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# Fractional Complexified Field Theory from Saxena-Kumbhat Fractional Integral, Fractional Derivative of Order $(\alpha, \beta)$ and Dynamical Fractional Integral Exponent

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**Abstract**

Fractional complexified field theory based on Saxena-Kumbhat fractional integrals with the presence of fractional derivative of order ($\alpha$,$\beta$)and dynamical fractional exponent is considered. Some interesting results are explored and discussed in some details.

**AMS Subject Classification:**49S05; 37K05.

**Keywords**: fractional action-like variational approach, fractional Euler-Lagrange equations, Saxena-Kumbhat fractional integral, fractional derivative, dynamical fractional exponent, hypergeometric function, complexification.

## 1 Introduction

Fractional Calculus (FC) goes back to the foundation of the theory of differential calculus. However, the application of FC just emerged in the last two decades, due to the improvement in the area of chaos that revealed subtle associations with the FC concepts. In the field of complex dynamical systems theory, some work has been carried out but the proposed models and algorithms are still in a groundwork stage of establishment. Many real complex dynamical systems are better characterized using a non-integer order dynamic model

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based on FC. The concept of FC has tremendous potential to change the way we see, model, and control the nature around us. It has become an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering including hydrology, viscoelastivity, heat conduction, polymer physics, chaos and fractals, control theory, plasma physics, wave propagation in complex and porous media, astrophysics, cosmology, quantum field theory, potential theory, and so on [1–23]. Having these ideas in mind, the following paper discusses a FC perspective in the study of quantum field theory.

A subject of current strong research concerns the fractional problems of the Calculus of Variations (COV) [16-43]. Its basic ingredient is the fractional Euler-Lagrange equation (FELE) which plays a central role in the study of complex Lagrangian and Hamiltonian dynamical systems. The fractional theory of the COV deals in reality with more general systems containing noninteger derivatives and therefore provides a more reasonable advance to physics, permitting to consider nonconservative dynamical systems in an unusual way-a very important issue since closed systems do not exist: forces that do not store energy, so-called nonconservative or weak dissipative forces, are constantly present in real dynamical systems. In most of the recent studies, fractional Euler-Lagrange equations depend on left and right fractional Riemann-Liouville derivatives, even when the problem depends only on one type of them. Accordingly, we argued that the fractional problem of the COV still needs more amplification as the problem is profoundly related to the fractional quantization process and to the presence of non-local fractional differential operators. One simple and realistic approach is the Fractional Action-Like Variational Approach (FALVA) based on the concept of left Riemann-Liouville fractional integral functionals with one parameter but not on fractional-order derivatives of the same order [31,32]. The aim of FALVA was to better model non-conservative and weak dissipative real and quantum dynamical systems (including quantum field theory). Many encouraging results were effectively obtained and discussed [33-37]. Motivated by theses advancements, fractional problems of the calculus of variations where the Lagrangian depends on different types of fractional order derivatives, e.g. Erdelyi-Kober fractional operators [35], Saxena-Kumbhat hypergeometrical fractional derivative [23] (SKHGFD) and on variable order fractional derivatives and integrals were constructed and discussed in some details. However, to the best of the author knowledge, the extension of FALVA based on SKHGFD to quantum field theory is absent in literature in contrast to FALVA. It is noteworthy that the main interest to deal with the hypergeometric function is related to the fact that the later is intimately associated with the Laplace and wave equations in four-dimensional space and their complexifications [44]. Furthermore, many fundamental problems in physics and mathematical physics require a conformal symmetry groups which in its turn accounts for many of the properties of ${}_2F_1$. Besides, the central argument for dealing with fractional differential operators within the concept of quantum field theory concerns the fact that fractional operators will lead to a much more elegant and effective way of treating problems in particle and high-energy physics [45-47]. It is notable that a time-dependent fractional exponent plays a significant role in various branches of complex dynamical systems, e.g. self-affine time-sequential data [48], it will be considered in this work. Others applications are found in [49-57]. Some applications of fractional calculus in field theories are found in [58-67].

We follow the rationale of [30] where it is assumed that at least one stationary point for the fractional functional exists. We introduce the main notations, conventions and assumptions that underlie the remainder of the present work.

[1] In the notation $t \to f(t)$, $t$ is a dummy variable.

