and integrating by parts over Q

$$\int_U\int_O h_0 y_\tau dtdadx + \int_U\int_\omega w y_\tau dtdadx = \int_0^A\int_\Omega q(0,a,x)\widehat{y}^0 dadx\ ,$$

$$\forall \widehat{y}^0 \in L^2(Q_A),\ \|\widehat{y}^0\|_{L^2(Q_A)} \le 1.$$

Thus, the condition (1.4) (or (2.1)) holds if and only if

$$q(0,a,x) = 0,\ \text{a.e } (a,x) \in (0,A)\times\Omega. \tag{2.4}$$

Then, multiplying both sides of the differential equation in (2.3) by $y_{\lambda_i} \in L^2(Q)$ which is solution of (1.9), and integrate by parts over Q

$$\int_U\int_\Omega q\widehat{f_i} dtdadx = \int_U\int_O h_0 y_{\lambda_i} dtdadx + \int_U\int_\omega w y_{\lambda_i} dtdadx,\ 1 \le i \le M. \tag{2.5}$$

Thus, the condition (1.5) (or (2.2)) is equivalent to

$$\int_U\int_\Omega q\widehat{f_i} dtdadx = c_i,\ 1 \le i \le M. \tag{2.6}$$

Therefore, the above considerations show that the existence of the sentinel defined by (1.3)-(1.5) holds if and only if, the following null controllability problem with constraints on the state q holds: *Given $h_0 \in L^2(U \times O)$, find w of minimal norm in $L^2(U \times \omega)$ such that the pair (w,q) verifies (2.3), (2.4) and (2.6).*

Actually, condition (1.5)(or the constraints (2.6) on the state q) is equivalent to constraint on the control. Indeed, let Y_λ be the real vector subspace of $L^2(U \times \omega)$ defined in (1.10). Since Y_λ is finite dimensional, there exists a unique $w_0 \in Y_\lambda$ such that

$$c_i - \int_U \int_O h_0 y_{\lambda_i} dtdadx = \int_U \int_\omega w_0 y_{\lambda_i} dtdadx \ 1 \le i \le M.$$

Therefore, the condition (2.2) or (2.6) holds if and only if

$$w - w_0 = v \in Y_\lambda^\perp. \tag{2.7}$$

Consequently, replacing w by $v + w_0$ in $(2.3)_1$, then setting

$$h = h_0 \chi_O + w_0 \chi_\omega \in L^2(Q), \tag{2.8}$$

we finally deduce that we have the existence of the sentinel (1.3)-(1.5) if and only if, null controllability with constraint on the control (1.12)-(1.15) holds ■

2.2 An adapted Carleman inequality

The observability inequality we are looking for is a consequence of Carleman's inequality. We consider an auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) > 0 \ \forall x \in \Omega, \ \psi(x) = 0 \ \forall x \in \Gamma, \ |\nabla \psi(x)| \neq 0 \ \forall x \in \overline{\Omega} - \omega_0, \tag{2.9}$$

where ω_0 denotes any open set such that $\overline{\omega}_0 \subset \omega$ (for example ω_0 can be some small enough open ball). Such a function ψ exists according to A.Fursikov and O.Yu.Imanuvilov [7].

We define for any positive parameter λ the following weight functions:

$$\varphi(t,a,x) = \frac{e^{\lambda\psi(x)}}{at(T-t)}, \ \alpha(t,a,x) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{at(T-t)}. \tag{2.10}$$

Since φ does not vanish on Q, we set

$$\theta = \frac{e^{s\alpha}}{\varphi\sqrt{\varphi}} \text{ or } \frac{1}{\theta} = \varphi\sqrt{\varphi}e^{-s\alpha}. \tag{2.11}$$

Remark 2.1. $\frac{1}{\theta} = \varphi\sqrt{\varphi}e^{-s\alpha}$ is defined on $\overline{Q} = [0;T] \times [0;A] \times \overline{\Omega}$ by:

$$\frac{1}{\theta}(t,a,x) = \begin{cases} \varphi^{\frac{3}{2}}(t,a,x)e^{-s\alpha(t,a,x)} \text{ on }]0,T[\times]0,A[\times\overline{\Omega}, \\ 0 \text{ on } \overline{Q} - (]0,T[\times]0,A[\times\overline{\Omega}); \end{cases}$$

and we have the following limits: $\lim_{(t,a,x)\to(0,0,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta}(0,0,x)$;
$\lim_{(t,a,x)\to(0,a,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta}(0,a,x)$;

$\lim\limits_{(t,a,x)\to(t,0,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta}(t,0,x)$;
$\lim\limits_{(t,a,x)\to(T,a,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta}(T,a,x)$;
$\lim\limits_{(t,a,x)\to(T,0,x)} \frac{1}{\theta}(t,a,x) = 0 = \frac{1}{\theta}(T,0,x)$.

Thus $\frac{1}{\theta}$ is continuous on $\overline{Q}$ and since $\overline{Q}$ is bounded in $\mathbb{R}^{N+2}$ then $\frac{1}{\theta}$ is bounded.

