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A Regularization Proximal Point Algorithm for Zeros of Accretive Operators in Banach Spaces

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Abstract

In this paper, we study the strong convergence of a regularization proximal point algorithm for the problem of finding a zero of $m-$accretive operators in a uniformly smooth Banach space E, and the stability of the regularization algorithms are considered.

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1 Introduction

Let E be a Banach space, let $A: E \longrightarrow 2^E$ be an $m-$accretive operator. It is well known that many problems in nonlinear analysis and optimization can be formulated as the problem:

$$\text{find an } x \text{ such that } 0 \in A(x).$$

This problem has been investigated by many researchers: see, for instance, Benavides et al. [6], Brézis and Lions [8], Ha and Jung [13], Jung and Takahashi [14, 15], Reich [22], Rockafellar [23], Xu [27, 28] and others. One popular method of solving equation $0 \in A(x)$ is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in E$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J_{r_n}^A(x_n), n \geq 0, \tag{1.1}$$

where $\{r_n\}$ is a sequence of positive real numbers and $J_{r_n}^A = (I + r_n A)^{-1}$ is the resolvent of A. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (1.1) and others proved strong convergence theorems by imposing assumptions on A.

Note that, algorithm (1.1), can be rewritten as

$$x_{n+1} - x_n + r_n A(x_{n+1}) \ni 0. \tag{1.2}$$

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This proximal iteration may be interpreted as an implicit one-step discretization method for the evolution differential inclusion

$$\frac{dx}{dt}(t) + A(x(t)) \ni 0, \text{ a.e. } t \geq 0, \tag{1.3}$$

where the parameter r_n is a (variable) stepsize. Let H be a real Hilbert space and A be a maximal monotone on H. When $A^{-1}(0) \neq \emptyset$ and A is demipositive, R. Bruck [9] proved the following convergence result: every solution trajectory $\{x(t) : t \longrightarrow \infty\}$ of (1.3) converges weakly in H to an element of $A^{-1}(0)$. To know more informations and new results for the evolution differential inclusion, we can see in [4, 7, 24, 29]...

In particular, in 1976, Rockafellar [23] devised the proximal point algorithm which generates, starting with an arbitrary initial x_0 in Hilbert space H, a sequence $\{x_n\}$ satisfying:

$$x_{n+1} = J^A_{r_n}(x_n) + e_n, \; n \geq 0, \tag{1.4}$$

where A is a maximal monotone operator in H, $r_n > 0$ is a real number, and e_n is an error vector. Rockafellar proved the weak convergence of algorithm (1.4) if the sequence $\{r_n\}$ is bounded away from zero and if the sequence of the errors satisfies the condition: $\sum_n \|e_n\| < \infty$. An analogous result was established by O. Nevanlinna and S. Reich [19] for the problem of finding a zero of the accretive operator A in Banach spaces. They considered the sequence $\{x_n\}$ defined by

$$x_{n+1} + \lambda_{n+1} A x_{n+1} \ni x_n + e_{n+1}, \; n \geq 0, \tag{1.5}$$

where $\{\lambda_n\}$ is a positive sequence, and they obtained the strong convergence of the sequence $\{x_n\}$ to an element of $A^{-1}0$ when $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and the operator A satisfies the converge condition. In 1991, Güler [11] gave an example showing that Rockafellar's proximal point algorithm does not converge strongly. An example of the authors Bauschke, Matoušková and Reich [5] also showed that the proximal algorithm only converges weakly but not in norm. Solodov and Svaiter [25] in 2000 proposed a modified proximal point algorithm which converges strongly to a solution of equation $0 \in A(x)$ by using projection method. Motivated by iterative algorithms of Halpern's type [12] and Mann's type [18], Kamimura and Takahashi [16] introduced the iterative algorithms in Hilbert spaces and Banach spaces:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) J^A_{r_n}(x_n), \; n \geq 0, \tag{1.6}$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J^A_{r_n}(x_n), \; n \geq 0, \tag{1.7}$$

and showed that the sequence $\{x_n\}$ generated by (1.6) converges strongly to some $v \in A^{-1}(0)$ and the sequence $\{x_n\}$ generated by (1.7) converges weakly to some $v \in A^{-1}(0)$. Lehdili and Moudafi [17] obtained the convergence of the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = J^{A_n}_{c_n}(x_n), \tag{1.8}$$

where $A_n = \mu_n I + A$ is viewed as a Tikhonov regularization of A.