[2] Exactly, the same function can be written, for example $(\dot{q}, q, \tau) \to f(\dot{q},\ q,\ \tau)$; $\dot{q},\ q,\ \tau$ are here dummy variables.

[3]For $(\dot{q}, q, \tau) \to f(\dot{q}, q, \tau)$, the partial derivative of $f$ with respect to the first argument is denoted by $\partial L/\partial \dot{q}$.

[4] For any $q$, $q_X \equiv \partial q/\partial X$.

[5] The left and right Riemann-Liouville temporal and spatial fractional derivatives are defined by:

$$D^{(\beta)}_{(a_+)} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_{-\infty}^{t} f(\tau)(t-\tau)^{-\beta} d\tau,$$

$$D^{(\beta)}_{(a_-)} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_{t}^{\infty} f(\tau)(\tau-t)^{-\beta} d\tau,$$

$$D^{(\beta)}_{(a_+)} f(x) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial x} \int_{-\infty}^{x} f(\xi)(x-\xi)^{-\beta} d\xi,$$

$$D^{(\beta)}_{(a_-)} f(x) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial x} \int_{x}^{\infty} f(\xi)(\xi-x)^{-\beta} dx,$$

which may be combined via

$$\partial^{(\alpha)}_{(t)} f(t) = \frac{D^{(\alpha)}_{(a_+)} - D^{(\alpha)}_{(a_-)}}{2\sin\left(\frac{\alpha\pi}{2}\right)} \int_{0}^{\infty} \frac{f(t+\tau)-f(t-\tau)}{\tau^{\alpha+1}} d\tau,\ \ 0<\alpha<1,$$

$$\partial^{(\beta)}_{(x)} f(x) = \frac{D^{(\beta)}_{(a_+)} - D^{(\beta)}_{(a_-)}}{2\sin\left(\frac{\beta\pi}{2}\right)} \int_{0}^{\infty} \frac{f(x+\xi)-f(x-\xi)}{\xi^{\alpha+1}} d\xi,\ 0<\beta<1.$$

The paper is organized as follows: In Sec. 2, we review rapidly the basic concepts of FALVA with Saxena-Kumbhat hypergeometric fractional operator augmented by a fractional derivative and a dynamical fractional exponent and its complexification counterpart. In Sec. 3, we introduce the basic concepts of the fractional quantum field theory and we discuss the case of a complexified quantum field theory. The paper concludes in Sec. 4 with a brief summary of main results and future challenge and perspectives.

## 2 GENERALIZED FALVA WITH SAXENA-KUMBHAT HYPERGEOMETRIC FRACTIONAL OPERATOR

For any function $f \in L_p(0,\infty)$, we will use the fractional derivative operator of order $(\alpha,\beta)$, defined by Cresson [39],

$$D^{(\alpha,\beta)}_{(\gamma)} f(\bullet) = \frac{1}{2}\left[D^{(\alpha)}_{(a_+)} f(\bullet) - D^{(\beta)}_{(b_-)} f(\bullet)\right] + \frac{i\gamma}{2}\left[D^{(\alpha)}_{(a_+)} f(\bullet) + D^{(\beta)}_{(b_-)} f(\bullet)\right], \tag{2.1}$$

where $\gamma \in \mathbb{C}$, $i = \sqrt{-1}$ and $0 < \alpha, \beta < 1$.

For mathematical convenience, we define the new variable

$$X \stackrel{\Delta}{=} 1 - \frac{\tau}{t} \in \mathbb{R}. \tag{2.2}$$

**Definition 2.1.** The modified left and right Riemann-Liouville fractional derivatives of order $0 < \alpha < 1$ of a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ are defined by

$$D_{(a_+)}^{(\alpha)} f(t(1-X)) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^{1-a/t} f(t(1-X)) X^{-\alpha} dX, \tag{2.3}$$

$$D_{(a_-)}^{(\beta)} f(t(1-X)) = -\frac{t^{1-\beta}}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_{1-b/t}^{0} f(t(1-X)) X^{-\beta} dX. \tag{2.4}$$