We adopt the following notations

$$\begin{cases} L = \frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \Delta + \mu I, \\ L^* = -\frac{\partial}{\partial t} - \frac{\partial}{\partial a} - \Delta + \mu I, \\ \mathcal{V} = \left\{\rho \in C^\infty(\overline{Q}),\ \rho = 0 \text{ on } \Sigma\right\}. \end{cases} \tag{2.12}$$

Lemma 2.2. *Assume that (1.11) holds. Let θ be the function given by (2.11) and P be the operator defined as in Theorem 1.5. Then there exists a positive constant C such that for any $\rho \in \mathcal{V}$:*

$$\int_U \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dtdadx \leq C(\int_U \int_\Omega |L\rho|^2 \, dtdadx + \int_U \int_\omega |\rho - P\rho|^2 \, dtdadx) \tag{2.13}$$

The proof of this lemma requires what we call the global Carleman's inequality.

Proposition 2.3 (Global Carleman's inequality)**.** *Let ψ, φ and α be the functions defined respectively as in (2.9)-(2.10). Then, there exists $\lambda_o > 1$ and $s_o > 1$ and there exists $C > 0$ such that, for any $\lambda \geq \lambda_o$, for any $s \geq s_o$ and for any $\rho \in \mathcal{V}$ the following inequality holds:*

$$\begin{aligned} &\int_Q \frac{e^{-2s\alpha}}{s\varphi} \left(|\rho_t + \rho_a|^2 + |\Delta\rho|^2\right) dtdadx + \int_Q s\lambda^2 \varphi e^{-2s\alpha} |\nabla\rho|^2 \, dtdadx \\ &+ \int_Q s^3 \lambda^4 \varphi^3 e^{-2s\alpha} |\rho|^2 \, dtdadx \\ \leq \;& C\left(\int_Q e^{-2s\alpha} |L\rho|^2 \, dtdadx + \int_0^T \int_0^A \int_\omega s^3 \lambda^4 \varphi^3 e^{-2s\alpha} |\rho|^2 \, dtdadx\right). \end{aligned} \tag{2.14}$$

Proof. We refer to [1] and [14]. □

According to the definition of φ and α given by (2.10), the function θ given by (2.11) is positive and $\frac{1}{\theta} = \varphi\sqrt{\varphi} e^{-s\alpha}$ is bounded. So, replacing $\frac{e^{s\alpha}}{\varphi\sqrt{\varphi}}$ by θ in (2.14) the following inequality holds:

$$\int_Q \frac{1}{\theta^2} |\rho|^2 \, dtdadx \leq C(\int_Q \frac{1}{\theta^2 \varphi^3 s^3 \lambda^4} |L\rho|^2 \, dtdadx + \int_U \int_\omega \frac{1}{\theta^2} |\rho|^2 \, dtdadx).$$

As a consequence of the boundedness of $\frac{1}{\theta}$ and $\frac{1}{\varphi^3 s^3 \lambda^4}$, we get the next observability inequality:

$$\int_Q \frac{1}{\theta^2} |\rho|^2 \, dtdadx \leq C(\int_Q |L\rho|^2 \, dtdadx + \int_U \int_\omega |\rho|^2 \, dtdadx). \tag{2.15}$$

Proof of Lemma 2.1. The proof uses a well known compactness-uniqueness argument and the inequality (2.15). Indeed suppose that (2.13) does not hold. Then

$$\begin{cases} \forall j \in \mathbf{N}^*, \exists \rho_j \in \mathcal{V}, \int_U \int_\Omega \frac{1}{\theta^2} \left|\rho_j\right|^2 dtdadx = 1, \\ \int_U \int_\Omega \left|L\rho_j\right|^2 dtdadx \leq \frac{1}{j} \text{ and } \int_U \int_\omega \left|\rho_j - P\rho_j\right|^2 dtdadx \leq \frac{1}{j}. \end{cases} \tag{2.16}$$

The forthcomming proof consists of extracting some subsequence, still denoted $(\rho_j)_j$ such that the following contradiction holds

$$\lim_{j\to+\infty} \int_U \int_\Omega \frac{1}{\theta^2} \left|\rho_j\right|^2 dtdadx = 0.$$

Denote by $(h|g)_{L^2(U\times\omega)}$ the natural scalar product in the Hilbert space $L^2(U\times\omega)$. Let $\{k_1, k_2, ..., k_M\}$ be some orthonormal basis of Y_λ.

Step 1. We show first that for any $i = 1, 2, ..., M$ the numerical sequence $((\rho_j|k_i)_{L^2(U\times\omega)})_{j\in\mathbf{N}^*}$ is bounded or equivalently that the sequence $(\left\|P\rho_j\right\|^2_{L^2(U\times\omega)})_j$ is bounded.

Start with the norm inequality

$$\begin{aligned} \left(\int_U \int_\omega \frac{1}{\theta^2} \left|P\rho_j\right|^2 dtdadx\right)^{\frac{1}{2}} \quad &\leq \quad \left(\int_U \int_\omega \frac{1}{\theta^2} \left|\rho_j\right|^2 dtdadx\right)^{\frac{1}{2}} \\ &\quad + \left(\int_U \int_\omega \frac{1}{\theta^2} \left|\rho_j - P\rho_j\right|^2 dtdadx\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\dfrac{1}{\theta^2}$ is bounded and by (2.16) it follows that there is some number γ

$$\forall j \in \mathbf{N}^*, \int_U \int_\omega \frac{1}{\theta^2} \left|P\rho_j\right|^2 dtdadx \leq \gamma. \tag{2.17}$$