When A is maximal monotone in Hilbert space H, in 2006, Xu [27]; in 2009, Song and Yang [26] used the technique of nonexpansive mappings to get convergence theorems for $\{x_n\}$ defined by the perturbed version of the algorithm (1.4)

$$x_{n+1} = J^A_{r_n}(t_n u + (1-t_n)x_n + e_n). \tag{1.9}$$

Note that, the algorithm (1.9) can be rewritten as

$$r_n A(x_{n+1}) + x_{n+1} \ni t_n u + (1-t_n)x_n + e_n,\ n \geq 0. \tag{1.10}$$

In this paper, we use the regularization proximal point algorithm (1.10) and the technique of accretive operators to get convergence theorems for the problem of finding a zero of $m-$accretive operator in Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $||.||$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

We know that if C is a closed convex subset of a reflexive strictly convex Banach E, then for each $x \in E$, there exists a unique element $u = P_C x \in C$ with $||x-u|| = \inf\{||x-y||\ :\ y \in C$. Such a P is called the metric projection of E onto C.

The function

$$\rho_E(\tau) = \sup\{2^{-1}(||x+y|| + ||x-y||) - 1 :\ ||x|| = 1,\ ||y|| = \tau\}, \tag{2.1}$$

is called the modulus of smoothness of the space E. The function $\rho_E(\tau)$ defined on the interval $[0,+\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.2}$$

It is well known that every uniformly smooth Banach space is reflexive.

A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 \text{ and } ||f|| = ||x||\} \tag{2.3}$$

is called the normalized duality mapping of E. In any smooth Banach space $J(x) = 2^{-1}\text{grad}||x||^2$, and if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J is single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$. We know that every Hilbert spaces and the l^p spaces with $1 < p < \infty$ are uniformly smooth spaces and have a weakly sequentially continuous duality mappings [1]. We denote the single valued normalized duality mapping by j.

Lemma 2.1. [2] *In an uniformly smooth Banach space E, for all $x, y \in E$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + c\rho_E(\|y\|), \tag{2.4}$$

where $c = 48\max(L, \|x\|, \|y\|)$.

Remark 2.2. Reich [21] established a similar inequality with inequality (2.4) in the form

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|), \tag{2.5}$$

where

$$\beta(t) = \sup\{(\|x+ty\|^2 - \|x\|^2)/t - 2\langle y, j(x)\rangle : \ \|x\| \le 1, \ \|y\| \le 1\}.$$

An operator $A : \ D(A) \subseteq E \longrightarrow 2^E$ is called accretive if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \ge 0, \ \forall u \in A(x), \ v \in A(y). \tag{2.6}$$

An operator $A : \ D(A) \subseteq E \longrightarrow 2^E$ is called $m-$accretive if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$. If A is a $m-$accretive operator in Banach space E with E has a weakly sequentially continuous duallity mapping J, then it is a demiclosed operator, i.e., if the sequence $\{x_n\} \subset D(A)$ satisfies $x_n \rightharpoonup x$ and $A(x_n) \ni y_n \longrightarrow f$, then $A(x) = f$ [3].

A mapping Q of C into C is said to be a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let D be a subset of E and let Q be a mapping of C into D. Then Q is said to be sunny if each point on the ray $\{Qx + t(x - Qx) : \ t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t > 0$ and $x \in C$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D [20].

Proposition 2.3. [10] *Let G be a nonempty closed convex subset of a smooth Banach space E. A mapping $Q_G : \ E \longrightarrow G$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_G x, j(\xi - Q_G x)\rangle \le 0, \ \forall x \in E, \ \forall \xi \in G. \tag{2.7}$$

Reich [22] showed that if E is uniformly smooth Banach and $A : \ D(A) \subseteq E \longrightarrow 2^E$ is an $m-$accretive mapping with $A^{-1}(0) \neq \emptyset$, then there exists a sunny nonexpansive retraction Q from E onto $A^{-1}(0)$.

Let $C_1, \ C_2$ be convex subsets of E. The quantity

$$\beta(C_1, C_2) = \sup_{u \in C_1} \inf_{v \in C_2} \|u - v\| = \sup_{u \in C_1} d(u, C_2)$$

is said to be semideviation of the set C_1 from the set C_2. The function

$$\mathcal{H}(C_1, C_2) = \max\{\beta(C_1, C_2), \ \beta(C_2, C_1)\}$$

is said to be a Hausdorff distance between C_1 and C_2.