**Corollary 2.2.** *: The following properties hold [68, 69]*

$$\int_a^b D_{(\gamma)}^{(\alpha,\beta)} f(X) g(X) dX = -\int_a^b f(X) D_{-(\gamma)}^{(\beta,\alpha)} g(X) dt, a \le X \le b, \tag{2.5}$$

$$\int_a^b D_{(a_+)}^{(\alpha)} f(X) g(X) dX = (-1)^\alpha \int_a^b f(X) D_{(b_-)}^{(\alpha)} g(X) dX, \tag{2.6}$$

$$D_{(a)}^{(\alpha)} D_{(a)}^{(\beta)} f(X) = D_{(a)}^{(\alpha+\beta)} f(X) - \sum_{i=1}^{k} D_{(a)}^{(\sigma-i)} f(X) \Big|_{X=a} \frac{(t-a)^{-\alpha-i}}{\Gamma(1-\alpha-i)}, \tag{2.7}$$

*with $0 \le k-1 \le q \le k$ provided that $f(a) = f(b) = 0$ and $g(a) = g(b) = 0$. $k$ is a whole number.*

The new concept of FALVA based on SKHGFD is based in fact on the following definition:

**Definition 2.3.** : The Saxena-Kumbhat hypergeometric fractional integrals for a function $f \in L_p(0,\infty)$ are defined by [23, 63, 69]:

$$I_{Left} f(t) \stackrel{\Delta}{=} \frac{1}{\Gamma(\alpha)} t^{-\alpha-\sigma} \int_0^t f(\tau) \tau^\sigma (t-\tau)^{\alpha-1} {}_2F_1\left(a,b;c;\zeta\left(1-\frac{\tau}{t}\right)\right) d\tau, \tag{2.8}$$

$$I_{Right} f(t) \stackrel{\Delta}{=} \frac{1}{\Gamma(\alpha)} t^{\delta} \int_t^\infty f(\tau) \tau^{-\delta-\alpha} (\tau-t)^{\alpha-1} {}_2F_1\left(a,b;c;\zeta\left(1-\frac{\tau}{t}\right)\right) d\tau. \tag{2.9}$$

${}_2F_1$ is the Gaussian hypergeometric function defined around $y = 0$ by the series expansion:

$${}_2F_1(a,b;c;y) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n n!} y^n = \sum_{n=0}^{+\infty} t_n y^n, \tag{2.10}$$

where $t_1 = 1$ and

$$t_{n+1} = \frac{(a+n)(b+n)}{(c+n)(n+1)} t_n, n \geq 0, \tag{2.11}$$

and where

$$(x)_n = 1 \cdot x \cdot \ldots \cdot (x+n-1) \tag{2.12}$$

is the standard Appell or Pochhammer symbol

$\Re(\alpha) > 0,\ \Re(\sigma) > -1/q,\ \Re(\delta) > -1/p,\ p^{-1} + q^{-1} = 1, p \geq 1, c \neq 0, -1, -2, ..., \Re(c-b-a) > 0$

and

$$\left|\arg(1-\zeta)\right| < \pi. \tag{2.13}$$

Normally, the series always converges for $|y| < 1$.

*Remark* 2.4. : We will perform our calculation in the real plane and we set $\zeta = 1$ for convenience. As the fractional problem of the COV is based on the left fractional integral, the Saxena-Kumbhat hypergeometric fractional left integral for a function $f \in L_p(0, \infty)$ with respect to the new variable is now defined on the interval by :

$$I_{Left} f \triangleq \int_0^1 f(t(1-X))(1-X)^{\alpha+\sigma-1} \frac{{}_2F_1(a,b;c;X)}{\Gamma(\alpha)} dX. \tag{2.14}$$

**Problem II-1**: Consider a smooth n-dimensional manifold $M$ and let $L : C^1([0,1] \times \mathbb{R}^n \times \mathbb{R}); \mathbb{R}$ be the smooth Lagrangian function satisfying fixed boundary conditions $q(0) = q_0$ and $q(1) = q_1$. Find the stationary points of the integral functional on multifractal time and space sets

$$S\,[q(\bullet)] = \int_0^1 L(D_{(\gamma)}^{(\alpha,\beta)} q(t(1-X)), q(t(1-X)), t(1-X)) \frac{(1-X)^{\alpha(X)+\sigma(X)-1}}{\Gamma(\alpha(t(1-X)))} {}_2F_1(a,b;c;X) e^{\chi \cdot t(1-X)} dX, \tag{2.15}$$