Since Y_λ is finite dimensional, norms are equivalent. Particularly the mappings

$$k \longmapsto \int_U \int_\omega |k|^2 dtdadx \text{ and } k \longmapsto \int_U \int_\omega \frac{1}{\theta^2} |k|^2 dtdadx,$$

are equivalent norms on Y_λ. There is then some number γ'

$$\forall j \in \mathbf{N}^*, \int_U \int_\omega \left|P\rho_j\right|^2 dtdadx \leq \gamma'.$$

The relation $(\rho_j - P\rho_j) \in Y_\lambda^\perp, \forall j \in \mathbf{N}^*$ means the following

$$(\rho_j - P\rho_j|k_i)_{L^2(U\times\omega)} = 0 \ \forall i, 1 \leq i \leq M, \ \forall j \in \mathbf{N}^*.$$

Thus

$$P\rho_j = \sum_{i=1}^{M} (P\rho_j|k_i)_{L^2(U\times\omega)} k_i = \sum_{i=1}^{M} (\rho_j|k_i)_{L^2(U\times\omega)} k_i, \tag{2.18}$$

and from orthonormality

$$\int_U \int_\omega \left|P\rho_j\right|^2 dtdadx = \sum_{i=1}^{M} \left|(\rho_j|k_i)_{L^2(U\times\omega)}\right|^2 = \left\|P\rho_j\right\|^2_{L^2(U\times\omega)}. \tag{2.19}$$

Thus

$$\left\|P\rho_j\right\|^2_{L^2(U\times\omega)} \leq \gamma'. \tag{2.20}$$

Step 2. Since $(P\rho_j)_{j\in\mathbb{N}^*}$ is bounded and

$$\left\|\rho_j - P\rho_j\right\|^2_{L^2(U\times\omega)} = \int_U\int_\omega \left|\rho_j - P\rho_j\right|^2 dtdadx \to 0,$$

then the sequence $(\rho_j)_{j\in\mathbb{N}^*}$ is bounded. There is some weakly convergent subsequence still denoted by $(\rho_j)_{j\in\mathbb{N}^*}$ such that:

$$\rho_j \rightharpoonup g \text{ weakly in } L^2(U\times\omega). \tag{2.21}$$

Since subsequences have the same limit as convergent sequence

$$\rho_j - P\rho_j \to 0 \text{ strongly in } L^2(U\times\omega). \tag{2.22}$$

Next, we deduce from the compactness of P (because Y_λ is finite dimensional) that there exists $\zeta\in Y_\lambda$ such that

$$P\rho_j \to \zeta \text{ strongly in } L^2(U\times\omega). \tag{2.23}$$

We deduce from (2.22) and (2.23) that $\rho_j \longrightarrow g = \zeta$ strongly in $L^2(U\times\omega)$. Thanks to the continuity of P, we have $P\rho_j \to Pg$ strongly in $L^2(U\times\omega)$. Therefore, $Pg = g$ and so $g\in Y_\lambda$.

Step 3. In fact, we have $g = 0$. Indeed, from (2.16), we also have $L\rho_j \to 0$ strongly in $L^2(Q)$. Thus $L\rho_j \longrightarrow 0$ strongly in $L^2(U\times\omega)$. We conclude that $L\rho_j \rightharpoonup 0$ weakly in $\mathcal{D}'(U\times\omega)$ and so $Lg = 0$. The assumption (1.11) implies $g = 0$ on $U\times\omega$. Finally, $\rho_j \to 0$ strongly in $L^2(U\times\omega)$.

Step 4. Since $\rho_j\in\mathcal{V}$, it follows from the observability inequality (2.15) that

$$\int_U\int_\Omega \frac{1}{\theta^2}\left|\rho_j\right|^2 dtdadx \leq C\left(\int_U\int_\Omega \left|L\rho_j\right|^2 dtdadx + \int_U\int_\omega \left|\rho_j\right|^2 dtdadx\right).$$

Then, the conclusions in the third step, yield that $\int_U\int_\Omega \frac{1}{\theta^2}\left|\rho_j\right|^2 dtdadx \to 0$ when $j\to +\infty$. The proof is now completed. □

3 Null controllability with constraint on the control

The main tool used is the observability inequality (2.13), adapted to the constraint.

3.1 Existence of optimal control variable for null controllability

Consider now the following symetric bilinear form

$$\forall\rho\in\mathcal{V}, \forall\widehat{\rho}\in\mathcal{V},\ a(\rho,\widehat{\rho}) = \int_U\int_\Omega L\rho L\widehat{\rho}dtdadx + \int_U\int_\omega (\rho - P\rho)(\widehat{\rho} - P\widehat{\rho})dtdadx. \tag{3.1}$$

According to Lemma 2.1, this symetric bilinear form is a scalar product on $\mathcal{V}$. Let V be the completion of $\mathcal{V}$ with respect to the related norm:

$$\rho \longmapsto \|\rho\|_V = \sqrt{a(\rho,\rho)}. \tag{3.2}$$

The closure of $\mathcal{V}$ is the Hilbert space V.