Finally, we need the following lemma:

Lemma 2.4. [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\beta_n + \sigma_n, \ \forall n \geq 0$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ satisfy the conditions

i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

ii) $\limsup_{n\longrightarrow\infty} \beta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\lambda_n\beta_n| < \infty$;

iii) $\sigma_n \geq 0, \ \forall n \geq 0$ *and* $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\{a_n\}$ converges to zero as $n \longrightarrow \infty$.

3 Main results

Let E be an uniformly smooth Banach space and $A: \ D(A) \subseteq E \longrightarrow 2^E$ be an $m-$accretive operator with $S = A^{-1}(0) \neq \emptyset$.

Now we study the strong convergence of sequence $\{x_n\}$ generated by the following algorithm: $u, \ x_0 \in E$,

$$r_nA(x_{n+1}) + x_{n+1} \ni t_nu + (1-t_n)x_n, \ n \geq 0, \tag{3.1}$$

where $\{t_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$.

Theorem 3.1. *Let E be an uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $A: \ D(A) \subseteq E \longrightarrow 2^E$ be an $m-$ accretive operator with $S = A^{-1}(0) \neq \emptyset$. If the sequences $\{r_n\} \subset (0,+\infty)$ and $\{t_n\} \subset (0,1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;

ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ generated by (3.1) *converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction of E onto S.*

Proof. Since A is an $m-$accretive operator, equation (3.1) has solution, i.e., there exists x_{n+1} such that

$$r_nA(x_{n+1}) + x_{n+1} \ni t_nu + (1-t_n)x_n. \tag{3.2}$$

Hence, for each n, there exists $y_{n+1} \in A(x_{n+1})$ such that

$$r_ny_{n+1} + x_{n+1} = t_nu + (1-t_n)x_n. \tag{3.3}$$

For each $x^* \in S$, we have

$$\langle r_ny_{n+1}, j(x_{n+1}-x^*)\rangle \geq 0, \ \forall n \geq 0. \tag{3.4}$$

Therefore,

$$\langle t_nu + (1-t_n)x_n - x_{n+1}, j(x_{n+1}-x^*)\rangle \geq 0, \ \forall n \geq 0. \tag{3.5}$$

It gives the inequality

$$\|x_{n+1} - x^*\|^2 \leq [t_n\|u - x^*\| + (1 - t_n)\|x_n - x^*\|].\|x_{n+1} - x^*\|, \ \forall n \geq 0.$$

Since $\|x_{n+1} - x^*\| \geq 0, \ \forall n \geq 0$, we obtain

$$\|x_{n+1} - x^*\| \leq t_n\|u - x^*\| + (1 - t_n)\|x_n - x^*\|, \ \forall n \geq 0. \tag{3.6}$$

Consequently,

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq t_n \max(\|u - x^*\|, \|x_n - x^*\|) + (1 - t_n) \max(\|u - x^*\|, \|x_n - x^*\|) \\
&= \max(\|u - x^*\|, \|x_n - x^*\|) \\
&\leq \max(\|u - x^*\|, \|x_{n-1} - x^*\|) \\
&\ \ \vdots \\
&\leq \max(\|u - x^*\|, \|x_0 - x^*\|), \ \forall n \geq 0.
\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which converges weakly to a limit point $\overline{x} \in E$.

From equation (3.3) and the sequence $\{x_n\}$ is bounded, we get

$$\|y_{n+1}\| = \frac{1}{r_n}\|t_n u + (1 - t_n)x_n\| \longrightarrow 0, \ n \longrightarrow \infty. \tag{3.7}$$

It is clear that $\overline{x} \in S$ because the operator A is demiclosed. Hence, noting the inequality (2.7), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - Q_S u, j(x_n - Q_S u)\rangle &= \lim_{k\to\infty} \langle u - Q_S u, j(x_{n_k} - Q_S u)\rangle \\
&= \langle u - Q_S u, j(\overline{x} - Q_S u)\rangle \leq 0.
\end{aligned} \tag{3.8}$$