$X \neq 1, \chi \in \mathbb{R}$, $q' = dq/dX$ and the smooth Lagrangian function $L(\dot{q}(t(1-X)),\ q(t(1-X)),\ t(1-X))$ is actually weighted with

$$\frac{(1-X)^{\alpha(X)+\sigma(X)-1} {}_2F_1(a,b;c;X)}{\Gamma(\alpha(t(1-X)))}. \tag{2.16}$$

Here

$$\Gamma(\alpha(t(1-X))) = t^{\alpha(t(1-X))} \times \int_{-\infty}^1 (1-X)^{\alpha(t(1-X))-1} \exp(-t(1-X)) dX \tag{2.17}$$

is the modified Euler gamma function.

**Theorem 2.5.** *: If $q(t(1-X))$ are solutions to the previous problem, then it satisfies the following generalized fractional Euler-Lagrange equation:*

$$\frac{\partial L(D_{(\gamma)}^{(\alpha,\beta)}q(t(1-X)),q(t(1-X)),t(1-X))}{\partial q}-D_{-(\gamma);X}^{(\beta,\alpha)}\left(\frac{\partial L(D_{(\gamma)}^{(\alpha,\beta)}q(t(1-X)),q(t(1-X)),t(1-X))}{\partial q_X}\right)$$

$$=\left[\frac{d(\alpha+\sigma)}{dX}\ln(1-X)-\frac{\alpha(X)+\sigma(X)-1}{1-X}-\frac{1}{t}\psi(t(1-X))-\chi t\right]\frac{\partial L(D_{(\gamma)}^{(\beta,\alpha)}q(t(1-X)),q(t(1-X)),t(1-X))}{\partial q_X}$$

$$+\frac{1}{{}_2F_1}\frac{d_2F_1}{dX}\frac{\partial L(D_{(\gamma)}^{(\beta,\alpha)}q(t(1-X)),q(t(1-X)),t(1-X))}{\partial q_X}. \tag{2.18}$$

We have used the logarithmic derivative of the gamma function [70, 71]

$$\frac{1}{\Gamma(\alpha(t(1-X)))}\frac{d}{dX}\left(\frac{1}{\Gamma(\alpha(t(1-X)))}\right)=\frac{d}{dX}\ln\Gamma(\alpha(t(1-X)))=\psi(t(1-X)) \tag{2.19}$$

which is the first of the polygamma functions and has the integral representation:

$$\psi(t(1-X))=-\gamma-\frac{1}{t(1-X)}+t(1-X)\sum_{k=1}^{\infty}\frac{1}{k(t(1-X)+k)}. \tag{2.20}$$

$\gamma$ is the Euler-Mascheroni constant defined by:

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\approx 0.57721566490153286.... \tag{2.21}$$

To cover the $N$-dimensional problem, we assume as previously, that the admissible paths are smooth functions $q:\Omega\subset\mathbb{R}^N\to M$ filling specified Dirichlet boundary conditions on $\partial\Omega$. For this, we propose the following definition:

**Definition 2.6.** : The Lagrangian function $(q_{X_1},...,q_{X_N},q,X_1,...,X_N)\to L(q_{X_1},...,q_{X_N},q,X_1,...,X_N)$ is supposed to be sufficiently smooth with respect to all its arguments. The N-dimensional fractional generalized-FALVA action integral is defined by

$$S_{(\chi)}^{(\alpha,\beta)}[q]=\int...\int_{\Omega(\xi)}L(\nabla_{\gamma}^{\alpha,\delta}q(t(1-X)),q(t(1-X)),t(1-X))$$

$$\times\prod_{i=1}^{N}\frac{(1-X_i)^{\alpha(X_i)+\sigma(X_i)-1}}{\Gamma(\alpha_i(t(1-X_i)))}{}_2F_1(a,b;c;X_i)\,e^{\chi_i\cdot t(1-X_i)}dX, \tag{2.22}$$

where $\nabla_{\gamma}^{\alpha,\delta}=(D_{\gamma;X_1}^{(\alpha_1,\delta_1)},...,D_{\gamma;X_N}^{(\alpha_N,\delta_N)})$ and $dx=dx_1...dx_N$.

