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Projective Motion in Special Finsler Spaces

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Abstract: The present paper deals with the differential geometry of a Finsler space whose projective deviation tensor satisfies certain conditions. It discusses the projective motion in such Finsler space. A sufficient condition has been obtained for the projective motion to be an affine motion in a Finsler space whose projective deviation tensor is recurrent. Similar problems have been discussed for recurrent and projective recurrent Finsler spaces. It is established that the projective motion, in a Finsler space for which the covariant derivative of the projective deviation tensor is symmetric, is an affine motion or the space is of scalar curvature.

AMS Subject Classification: 53B40.

Keywords: Projective motion, Affine motion, Projective deviation tensor, Recurrent and Projective recurrent Finsler spaces.

1 Introduction

Takano [4] discussed certain types of affine motion generated by contra, concurrent, special concircular, torse forming and birecurrent vector fields in a non-Riemannian manifold of recurrent curvature. Following Takano's techniques, Sinha [16], Misra and Meher [15] and Kumar [1] studied the above types of affine motion in a Finsler space of recurrent curvature and obtained various results. The first author [9], for the first time, obtained necessary and sufficient conditions for these vector fields to generate an affine motion in a general Finsler space. Also, he [10, 11] obtained the necessary and sufficient conditions for the projective motion generated by these vector fields to be an affine motion in a general Finsler space. He [13] discussed projective motion in symmetric and projectively symmetric Finsler spaces and proved that every projective motion in a non-Riemannian symmetric Finsler space as well as in a non-trivial projectively symmetric space is necessarily an affine motion. S. P. Singh [17] studied projective motion in Finsler spaces and investigated that if the projective motion becomes an affine motion, then the partial derivative and covariant derivative of the

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Berwald curvature tensor are Lie-invariant. C. K. Mishra, *et al.* [2] discussed some special types of projective motions in a projective Finsler space.

The purpose of the present paper is to study projective motion in some special type of Finsler spaces. First, we consider a Finsler space whose projective deviation tensor is recurrent. We discuss projective motion in such Finsler space and obtain a sufficient condition for the projective motion to be an affine motion. We prove that the projective motion, in a Finsler space whose projective deviation tensor is recurrent, is necessarily an affine motion if the recurrence vector is Lie-invariant. We investigate that the same condition holds for the projective motion to be an affine motion in recurrent and projective recurrent Finsler spaces. Next, we consider a Finsler space for which the covariant derivative (in Berwald sense) of the projective deviation tensor is symmetric. For such space, we obtain a very interesting result which states that if a Finsler space $F^n(n>2)$ for which the covariant derivative of projective deviation tensor is symmetric, admits an infinitesimal projective motion, then the projective motion is an affine motion or the space is of scalar curvature.

2 Preliminaries

Let $F^n(n>2)$ be an n-dimensional Finsler space of class C^6 equipped with a metric function F satisfying the requisite conditions [3, 6]. Let g_{ij}, G^i_{jk}, H^i_{jkh} and W^i_{jkh} be the components of corresponding metric tensor, Berwald's connection parameters, components of Berwald's curvature tensor and components of projective curvature tensor respectively. The curvature tensor H^i_{jkh} and projective curvature tensor W^i_{jkh} are skew-symmetric in their last two lower indices and are positively homogeneous of degree zero in $\dot{x}^i$'s.
Transvecting these tensors by $\dot{x}$'s, we obtain the following tensors:

$$(a)\ H^i_{kh}=H^i_{jkh}\dot{x}^j,\ (b)\ W^i_{kh}=W^i_{jkh}\dot{x}^j,\ (c)\ H^i_h=H^i_{kh}\dot{x}^k,\ (d)\ W^i_h=W^i_{kh}\dot{x}^k. \tag{2.1}$$

The tensors H^i_h and W^i_h satisfy

$$\begin{aligned}
a)&\quad H^i_{kh}=\frac{1}{3}(\dot{\partial}_k H^i_h-\dot{\partial}_h H^i_k),\\
b)&\quad H^i_i=(n-1)H,\\
c)&\quad W^i_h\dot{x}^h=0,\\
d)&\quad \dot{x}^m\dot{\partial}_m W^i_h=2W^i_h\\
e)&\quad W^i_h=H^i_h-H\delta^i_h-\frac{\dot{x}^i}{n+1}(\dot{\partial}_r H^r_h-\dot{\partial}_h H),\\
f)&\quad y_iW^i_h=0,\\
g)&\quad y_iH^i_h=0,
\end{aligned} \tag{2.2}$$

where H is scalar curvature, $y_i=g_{ik}\dot{x}^k$ and $\dot{\partial}_k\equiv\frac{\partial}{\partial\dot{x}^k}$.
Commutation formula for the operators $\dot{\partial}_k$ and Berwald covariant differential operator $\mathcal{B}_h$ is given by