Remark 3.1. 1. The norm $||.||_V$ is related to the right side of the inequality (2.13) while the left member of (2.13) leads to the norm

$$\forall \rho \in \mathcal{V},\ |\rho|_\theta = \left(\int_U \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dtdadx \right)^{\frac{1}{2}}.$$

2. The completion of $\mathcal{V}$ is the weigthed Hilbert space usually denoted by $L^2_{\frac{1}{\theta}}$.
3. The inequality (2.13) shows that

$$|\rho|_\theta \leq C\, ||\rho||_V. \tag{3.3}$$

Let θ be defined by (2.11) and $h \in L^2(Q)$ be such that $\theta h \in L^2(Q)$. Then, thanks to Cauchy-Schwartz's inequality and (2.13), the following linear form defined on V by:

$$\rho \longrightarrow \int_U \int_\Omega h\rho dtdadx,$$

is continuous. Therefore, Lax-Milgram's Theorem [6], allows us to say that, for every function $h \in L^2(Q)$ such that $\theta h \in L^2(Q)$, there exists one and only one solution ρ_θ in V of the variational equation:

$$a(\rho_\theta, \rho) = \int_U \int_\Omega h\rho dtdadx \ \forall \rho \in V. \tag{3.4}$$

Remark 3.2. In the statement of the null-controllability problem, there are boundary and initial or end conditions. These conditions concern the values of the control or state functions at the points of the boundary for example. The solutions dealt by means of functionnal analysis are not functions but elements of function spaces which are equivalence classes. As a consequence boundary or initial or end values of the solutions have to be considered in function spaces. Such a question has been adressed by Lions-Magenes. We refer to [11] to derive the following trace theorems in regular open set Ω. Let's assume that $q \in L^2(U\times\Omega) \simeq L^2(U, L^2(\Omega))$ and $\Delta q \in H^{-1}(U, L^2(\Omega))$. Then $q|_{U\times\Gamma} \in H^{-1}(U, H^{-\frac{1}{2}}(\Gamma))$. The meaning of $q|_\Sigma$, the trace of q on Σ, is clear. Let's assume that $q \in L^2(U\times\Omega) \simeq L^2([0,T]\times[0,A], L^2(\Omega))$ and $\dfrac{\partial q}{\partial t} + \dfrac{\partial q}{\partial a} \in L^2(U, H^{-2}(\Omega)$. Then

$$q \in C([0,A], L^2([0,T], H^{-2}(\Omega))) \cap C([0,T], L^2([0,A], H^{-2}(\Omega))).$$

That means there exists some function $\widetilde{q} : [0,T]\times[0,A] \longrightarrow L^2([0,T], H^{-2}(\Omega))$ standing for $q \in L^2(U\times\Omega)$ which is separately continuous, so that the following values in $L^2([0,T], H^{-2}(\Omega))$ get sense

$$\forall (t,a) \in U,\ q(t,a) = \widetilde{q}(t,a),$$

and

$$\begin{array}{rcl} q(T) & \in & L^2([0,A], H^{-2}(\Omega)), \\ q(0) & \in & L^2([0,T], H^{-2}(\Omega)), \\ q(A) & \in & L^2([0,T], H^{-2}(\Omega)). \end{array}$$

Proposition 3.3. *Assume (1.11) holds. For $h \in L^2(Q)$ such that $\theta h \in L^2(Q)$, let ρ_θ be the unique solution of (3.4),*

$$v_\theta = -(\rho_\theta \chi_\omega - P\rho_\theta), \tag{3.5}$$

and

$$q_\theta = L\rho_\theta. \tag{3.6}$$

Then, the pair (v_θ, q_θ) is such that (1.12)-(1.14) hold.

Proof. We prove that (v_θ, q_θ) is a solution of (1.12)-(1.14). According to (3.4), we have $\rho_\theta \in V$. Consequently $q_\theta \in L^2(Q)$ and since $P\rho_\theta \in Y_\lambda$, the function $v_\theta = -(\rho_\theta\chi_\omega - P\rho_\theta) \in Y_\lambda^\perp$. Next, replacing $L\rho_\theta$ by q_θ and $-(\rho_\theta\chi_\omega - P\rho_\theta)$ by v_θ in (3.4), we obtain

$$\int_U \int_\Omega q_\theta L\rho dtdadx - \int_U \int_\omega v_\theta(\rho - P\rho) dtdadx = \int_U \int_\Omega h\rho dtdadx, \ \forall \rho \in V.$$

Since $P\rho \in Y_\lambda$ and $v_\theta \in Y_\lambda^\perp$, this latter equality is reduced to

$$\int_U \int_\Omega q_\theta L\rho dtdadx = \int_U \int_\Omega h\rho dtdadx + \int_U \int_\omega v_\theta \rho dtdadx, \ \forall \rho \in V. \tag{3.7}$$

In the duality frame $\mathcal{D}(Q)$, $\mathcal{D}'(Q)$ (3.7) means that

$$L^* q_\theta = h + v_\theta \chi_\omega \text{ in } \mathcal{D}'(Q). \tag{3.8}$$

Besides $h + v_\theta\chi_\omega \in L^2(Q)$, then $L^* q_\theta \in L^2(Q)$.