Next, we have

$$\begin{aligned}
\|x_{n+1} - Q_S u\|^2 &= \langle -r_n y_{n+1} + t_n u + (1 - t_n)x_n - Q_S u, j(x_{n+1} - Q_S u)\rangle \\
&= -\langle r_n y_{n+1}, j(x_{n+1} - Q_S u)\rangle \\
&\quad + \langle t_n u + (1 - t_n)x_n - Q_S u, j(x_{n+1} - Q_S u)\rangle \\
&\leq \langle t_n(u - Q_S u) + (1 - t_n)(x_n - Q_S u), j(x_{n+1} - Q_S u)\rangle \\
&\leq \frac{1}{2}[\|t_n(u - Q_S u) + (1 - t_n)(x_n - Q_S u)\|^2 + \|x_{n+1} - Q_S u\|^2].
\end{aligned}$$

By the Lemma 2.1 and the estimate above, we conclude that

$$\begin{aligned}
\|x_{n+1} - Q_S u\|^2 &\leq \|t_n(u - Q_S u) + (1 - t_n)(x_n - Q_S u)\|^2 \\
&\leq (1 - t_n)^2\|x_n - Q_S u\|^2 + 2t_n(1 - t_n)\langle u - Q_S u, j(x_n - Q_S u)\rangle \\
&\quad + c\rho_E(t_n\|u - Q_S u\|).
\end{aligned}$$

Consequently,

$$\|x_{n+1} - Q_S u\|^2 \leq (1 - t_n)\|x_n - Q_S u\|^2 + t_n\beta_n, \tag{3.9}$$

where

$$\beta_n = 2(1-t_n)\langle u - Q_S u, j(x_n - Q_S u)\rangle + c\frac{\rho_E(t_n\|u - Q_S u\|)}{t_n}.$$

Since E is the uniformly smooth Banach space, $\frac{\rho_E(t_n\|u - Q_S u\|)}{t_n} \longrightarrow 0,\ n \longrightarrow \infty$. By (3.8), we obtain $\limsup_{n\to\infty}\beta_n \le 0$. So, an application of Lemma 2.4 to (3.9) yields the desired result. □

Remark 3.2. If for some n_0, $\|x_{n_0} - x^*\| = 0$, then $\|x_{n_0+k} - x^*\| = 0$, $\forall k \ge 1$ in proximal point algorithm (because $\|x_{n+1} - x^*\| \le \|x_n - x^*\|$, $\forall n \ge 0$), but this property is not necessarity true in a regularization proximal point algorithm.

Remark 3.3. The sequences $\{r_n\}$ and $\{t_n\}$ defined by $r_n = n,\ t_n = \frac{1}{n}$ satisfy all conditions in Theorem 3.1.

Theorem 3.4. *Let E be an uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $A:\ D(A) \subseteq E \longrightarrow 2^E$ be an $m-$ accretive operator with $S = A^{-1}(0) \neq \emptyset$. If the sequences $\{r_n\} \subset (0,+\infty)$ and $\{t_n\} \subset (0,1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$, $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;

ii) $\inf_n r_n = r > 0$, $\sum_{n=0}^{\infty}\left|1 - \frac{r_n}{r_{n+1}}\right| < +\infty$,

then the sequence $\{x_n\}$ generated by (3.1) *converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction of E onto S.*

Proof. From the proof of Theorem 3.1, we obtain the sequence $\{x_n\}$ is bounded and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \overline{x} \in E$. Now, we show that $\overline{x} \in S$.

In equation (3.3) replacing n by $n+1$, we get

$$r_{n+1}y_{n+2} + x_{n+2} = t_{n+1}u + (1 - t_{n+1})x_{n+1}. \tag{3.10}$$

From (3.3) and (3.10) and by the accretiveness of A, we have

$$\begin{aligned} r_{n+1}\langle x_{n+2} - x_{n+1}, j(x_{n+2} - x_{n+1})\rangle - (r_{n+1} - r_n)\langle x_{n+2}, j(x_{n+2} - x_{n+1})\rangle \\ \le \langle r_n[t_{n+1}u + (1-t_{n+1})x_{n+1}] - r_{n+1}[t_n u + (1-t_n)x_n], j(x_{n+2} - x_{n+1})\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} r_{n+1}\|x_{n+2} - x_{n+1}\| \le& |r_{n+1} - r_n|.\|x_{n+2}\| \\ &+ \|r_n[t_{n+1}u + (1-t_{n+1})x_{n+1}] - r_{n+1}[t_n u + (1-t_n)x_n]\| \\ \le& r_{n+1}(1-t_{n+1})\|x_{n+1} - x_n\| + |r_{n+1} - r_n|.\|x_{n+2}\| \\ &+ r_{n+1}|t_{n+1} - t_n|(\|x_n\| + \|u\|) \\ &+ |r_{n+1} - r_n|.[(1-t_{n+1})\|x_{n+1}\| + t_{n+1}\|u\|]. \end{aligned}$$