$$\dot{\partial}_k\mathcal{B}_hT^i_j-\mathcal{B}_h\dot{\partial}_kT^i_j=T^r_jG^i_{khr}-T^i_rG^r_{khj}, \tag{2.3}$$

where T^i_j is an arbitrary tensor and $G^i_{jkh} = \dot{\partial}_h G^i_{jk}$ is symmetric in all its lower indices and satisfies

$$G^i_{jkh}\dot{x}^h = 0\,. \tag{2.4}$$

A Finsler Space F^n is said to be recurrent or projective recurrent according as

$$\mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh}, \quad H^i_{jkh} \neq 0\,, \tag{2.5}$$

or

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh}, \quad W^i_{jkh} \neq 0\,, \tag{2.6}$$

where λ_m is a non-zero vector, called the recurrence vector [1, 7, 15]. Pandey [8] proved that λ_m is independent of directional arguments.

The tensor W^i_{kh} and the projective deviation tensor W^i_h are recurrent in recurrent and projective recurrent Finsler Spaces, i.e.

$$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh}, \quad W^i_{kh} \neq 0\,, \tag{2.7}$$

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h, \quad W^i_h \neq 0\,. \tag{2.8}$$

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i(x^j)\,, \tag{2.9}$$

where v^i is a contravariant vector field and ϵ is an infinitesimal constant.

The infinitesimal transformation (2.9) is called an affine motion if it preserves parallelism of vectors while it is a projective motion if it preserves geodesics.
The necessary and sufficient condition for the transformation (2.9) to be an affine motion is

$$\pounds G^i_{jk} = 0\,, \tag{2.10}$$

where £ denotes the operator of Lie differentiation.
The necessary and sufficient condition for (2.9) to be a projective motion is

$$\pounds G^i_{jk} = \delta^i_j P_k + \delta^i_k P_j + \dot{x}^i P_{jk}\,, \tag{2.11}$$

where

$$(a)\ P_j = \dot{\partial}_j P, \quad (b)\ P_{jk} = \dot{\partial}_j \dot{\partial}_k P\,, \tag{2.12}$$

P being a scalar invariant positively homogeneous of degree one in $\dot{x}^i$[5]. The homogeneity of P gives rise to

$$(a)\ P_j\dot{x}^j = P, \quad (b)\ P_{jk}\dot{x}^k = 0\,. \tag{2.13}$$

The integrability condition for (2.11) is given by

$$\pounds W^i_{jkh} = 0\,. \tag{2.14}$$

It is well known that every affine motion is a projective motion. A non-affine projective motion is characterized by (2.11) and $P \neq 0$.

3 Projective Motion in Special Finsler Spaces

Let a Finsler space $F^n(n > 2)$ admits a projective motion. Then all equations from (2.11) to (2.14) hold. Transvecting (2.14) by $\dot{x}^j \dot{x}^k$ and using (2.1 b) and (2.1 d), we have

$$\pounds W^i_h = 0, \tag{3.1}$$

for $\pounds \dot{x}^i = 0$.
The commutation formula for the operators £ and $\mathcal{B}_m$ is given by

$$\pounds \mathcal{B}_k W^i_h - \mathcal{B}_k \pounds W^i_h = W^r_h \pounds G^i_{rk} - W^i_r \pounds G^r_{hk} - (\dot{\partial}_r W^i_h) \pounds G^r_{km} \dot{x}^m. \tag{3.2}$$

Suppose that the space considered satisfies (2.8). Then (3.2), in view of (2.2 c), (2.8), (2.11), (2.13) and (3.1), gives

$$(\pounds \lambda_k) W^i_h = \dot{x}^i W^r_h P_{rk} + \delta^i_k P_r W^r_h - P_h W^i_k - P \dot{\partial}_k W^i_h - P_k \dot{x}^r \dot{\partial}_r W^i_h. \tag{3.3}$$

Using (2.2 d) in (3.3), we obtain

$$W^i_h (\pounds \lambda_k) = \dot{x}^i W^r_h P_{rk} + \delta^i_k P_r W^r_h - P_h W^i_k - P \dot{\partial}_k W^i_h - 2 P_k W^i_h. \tag{3.4}$$

Transvection of (3.4) by $\dot{x}^k$ followed by use of (2.2 c), (2.2 d) and (2.13), gives

$$W^i_h \pounds (\lambda_k \dot{x}^k) = \dot{x}^i W^r_h P_r - 4 P W^i_h. \tag{3.5}$$