Since $q_\theta \in L^2(Q)$ and $\Delta q_\theta \in H^{-1}(U, L^2(\Omega))$ and by the above Remark $q_\theta|_{U\times\Gamma} \in H^{-1}(U, H^{-\frac{1}{2}}(\Gamma))$. Similarly, since $q_\theta \in L^2(Q)$) and $\dfrac{\partial q_\theta}{\partial t} + \dfrac{\partial q_\theta}{\partial a} \in L^2(U, H^{-2}(\Omega))$, $q_\theta(0,a,x) \in L^2([0,A], H^{-2}(\Omega))$, $q_\theta(T,a,x) \in L^2([0,A], H^{-2}(\Omega))$;

$q_\theta(t,0,x) \in L^2([0,T], H^{-2}(\Omega))$ and $q_\theta(t,A,x) \in L^2([0,T], H^{-2}(\Omega))$. Taking into account (3.8), integrate by parts

$$\begin{aligned}
&\forall \rho \in \mathcal{V}, \int_U \int_\Omega q_\theta L\rho dtdadx + \int_U \left\langle q_\theta, \frac{\partial \rho}{\partial \nu} \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} dtda \\
&+ \int_0^T [\langle q_\theta(t,0,.), \rho(t,0,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(t,A,.), \rho(t,A,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}] dt \\
&+ \int_0^A [\langle q_\theta(0,a,.), \rho(0,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(T,a,.), \rho(T,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}] da \\
= \ & \int_Q (h + v_\theta \chi_\omega)\rho dtdadx.
\end{aligned}$$

By (3.7) since $\mathcal{V} \subset V$, it follows

$$\begin{aligned}
&\forall \rho \in \mathcal{V}, \int_U \left\langle q_\theta, \frac{\partial \rho}{\partial \nu} \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} dtda \\
&+ \int_0^T [\langle q_\theta(t,0,.), \rho(t,0,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(t,A,.), \rho(t,A,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}] dt \\
&+ \int_0^A [\langle q_\theta(0,a,.), \rho(0,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)} - \langle q_\theta(T,a,.), \rho(T,a,.)\rangle_{H^{-2}(\Omega),H^2(\Omega)}] da \\
= \ & 0.
\end{aligned}$$

Then, successively, we get $q_\theta = 0$ on Σ, $q_\theta(0,a,x) = 0$ and $q_\theta(T,a,x) = 0$ in Q_A; $q_\theta(t,0,x) = 0$ and $q_\theta(t,A.x) = 0$in Q_T. Since $q_\theta(t,0,x) = 0$ we have

$$L^* q_\theta = \beta q_\theta(t,0,x) + h + v_\theta \chi_\omega.$$

Hence the proof is completed. □

Proposition 3.4. *Under the assumptions of the Proposition 3.1, there exists a control variable v such that the pair (v,q) satisfies (1.12)-(1.14). Moreover, we can get a unique control $\widehat{v_\theta}$ such that (1.15) holds.*

Proof. We have proved in Proposition 3.1 that (v_θ, q_θ) satisfies (1.12)-(1.14). Consequently, the set $\mathcal{E}$ of the control variables $v \in L^2(U \times \omega)$ such that $(v, q(t,a,x,v))$ verifies (1.12)-(1.14) is non-empty. Moreover, adapted observability inequality (2.13) shows that the choice of the scalar product on $\mathcal{V}$ is not unique. Thus, proceeding as in Proposition 3.1, we can construct infinitely many control functions v which belong to $\mathcal{E}$. It is then clear that $\mathcal{E}$ is a nonempty closed convex subset of $L^2(U \times \omega)$. Therefore, there exists a unique control variable $\widehat{v_\theta}$ of minimal norm in $L^2(U \times \omega)$ such that $(\widehat{v_\theta}, \widehat{q_\theta} = q(t,a,x,\widehat{v_\theta}))$ solves (1.12)-(1.15). □

3.2 Proof of Theorem 1.5

In this subsection, we are concerned with the proof of Theorem 1.5. That is, the optimality system for the control $\widehat{v_\theta}$ such that the pair $(\widehat{v_\theta}, \widehat{q_\theta})$ satisfies (1.12)-(1.15). As a classical way to derive this optimality system is the method of penalization due to J.L.Lions [10], the proof of Theorem 1.5 requires some preliminary results.

Let $\epsilon > 0$. We define the functional

$$J_\epsilon(v,q) = \frac{1}{2}\|v\|^2_{L^2(U\times\omega)} + \frac{1}{2\epsilon}\left\|-\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q - \beta q(t,0,x) - h - v\chi_\omega\right\|^2_{L^2(Q)}, \tag{3.9}$$

for any pair (v,q) such that

$$\begin{cases} v \in Y_\lambda^\perp,\ q \in L^2(Q), \\ -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q - \beta q(t,0,x) \in L^2(Q), \\ q = 0 \text{ on } \Sigma,\ q(T,a,x) = 0 \text{ in } Q_A,\ q(t,A,x) = 0 \text{ in } Q_T, \\ q(0,a,x) = 0 \text{ in } Q_A. \end{cases} \tag{3.10}$$

and we consider the minimization problem

$$\inf\{J_\epsilon(v,q) \mid (v,q) \text{ subject to } (3.10)\}. \tag{3.11}$$

Proposition 3.5. *Under the assumptions of Proposition 3.1, the problem (3.11) has an optimal solution. In other words, there exists a unique pair (v_ϵ, q_ϵ) such that*

$$J_\epsilon(v_\epsilon, q_\epsilon) = \inf\{J_\epsilon(v,q) \mid (v,q) \textit{ subject to (3.10)}\} \tag{3.12}$$