By $\{t_n\} \subset (0,1)$ and $r_n > 0$ for all n, we deduce

$$\|x_{n+2} - x_{n+1}\| \le (1-t_{n+1})\|x_{n+1} - x_n\| + \left(2\left|t_{n+1} - t_n\right| + 3\left|1 - \frac{r_n}{r_{n+1}}\right|\right)K, \tag{3.11}$$

where $K = \max\{\|u\|, \sup\|x_n\|\} < +\infty$. By Lemma 2.4, $\|x_{n+1} - x_n\| \longrightarrow 0$, as $n \longrightarrow \infty$. Therefore,

$$\begin{aligned} \|y_{n+1}\| &= \frac{1}{r_n}\|t_n(u - x_n) + (x_n - x_{n+1})\| \\ &\leq \frac{1}{r}(2Kt_n + \|x_{n+1} - x_n\|) \longrightarrow 0,\ n \longrightarrow \infty. \end{aligned} \tag{3.12}$$

Since A is demiclosed, we obtain $\overline{x} \in S$.
The rest of the proof follows the pattern of Theorem 3.1. □

Remark 3.5. The sequences $\{r_n\}$ and $\{t_n\}$ defined by $r_n = 1 + \frac{1}{n}$, $t_n = \frac{1}{n}$ satisfy all conditions in Theorem 3.4.

Next, we study stability of algorithm (3.1) in the form

$$r_n A^n(x_{z+1}) + z_{n+1} \ni t_n u + (1 - t_n)z_n,\ u,\ z_0 \in E,\ n \geq 0, \tag{3.13}$$

where $A^n : D(A^n) \subseteq E \longrightarrow 2^E$ are $m-$accretive operators with $D(A^n) = D(A)$ such that

$$\mathcal{H}(A^n(x), A(x)) \leq g(\|x\|)h_n, \tag{3.14}$$

where g is real bounded (image of a bounded set is bounded) function for $t \geq 0$ with $g(0) = 0$ and $\{h_n\}$ is positive sequence.

We have the following results:

Theorem 3.6. *Let E be an uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $A : D(A) \subseteq E \longrightarrow 2^E$ and $A^n : D(A^n) \subseteq E \longrightarrow 2^E$ be $m-$ accretive operators with $S = A^{-1}(0) \neq \emptyset$ and $D(A) = D(A^n)$ for all n. If the condition* (3.14) *is fulfilled and the sequences $\{r_n\} \subset (0, +\infty)$, and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;

ii) $\lim_{n\to\infty} r_n = +\infty$;

iii) $\sum_{n=1}^{\infty} r_n h_n < +\infty$,

then the sequence $\{z_n\}$ generated by (3.13) *converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction of E onto S.*

Proof. For each n, by A^n is an $m-$accretive operator, the equation (3.13) has solution, i.e., there exists z_{n+1} such that

$$r_n A^n(z_{n+1}) + z_{n+1} \ni t_n u + (1 - t_n)z_n. \tag{3.15}$$

Hence, there exists $w_{n+1} \in A^n(z_{n+1})$ such that

$$r_n w_{n+1} + z_{n+1} = t_n u + (1 - t_n)z_n. \tag{3.16}$$

By the condition (3.14), for each $y_{n+1} \in A(x_{n+1})$, there exists $b_{n+1} \in A^n(x_{n+1})$ such that

$$\|y_{n+1} - b_{n+1}\| \leq g(\|x_{n+1}\|)h_n \leq g(K)h_n. \tag{3.17}$$

From (3.3) and (3.16), we have

$$\begin{aligned}\langle r_n(w_{n+1} - b_{n+1}), j(z_{n+1} - x_{n+1})\rangle + \langle r_n(b_{n+1} - y_{n+1}), j(z_{n+1} - x_{n+1})\rangle \\ + \|z_{n+1} - x_{n+1}\|^2 = (1 - t_n)\langle z_n - x_n, j(z_{n+1} - x_{n+1})\rangle.\end{aligned}$$