**Theorem 2.7.** *The N-dimensional associated fractional Euler-Lagrange equation is given by*

$$\sum_{i=1}^{N}\left[D_{-\gamma;X_i}^{(\alpha_i,\delta_i)}\left(\frac{\partial L}{\partial q_{X_i}}\right)+\left[\frac{d(\alpha+\sigma)}{dX_i}\ln(1-X_i)-\frac{\alpha(X_i)+\sigma(X_i)-1}{1-X_i}-\frac{1}{t}\psi(t(1-X_i))-\chi_i t\right]\frac{\partial L}{\partial q_{X_i}}\right.$$

$$\left.+\frac{1}{{}_2F_1}\frac{d_2F_1}{dX_i}\frac{\partial L}{\partial q_{X_i}}\right]-\frac{\partial L}{\partial q}=0, \tag{2.23}$$

*where all partial derivatives of the Lagrangian are evaluated at* $(\nabla_{\gamma}^{\alpha,\ \delta}q(t(1-X)),\ q(t(1-X)),\ t(1-X)),\ X\in\Omega.$

For more details, the authors are refereed to [23].

# 3 FRACTIONAL QUANTUM FIELD THEORY FROM FALVA+ SKHGFD

In classical field theory, the classical fields are functions on spacetime $M$or sections of some vector bundle with the spacetime $M$ as the base. Let us denote all the fields collectively by $\phi_i, i=1,...,N$. The Lagrangian density of a classical, massless field established in electrodynamics, weak and strong interactions, and gravitation) depends on the field and their first covariant fractional derivatives and is given by $L(\phi_i,\nabla_{(x)}^{(\alpha,\beta)}\phi_i)$. We let $X_i, i=1,...,N$ and $X_0$ be respectively the space and the time components of the classical fields. The partial fractional derivatives of order $(\alpha,\beta)$ are defined by:

$$\bar{D}_{(\gamma)}^{(\alpha,\beta)}\phi_i(t)=\frac{1}{2}\left[\partial_{(a_+)}^{(\alpha)}\phi_i(t)-\partial_{(b_-)}^{(\beta)}\phi_i(t)\right]+\frac{i\gamma}{2}\left[\partial_{(a_+)}^{(\alpha)}\phi_i(t)+\partial_{(b_-)}^{(\beta)}\phi_i(t)\right], \tag{3.1}$$

$$\bar{D}_{(\gamma)}^{(\alpha,\beta)}\phi_i(x_i)=\frac{1}{2}\left[\partial_{(a_+)}^{(\alpha)}\phi_i(x_i)-\partial_{(b_-)}^{(\beta)}\phi_i(x_i)\right]+\frac{i\gamma}{2}\left[\partial_{(a_+)}^{(\alpha)}\phi_i(x_i)+\partial_{(b_-)}^{(\beta)}\phi_i(x_i)\right], \tag{3.2}$$

where $\partial^{(\alpha)}\big/\partial x^{(\alpha)}$ is a short hand notation for e.g. the fractional left and right Riemann-Liouville derivatives adopting the Einstein notation and summation such that:

$$\partial_{(a_+)}^{(\alpha)}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_a^t\phi(\tau)(t-\tau)^{-\alpha}d\tau, \tag{3.3}$$

$$\partial_{(b-)}^{(\beta)}\phi(t)=\frac{1}{\Gamma(1-\beta)}\left(-\frac{\partial}{\partial t}\int_t^b\phi(\tau)(\tau-t)^{-\beta}d\tau\right). \tag{3.4}$$

It is notable if $\phi:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism and $\Phi=\{\phi_s\},\ s\in\mathbb{R}$ be a one-parameter family of diffeomorphisms, then the Lagrangian density is invariant under the fractional action of $\Phi$ if $L\to L(\bar{\nabla}_{\gamma}^{\alpha,\delta}\phi_i(t(1-X_i)),\phi_i(t(1-X_i)),t(1-X_i))$. The $N$-dimensional fractional field action is defined by [72]:

$$S^{(\alpha,\beta)}[\phi](\xi)=\int\ldots\int_{\Omega_\phi(\xi)} L(\bar{\nabla}_\gamma^{\alpha,\delta}\phi_i(t(1-X)),\phi_i(t(1-X)),t(1-X))$$

$$\times\prod_{i=1}^{N}\frac{(1-X_i)^{\alpha(X_i)+\sigma(X_i)-1}}{\Gamma(\alpha_i(t(1-X_i)))}\,{}_2F_1\,(a,b;c;X_i)\,e^{\chi_i\cdot t(1-X_i)}dX. \tag{3.5}$$