Transvecting (3.5) by y_i and using (2.2 f), we have

$$P_r W^r_h = 0. \tag{3.6}$$

Using (3.6) in (3.5), we get

$$W^i_h \left(\pounds (\lambda_k \dot{x}^k) + 4P \right) = 0,$$

which implies

$$\pounds (\lambda_k \dot{x}^k) = -4P, \tag{3.7}$$

for $W^i_h \neq 0$. If we assume that $\pounds \lambda_l = 0$, then from (3.7), we find $P = 0$, which shows that the projective motion is necessarily an affine motion. Thus, we have:

Theorem 3.1. *If a Finsler space $F^n(n > 2)$, satisfying $\mathcal{B}_m W^i_h = \lambda_m W^i_h$, $W^i_h \neq 0$, admits an infinitesimal projective motion with respect to which the recurrence vector λ_m is Lie-invariant, i.e. $\pounds \lambda_m = 0$, then the projective motion is necessarily an affine motion.*

Let us consider a recurrent Finsler space $F^n(n > 2)$. Since $W^i_h = 0$ implies that the space is of scalar curvature and a recurrent space cannot be of scalar curvature [14], therefore in a recurrent space $W^i_h \neq 0$. Also, in a recurrent space W^i_h is recurrent, i.e. $\mathcal{B}_m W^i_h = \lambda_m W^i_h$ holds. Therefore from Theorem 1, we have:

Corollary 3.2. *If a recurrent Finsler space $F^n(n > 2)$ admits an infinitesimal projective motion with respect to which the recurrence vector is Lie-invariant, then the projective motion is necessarily an affine motion.*

Let us consider a projective recurrent Finsler space $F^n(n>2)$. Since in a projective recurrent Finsler space the condition $\mathcal{B}_m W^i_h = \lambda_m W^i_h$, $W^i_h \neq 0$ holds good. Therefore, we may state:

Corollary 3.3. *If a projective recurrent Finsler space $F^n(n>2)$ admits an infinitesimal projective motion with respect to which the recurrence vector is Lie-invariant, then the projective motion is necessarily an affine motion.*

Using (3.1) in (3.2), we have

$$\pounds\mathcal{B}_k W^i_h = W^r_h \pounds G^i_{rk} - W^i_r \pounds G^r_{hk} - (\dot{\partial}_r W^i_h)\pounds G^r_{km}\dot{x}^m\,,$$

which in view of (2.2 c), (2.11) and (2.13), gives

$$\pounds\mathcal{B}_k W^i_h = \dot{x}^i W^r_h P_{rk} + \delta^i_k P_r W^r_h - P_h W^i_k - P\dot{\partial}_k W^i_h - P_k \dot{x}^r \dot{\partial}_r W^i_h\,. \tag{3.8}$$

Using (2.2 d) in (3.8), we have

$$\pounds\mathcal{B}_k W^i_h = \dot{x}^i W^r_h P_{rk} + \delta^i_k P_r W^r_h - P_h W^i_k - P\dot{\partial}_k W^i_h - 2P_k W^i_h\,. \tag{3.9}$$

Transvecting (3.9) by $\dot{x}^k$ and using (2.2 c), (2.2 d) and (2.13), we have

$$\dot{x}^k(\pounds\mathcal{B}_k W^i_h) = \dot{x}^i W^r_h P_r - 4PW^i_h\,. \tag{3.10}$$

Now, if Berwald covariant derivative of projective deviation tensor is symmetric, i.e. $\mathcal{B}_k W^i_h = \mathcal{B}_h W^i_k$, then from (3.10) we have

$$\dot{x}^k(\pounds\mathcal{B}_h W^i_k) = \dot{x}^i W^r_h P_r - 4PW^i_h\,,$$

which implies that

$$\pounds\mathcal{B}_h(\dot{x}^k W^i_k) = \dot{x}^i W^r_h P_r - 4PW^i_h,\;\; for\, \pounds\dot{x}^k = 0 = \mathcal{B}_h\dot{x}^k\,. \tag{3.11}$$

Equation (3.11), in view of (2.2 c), reduces to

$$\dot{x}^i W^r_h P_r - 4PW^i_h = 0\,. \tag{3.12}$$

Transvecting (3.12) by y_i and using (2.2 f), we have

$$W^r_h P_r = 0\,. \tag{3.13}$$

Using (3.13) in (3.12), we get at least one of the conditions

$$(a)\;\; W^i_h = 0,\;\; (b)\;\; P = 0\,. \tag{3.14}$$

The condition (3.14 a) implies that the space is of scalar curvature [12, 18] while (3.14 b) implies that the projective motion is an affine motion. Thus, we conclude:

Theorem 3.4. *If a Finsler space $F^n(n>2)$, for which the covariant derivative of projective deviation tensor is symmetric, admits an infinitesimal projective motion, then the projective motion is an affine motion or the space is of scalar curvature.*

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