Proof. Let (v_n, q_n) be a minimizing sequence satisfying (3.10). The sequence $(J_\epsilon(v_n,q_n))_n$ is bounded from above

$$J_\epsilon(v_n, q_n) \leq \gamma(\epsilon), \tag{3.13}$$

then

$$\begin{cases} \|v_n\|_{L^2(U\times\omega)} \le C(\epsilon), \\ \left\| -\dfrac{\partial q_n}{\partial t} - \dfrac{\partial q_n}{\partial a} - \Delta q_n + \mu q_n - \beta q_n(t,0,x) - h - v_n\chi_\omega \right\|_{L^2(Q)} \le \sqrt{\epsilon}C(\epsilon). \end{cases} \tag{3.14}$$

There is some subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, such that

$$v_n \rightharpoonup v_\epsilon \text{ weakly in } L^2(U\times\omega). \tag{3.15}$$

As a consequence (3.10) the sequence $(q_n)_n$ is bounded

$$\|q_n\|_{L^2(Q)} \le C. \tag{3.16}$$

There is some subsequence of $(q_n)_n$, still denoted by $(q_n)_n$ such that

$$q_n \rightharpoonup q_\epsilon \text{ weakly in } L^2(Q). \tag{3.17}$$

Then

$$\liminf J_\epsilon(v_n,q_n) \ge J_\epsilon(v_\epsilon,q_\epsilon). \tag{3.18}$$

We deduce that (v_ϵ,q_ϵ) is a unique optimal control, from the strict convexity of J_ϵ. □

Proposition 3.6. *The assumptions are as in Proposition 3.1. Then, the pair (v_ϵ,q_ϵ) is optimal solution of the problem (3.12) if and only if there exists a function ρ_ϵ such that $(v_\epsilon,q_\epsilon,\rho_\epsilon)\in L^2(U\times\omega)\times L^2(Q)\times V$ satisfies the following approximate optimality system:*

$$\begin{cases} -\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon = \beta q_\epsilon(t,0,x) + h + v_\epsilon\chi_\omega + \epsilon\rho_\epsilon \text{ in } Q, \\ q_\epsilon = 0 \text{ on } \Sigma, \\ q_\epsilon(T,a,x) = 0 \text{ in } Q_A, \\ q_\epsilon(t,A,x) = 0 \text{ in } Q_T; \end{cases} \tag{3.19}$$

$$q_\epsilon(0,a,x) = 0 \text{ in } Q_A; \tag{3.20}$$

$$\begin{cases} \frac{\partial \rho_\epsilon}{\partial t} + \frac{\partial \rho_\epsilon}{\partial a} - \Delta\rho_\epsilon + \mu\rho_\epsilon = 0 \text{ in } Q, \\ \rho_\epsilon = 0 \text{ on } \Sigma, \\ \rho_\epsilon(t,0,x) = \int_0^A \beta(t,a,x)\rho_\epsilon(t,a,x)da \text{ in } Q_T; \end{cases} \tag{3.21}$$

$$v_\epsilon = -(\rho_\epsilon\chi_\omega - P\rho_\epsilon) \in Y_\lambda^\perp. \tag{3.22}$$

Proof. Express the Euler-Lagrange optimality conditions which characterize (v_ϵ,q_ϵ). For any (v,φ) such that (3.10) the following holds

$$\begin{aligned} &\int_U\int_\omega v_\epsilon v dtdadx + \\ &\frac{1}{\epsilon}\int_Q \left(-\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon - \beta q_\epsilon(t,0,x) - h - v_\epsilon\chi_\omega\right) \\ &\times\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - \Delta\varphi + \mu\varphi - \beta\varphi(t,0,x) - v\chi_\omega\right)dtdadx = 0. \end{aligned} \tag{3.23}$$

Define the adjoint state

$$\rho_\epsilon = -\frac{1}{\epsilon}\left(-\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon - \beta q_\epsilon(t,0,x) - h - v_\epsilon \chi_\omega\right). \tag{3.24}$$

Then (3.19) holds.

For any (v,φ) such that (3.10),(3.23) becomes

$$\int_U\int_\omega v_\epsilon v dtdadx + \int_Q \rho_\epsilon(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} - \Delta\varphi + \mu\varphi - \beta\varphi(t,0,x) - v\chi_\omega)dtdadx = 0. \tag{3.25}$$

Integrate by parts in (3.25). As a consequence the couple $(v_\epsilon,\rho_\epsilon)$ is shown to satisfy

$$\begin{cases} \frac{\partial\rho_\epsilon}{\partial t} + \frac{\partial\rho_\epsilon}{\partial a} - \Delta\rho_\epsilon + \mu\rho_\epsilon = 0 \text{ in } Q, \\ \rho_\epsilon = 0 \text{ on } \Sigma, \\ \rho_\epsilon(t,0,x) = \int_0^A \beta(t,a,x)\rho_\epsilon(t,a,x)da; \end{cases} \tag{3.26}$$

and

$$\int_U\int_\omega (v_\epsilon + \rho_\epsilon) v dtdadx = 0, \ \forall v \in Y_\lambda^\perp. \tag{3.27}$$

Hence $v_\epsilon + \rho_\epsilon\chi_\omega \in Y_\lambda$. Since $v_\epsilon \in Y_\lambda^\perp$ then $v_\epsilon + \rho_\epsilon\chi_\omega = P(v_\epsilon + \rho_\epsilon\chi_\omega) = P\rho_\epsilon$ and thus

$$v_\epsilon = -(\rho_\epsilon\chi_\omega - P\rho_\epsilon). \tag{3.28}$$

Hence the assertion follows. □

Remark 3.7. There is no available information concerning $\rho_\epsilon(t,A,x)$ in Q_T, $\rho_\epsilon(0,a,x)$ in Q_A, $\rho_\epsilon(T,a,x)$ in Q_A.