The $N$-dimensional associated fractional Euler-Lagrange field equation is given by:

$$\sum_{i=1}^{N}\left[\bar{D}_{-(\gamma);X_i}^{(\alpha_i,\delta_i)}\left(\frac{\partial L(\bar{\nabla}_\gamma^{\alpha,\delta}\phi_i(t(1-X_i)),\phi_i(t(1-X_i)),t(1-X_i))}{\partial\phi_i}\right)\right.$$

$$+\left[\frac{d(\alpha+\sigma)}{dX_i}\ln(1-X_i)-\frac{\alpha(X_i)+\sigma(X_i)-1}{1-X_i}-\frac{1}{t}\psi(t(1-X_i))-\chi_i t+\frac{1}{{}_2F_1}\frac{d_2F_1}{dX_i}\right]\times$$

$$\frac{\partial L(\bar{\nabla}_\gamma^{\alpha,\delta}\phi_i(t(1-X_i)),\phi_i(t(1-X_i)),t(1-X_i))}{\partial\phi_i}-\frac{\partial L(\bar{\nabla}_\gamma^{\alpha,\delta}\phi_i(t(1-X_i)),\phi_i(t(1-X_i)),t(1-X_i))}{\partial\phi}=0 \tag{3.6}$$

where all partial derivatives of the Lagrangian are evaluated at $(\bar{\nabla}_\gamma^{\alpha,\delta}\phi_i(t(1-X_i)),\phi_i(t(1-X_i))),X\in\Omega_\phi(\xi)$.

More particularly, in four-dimensions, Eq. (3.6) is reduced straightforwardly to:

$$\frac{\partial L}{\partial\phi}-\bar{D}_{-(\gamma);X_0}^{(\alpha,\delta)}\left(\frac{\partial L}{\partial(\bar{D}_{-(\gamma);X_0}^{(\alpha,\delta)}\phi)}\right)-\sum_{i=1}^{3}\bar{D}_{-(\gamma);X_i}^{(\alpha,\delta)}\left(\frac{\partial L}{\partial(\bar{D}_{-(\gamma);X_i}^{(\alpha,\delta)}\phi)}\right)$$

$$+\left[\frac{d(\alpha+\sigma)}{dX_0}\ln(1-X_0)-\frac{\alpha(X_0)+\sigma(X_0)-1}{1-X_0}-\frac{1}{t}\psi(t(1-X_0))-\chi_0 t+\frac{1}{{}_2F_1}\frac{d_2F_1}{dX_0}\right]\frac{\partial L}{\partial(\bar{D}_{-(\gamma);X_0}^{(\alpha,\delta)}\phi)}$$

$$+\left[\frac{d(\alpha+\sigma)}{dX_i}\ln(1-X_i)-\frac{\alpha(X_i)+\sigma(X_i)-1}{1-X_i}-\frac{1}{t}\psi(t(1-X_i))-\chi_i t+\frac{1}{{}_2F_1}\frac{d_2F_1}{dX_i}\right]\frac{\partial L}{\partial(\bar{D}_{-(\gamma);X_i}^{(\alpha,\delta)}\phi)}$$

$$-\frac{\partial L}{\partial\phi}=0. \tag{3.7}$$

All basic differential equations of physics have a variational structure. One important equation is the wave equation. One expects then a deformed wave equation when the fractional formalism is applied to the wave problem. In fact, by applying the fractional Euler-Lagrange equation (3.7) to the massless Lagrangian density:

$$L\left(\phi,\bar{D}_{(\gamma);X_0}^{(\alpha,\beta)}\phi,\bar{D}_{(\gamma);X}^{(\alpha,\beta)}\phi,\bar{D}_{(\gamma);Y}^{(\alpha,\beta)}\phi,\bar{D}_{(\gamma);Z}^{(\alpha,\beta)}\phi\right)$$