By A^n is an $m-$accretive operator and by (3.17), we obtain

$$\|z_{n+1} - x_{n+1}\| \leq (1 - t_n)\|z_n - x_n\| + g(K)r_nh_n. \tag{3.18}$$

By the assumption and Lemma 2.1, we conclude that $\|z_n - x_n\| \longrightarrow 0$, as $n \longrightarrow \infty$. In addition, by Theorem 3.1,

$$\|z_n - Q_S u\| \leq \|z_n - x_n\| + \|x_n - Q_S u\| \longrightarrow 0, \ n \longrightarrow \infty, \tag{3.19}$$

which implies that z_n converges strongly to $Q_S u$. □

Theorem 3.7. *Let E be an uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $A : D(A) \subseteq E \longrightarrow 2^E$ and $A^n : D(A^n) \subseteq E \longrightarrow 2^E$ be $m-$ accretive operators with $S = A^{-1}(0) \neq \emptyset$ and $D(A) = D(A^n)$ for all n. If the condition* (3.14) *is fulfilled and the sequences $\{r_n\} \subset (0, +\infty)$, and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$, $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;

ii) $\inf_n r_n = r > 0$, $\sum_{n=0}^{\infty} \left|1 - \frac{r_n}{r_{n+1}}\right| < +\infty$;

iii) $\sum_{n=1}^{\infty} r_nh_n < +\infty$,

then the sequence $\{z_n\}$ generated by (3.13) *converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction of E onto S.*

Corollary 3.8. *Let H be a Hilbert space. Let $A : D(A) \subseteq H \longrightarrow 2^H$ and $A^n : D(A^n) \subseteq H \longrightarrow 2^H$ be maximal monotone operators with $S = A^{-1}(0) \neq \emptyset$ and $D(A) = D(A^n)$ for all n. If the condition* (3.14) *is fulfilled and the sequences $\{r_n\} \subset (0, +\infty)$, and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;

ii) $\lim_{n\to\infty} r_n = +\infty$;

iii) $\sum_{n=1}^{\infty} r_nh_n < +\infty$,

then the sequence $\{z_n\}$ generated by (3.13) *converges strongly to $P_S u$, where P_S is a metric projection of E onto S.*

Corollary 3.9. *Let H be a Hilbert space. Let $A : D(A) \subseteq H \longrightarrow 2^H$ and $A^n : D(A^n) \subseteq H \longrightarrow 2^H$ be maximal monotone operators with $S = A^{-1}(0) \neq \emptyset$ and $D(A) = D(A^n)$ for all n. If the condition* (3.14) *is fulfilled and the sequences $\{r_n\} \subset (0, +\infty)$, and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$, $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;

ii) $\inf_n r_n = r > 0$, $\sum_{n=0}^{\infty} \left|1 - \frac{r_n}{r_{n+1}}\right| < +\infty$;

iii) $\sum_{n=1}^{\infty} r_n h_n < +\infty$,

then the sequence $\{z_n\}$ *generated by* (3.13) *converges strongly to* $P_S u$, *where* P_S *is a metric projection of* E *onto* S.

Remark 3.10. Corollary 3.8 and Corollary 3.9 are more general than the results of H. -K. Xu in [27].

Corollary 3.11. *Let* H *be a Hilbert space. Let* $T : H \longrightarrow H$ *be a nonexpansive mapping from* H *into itself with* $S = \{x \in H : T(x) = x\} \neq \emptyset$. *If the sequences* $\{r_n\} \subset (0, +\infty)$, *and* $\{t_n\} \subset (0, 1)$ *satisfy the conditions i) and ii) in Theorem 3.1 or the conditions i) and ii) in Theorem 3.4, then the sequence* $\{x_n\}$ *defined by* $u,\ x_0 \in E$ *and*

$$\begin{cases} y_n = t_n u + (1 - t_n) x_n, \\ x_{n+1} = \dfrac{r_n}{1 + r_n} T(x_{n+1}) + \dfrac{1}{1 + r_n} y_n,\ n \geq 0, \end{cases} \tag{3.20}$$

converges strongly to $P_S u$, *where* P_S *is a metric projection of* E *onto* S.

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