Proposition 3.8. *Let $(v_\epsilon, q_\epsilon, \rho_\epsilon)$ be defined as in Proposition 3.6. Then there exists a constant $C > 0$ independent on ϵ such that*

$$\begin{aligned} \|q_\epsilon\|_{L^2(Q)} &\leq C, &(3.29)\\ \|\rho_\epsilon - P\rho_\epsilon\|_{L^2(U\times\omega)} &\leq C, &(3.30)\\ \|\rho_\epsilon\|_{L^2(U\times\omega)} &\leq C, &(3.31)\\ \|\rho_\epsilon\|_V &\leq C. &(3.32) \end{aligned}$$

Proof. From (3.14), we have

$$\begin{aligned} \left\|-\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon - \beta q_\epsilon(t,0,x) - h - v_\epsilon\chi_\omega\right\|_{L^2(Q)} &\leq C\sqrt{\epsilon} &(3.33)\\ \|v_\epsilon\|_{L^2(U\times\omega)} &\leq C. &(3.34) \end{aligned}$$

Since q_ϵ verifies (3.10), we derive from (3.33), the relation (3.29). From (3.22) and (3.34), we obtain (3.30). Then as $L\rho_\epsilon = 0$, using the definition of the norm on V given by (3.2), we have (3.32) in one hand.

On the over hand, since $\rho_\epsilon \in \mathcal{V}$, applying the observability inequality (2.13) to ρ_ϵ, we have $\left\|\frac{1}{\theta}\rho_\epsilon\right\|_{L^2(U\times\omega)} \leq C$. Therefore, using (3.30) and the fact that $\frac{1}{\theta}$ is in $L^\infty(Q)$, we deduce that $\left\|\frac{1}{\theta}P\rho_\epsilon\right\|_{L^2(U\times\omega)} \leq C$. Since $P\rho_\epsilon$ is in Y_λ which is finite dimensional, we have $\|P\rho_\epsilon\|_{L^2(U\times\omega)} \leq C$. Hence using again (3.30), we obtain estimate (3.31). □

Proof of Theorem 1.5. We proceed in three steps:

Step 1. We study the convergence of $(v_\epsilon, q_\epsilon)_\epsilon$.

According to (3.34) and (3.29) we can extract subsequences, still denoted $(q_\epsilon)_\epsilon$ and $(v_\epsilon)_\epsilon$ such that

$$v_\epsilon \rightharpoonup v_0 \text{ weakly in } L^2(U\times\omega), \tag{3.35}$$

$$q_\epsilon \rightharpoonup q_0 \text{ weakly in } L^2(Q). \tag{3.36}$$

And, as v_ϵ belongs to $Y_\lambda^\perp$ which is a closed vector subspace of $L^2(U\times\omega)$, we have

$$v_0 \in Y_\lambda^\perp. \tag{3.37}$$

From (3.36), we have $q_\epsilon \rightharpoonup q_0$ weakly in $\mathcal{D}'(Q)$ and by the weak continuity of the operator L^* in $\mathcal{D}'(Q)$ it follows $L^* q_\epsilon \rightharpoonup L^* q_0$ weakly in $\mathcal{D}'(Q)$. Moreover the traces functions are continuous, then the pair (v_0, q_0) satisfies the system

$$\begin{cases} -\dfrac{\partial q_0}{\partial t} - \dfrac{\partial q_0}{\partial a} - \Delta q_0 + \mu q_0 = \beta q_0(t,0,x) + h + v_0\chi_\omega \text{ in } Q, \\ q_0 = 0 \qquad \text{on} \quad \Sigma, \\ q_0(T,a,x) = 0 \qquad \text{in} \quad Q_A, \\ q_0(t,A,x) = 0 \qquad \text{in} \quad Q_T. \end{cases} \tag{3.38}$$

$$q_0(0,a,x) = 0 \ \text{ in } \ Q_A. \tag{3.39}$$

Step 2. We prove that $(v_0, q_0 = q(t,a,x,v_0)) = (\widehat{v_\theta}, \widehat{q_\theta} = q(t,a,x,\widehat{v_\theta}))$.

From the expression of J_ϵ given by (3.9), we can write

$$\frac{1}{2}\|v_\epsilon\|^2_{L^2(U\times\omega)} \le J_\epsilon(v_\epsilon, q_\epsilon).$$

Since $(\widehat{v_\theta}, \widehat{q_\theta})$ satisfies (1.12)-(1.14) (or equivalently verifies (3.10)), this latter inequality becomes

$$\frac{1}{2}\|v_\epsilon\|^2_{L^2(U\times\omega)} \le J_\epsilon(v_\epsilon, q_\epsilon) \le \frac{1}{2}\left\|\widehat{v_\theta}\right\|^2_{L^2(U\times\omega)}. \tag{3.40}$$

Then using (3.35) while passing to the limit in (3.40), we obtain

$$\frac{1}{2}\|v_0\|^2_{L^2(U\times\omega)} \le \liminf_{\epsilon\to 0} J_\epsilon(v_\epsilon, q_\epsilon) \le \frac{1}{2}\left\|\widehat{v_\theta}\right\|^2_{L^2(U\times\omega)}.$$

Consequently,

$$\|v_0\|_{L^2(U\times\omega)} \le \left\|\widehat{v_\theta}\right\|_{L^2(U\times\omega)},$$

and thus,

$$\|v_0\|_{L^2(U\times\omega)} = \left\|\widehat{v_\theta}\right\|_{L^2(U\times\omega)}.$$

Hence, $v_0 = \widehat{v_\theta}$ and since (3.38) has a unique solution, it follows that $q_0 = \widehat{q_\theta}$.