$$= \frac{1}{2}\left[\bar{D}^{(\alpha,\beta)}_{(\gamma);X_0}\phi\bar{D}^{(\alpha,\beta)}_{(\gamma);X_0}\phi - \bar{D}^{(\alpha,\beta)}_{(\gamma);X}\phi\bar{D}^{(\alpha,\beta)}_{(\gamma);X}\phi - \bar{D}^{(\alpha,\beta)}_{(\gamma);Y}\phi\bar{D}^{(\alpha,\beta)}_{(\gamma);Y}\phi - \bar{D}^{(\alpha,\beta)}_{(\gamma);Z}\phi\bar{D}^{(\alpha,\beta)}_{(\gamma);Z}\phi\right], \tag{3.8}$$

with certain generalized coordinate $\phi(X_0, X_i)$ where $(X_0, \vec{X}_i) \in M^4$ (the 4-dimensional Minkowski space), the following equation arises consequently:

$$\bar{D}^{(\alpha,\beta)}_{(\gamma);X_0}\bar{D}^{(\alpha,\beta)}_{(\gamma);X_0}\phi + \left[\frac{d(\alpha+\sigma)}{dX_0}\ln(1-X_0) - \frac{\alpha(X_0)+\sigma(X_0)-1}{1-X_0} - \frac{1}{t}\psi(t(1-X_0)) - \chi_0 t\right.$$

$$\left. + \frac{1}{2F_1}\frac{d_2F_1}{dX_0}\right]\bar{D}^{(\alpha,\beta)}_{(\gamma);X_0}\phi - \left\{\sum_{i=1}^{3}\left\{\bar{D}^{(\alpha,\beta)}_{(\gamma);X_i}\bar{D}^{(\alpha,\beta)}_{(\gamma);X_i}\phi + \left[\frac{d(\alpha+\sigma)}{dX_i}\ln(1-X_i) - \frac{\alpha(X_i)+\sigma(X_i)-1}{1-X_i}\right.\right.\right.$$

$$\left.\left.\left. - \frac{1}{t}\psi(t(1-X_i)) - \chi_i t + \frac{1}{2F_1}\frac{d_2F_1}{dX_i}\right]\bar{D}^{(\alpha,\beta)}_{(\gamma);X_i}\phi\right\}\right\} = 0, \tag{3.9}$$

which is the fractional generalized Klein-Gordon equation.

It is noteworthy that the Lagrangian should be physically a scalar function with respect to some set of transformations. At the same time, this set exactly corresponds to the definite laws of conservation, e.g. translation in spacetime corresponds to momentum and energy conservation, rotation to angular momentum conservation and therefore from corresponding representation of the rotation group one can read the spin of the particle as it is well-known that any elementary particle in nature has definite spin and another quantum numbers. In the most general case, it can be shown that the Lagrangian density is invariant under the infinitesimal Lorentz transformation of coordinates given by: $x'^i = x^i + \varepsilon^i_j x^j + a^i$ where $a^i$ is a constant translation vector and $\varepsilon^{ij} = -\varepsilon^{ji}$ is a constant antisymmetric rotation tensor and where the field components respond as $\kappa'^m(x') = [\delta^m_n + \frac{1}{2}S^{mij}_n\omega_{ij}]\kappa^n(x)$ in which $S^{mij}_n$ denotes the spin tensor. The number of independent parameters characterizing a Lorentz transformation is six. As recognized, three of them correspond to spatial rotations, whereas the remaining three correspond to Lorentz boosts. In general, the relativistic fields are chosen to belong to a representation of the Lorentz group, e.g. the Klein-Gordon field belongs to the scalar representation. This already means that under a Lorentz transformation the components of the field mix together, as, for instance, a vector field does under rotations. Moreover, it is well-known that spin represents a quantum mechanical observable that has no classical analogue. However, it was lately showed that the notion of spin and the notion of fractional dimension are strictly related and that fractional dynamics has the effect of turning a scalar field into a spinor field [60]. From a phenomenological point of view and on account of Noether's theorem, one can conjecture that spin is an observable that may be interpreted as the conserved charge associated with the local variation of a definite fractal dimension which is a continuous function of coordinates.