Step 3. According to the inequalities (3.31) and (3.32), we can extract a subsequence, still denoted $(\rho_\epsilon)_\epsilon$ such that

$$\rho_\epsilon \rightharpoonup \widehat{\rho_\theta} \text{ weakly in } L^2(U\times\omega), \tag{3.41}$$

$$\rho_\epsilon \rightharpoonup \widehat{\rho_\theta} \text{ weakly in } V. \tag{3.42}$$

As P is a compact operator, we deduce from (3.41) that

$$P\rho_\epsilon \to P\widehat{\rho_\theta} \text{ strongly in } L^2(U\times\omega). \tag{3.43}$$

Therefore, combining (3.41) and (3.43), we get

$$v_\epsilon = \rho_\epsilon\chi_\omega - P\rho_\epsilon \rightharpoonup \widehat{v_\theta} = \widehat{\rho_\theta}\chi_\omega - P\widehat{\rho_\theta} \text{ weakly in } L^2(U\times\omega).$$

Thus, we have proved that there exists θ given by (2.11) such that for a given $h \in L^2(Q)$ with $\theta h \in L^2(Q)$, the unique pair $(\widehat{v_\theta},\widehat{q_\theta})$ satisfies (1.12)-(1.15) with $\widehat{v_\theta} = \widehat{\rho_\theta}\chi_\omega - P\widehat{\rho_\theta}$, and where $\widehat{\rho_\theta}$ is a solution of (1.18). Since the function h defined by (2.8) belongs to $L^2(Q)$ if $\theta h \in L^2(Q)$, the proof of Theorem 1.5 is complete. □

4 Expression of the sentinel with given sensitivity and identification of parameter λ_i

We can now give the expression of the sentinel S defined by (1.3)-(1.6) and identify the parameter λ_i.

4.1 Expression of the sentinel with given sensitivity

We consider the results obtained in the previous sections and we assume that h given by (2.8) and θ given by (2.11) are such that $\theta h \in L^2(U\times O)$. Let $(\widehat{\rho_\theta},\widehat{v_\theta})$ be defined as in Theorem 1.5. Since $\widehat{v_\theta} = -(\widehat{\rho_\theta}\chi_\omega - P\widehat{\rho_\theta})$ realizes the minimum in $L^2(U\times\omega)$ among all controls v such that the pair (v,q) satisfies (1.12)-(1.15), using (2.7), we deduce that $w = w_0 + \widehat{v_\theta} = w_0 - (\widehat{\rho_\theta}\chi_\omega - P\widehat{\rho_\theta})$. Consequently, replacing w by its expression in (1.3), the function S becomes:

$$S(\lambda,\tau) = \int_U\int_O h_0 y(\lambda,\tau)dtdadx + \int_U\int_\omega (w_0 - (\widehat{\rho_\theta} - P\widehat{\rho_\theta}\chi_\omega))y(\lambda,\tau)dtdadx, \tag{4.1}$$

and (w,S) is such that (1.4)-(1.6) hold.

4.2 Identification of the parameter λ_i

y_0 is the solution of the problem (1.1) when $\lambda = 0$ and $\tau = 0$. Hence, from (4.1) we have

$$S(0,0) = \int_U\int_O h_0 y_0 dtdadx + \int_U\int_\omega (w_0 - (\widehat{\rho_\theta} - P\widehat{\rho_\theta}\chi_\omega))y_0 dtdadx = 0.$$

Next, using (1.4), we obtain

$$S(\lambda,\tau) - S(0,0) \simeq \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial\lambda_i}(0,0) \text{ for } \lambda_i \text{ and } \tau \text{ small}.$$

Since get at our disposal the observation y_{obs}, we get

$$S(\lambda,\tau) - S(0,0) = \int_U\int_O h_0(y_{obs} - y_0)dtdadx + \int_U\int_\omega w(y_{obs} - y_0)dtdadx.$$

Thus, we also have the following information:

$$\sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0,0) \simeq \int_U \int_O h_0(y_{obs} - y_0)dtdadx + \int_U \int_\omega w(y_{obs} - y_0)dtdadx,$$

which, using (1.5) gives

$$\sum_{i=1}^{M} \lambda_i c_i \simeq \int_U \int_O h_0(y_{obs} - y_0)dtdadx + \int_U \int_\omega w(y_{obs} - y_0)dtdadx.$$

Now, fixing $i \in \{1,...,M\}$ and choosing $c_i \neq 0$ and $c_j = 0$, for all j in $\{1,...,M\}$ with $j \neq i$, we get this estimate of the parameter λ_i

$$\lambda_i \simeq \frac{1}{c_i}\left\{\int_U \int_O h_0(y_{obs} - y_0)dtdadx + \int_U \int_\omega w(y_{obs} - y_0)dtdadx\right\}.$$

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