Complexified quantum field theory are obtained if we suppose that the dynamical fractional exponent and accordingly the spacetime components are complex as $\gamma = a + ib \in \mathbb{C}, (a,b) \in \mathbb{R}$, $\alpha = F + iG$ and $\sigma = E + iD$. The following equations are useful:

$$\frac{\partial}{\partial q_{X_i}} = \frac{1}{2}\left(\frac{\partial}{\partial q_{1X_i}} - i\frac{\partial}{\partial q_{2X_i}}\right), \tag{3.10}$$

$$D^{\alpha_i,\delta_i}_{-\gamma;X_i} = D^{\alpha_i,\delta_i;+}_{-\gamma;X_i} + bD^{\alpha_i,\delta_i;-}_{-\gamma;X_i} - iaD^{\alpha_i,\delta_i;-}_{-\gamma;X_i} \equiv \bar{D}^{\alpha_i,\delta_i;+}_{-\gamma;X_i} - i\underline{D}^{\alpha_i,\delta_i;+}_{-\gamma;X_i}, \tag{3.11}$$

$$\bar{D}^{\alpha_i,\delta_i;+}_{-\gamma;X_i} = D^{\alpha_i,\delta_i;+}_{-\gamma;X_i} + bD^{\alpha_i,\delta_i;-}_{-\gamma;X_i} = \Re D^{\alpha_i,\delta_i;+}_{-\gamma;X_i}, \tag{3.12}$$

$$\underline{D}^{\alpha_i,\delta_i;+}_{-\gamma;X_i} = aD^{\alpha_i,\delta_i;-}_{-\gamma;X_i} = \Im D^{\alpha_i,\delta_i}_{-\gamma;X_i}, \tag{3.13}$$

$$D^{\alpha_i,\delta_i;+}_{-\gamma;X_i} \triangleq \frac{1}{2}\left[D^{(\alpha_i)}_{(a_+);X_i} - D^{(\beta_i)}_{(b_-);X_i}\right], \tag{3.14}$$

$$D^{\alpha_i,\delta_i;-}_{-\gamma;X_i} \triangleq \frac{1}{2}\left[D^{(\alpha_i)}_{(a_+);X_i} + D^{(\beta_i)}_{(b_-);X_i}\right]. \tag{3.15}$$

Therefore, Eq. (3.9) is complexified as follows:

$$\begin{aligned}
&\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0} - i\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\right]\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0} - i\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\right]\phi + \left[\frac{d(E+F+i(G+D))}{dX_0}\ln(1-X_0) - \right.\\
&\left.\frac{E+F-1+i(G+D)}{1-X_0} - \frac{1}{t}\psi(t(1-X_0)) - \chi_0 t + \frac{1}{{}_2F_1}\frac{d_2F_1}{dX_0}\right]\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0} - i\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\right]\phi\\
&-\left[\sum_{i=1}^{3}\left\{\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i} - i\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\right]\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i} - i\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\right]\phi + \left[\frac{d(E+F+i(G+D))}{dX_i}\ln(1-X_i)\right.\right.\right.\\
&\left.\left.\left.-\frac{E+F-1+i(G+D)}{1-X_i} - \frac{1}{t}\psi(t(1-X_i)) - \chi_i t + \frac{1}{{}_2F_1}\frac{d_2F_1}{dX_0}\right]\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i} - i\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\right]\phi\right\}\right] = 0,
\end{aligned} \tag{3.16}$$

which in its turn may be splitted into real and complex parts consecutively:

$$\begin{aligned}
&\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0} - \underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\right]\phi\\
&+\left[\frac{d(E+F)}{dX_0}\ln(1-X_0) - \frac{E+F-1}{1-X_0} - \frac{1}{t}\psi(t(1-X_0)) - \chi_0 t + \frac{1}{{}_2F_1}\frac{d_2F_1}{dX_0}\right]\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\phi\\
&+\left[\frac{d(G+D)}{dX_0}\ln(1-X_0) - \frac{G+D}{1-X_0}\right]\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\phi - \left[\sum_{i=1}^{3}\left\{\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i} - \underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\right]\phi\right.\right.
\end{aligned}$$

$$+\left[\frac{d(E+F)}{dX_i}\ln(1-X_i)-\frac{E+F-1}{1-X_i}-\frac{1}{t}\psi(t(1-X_i))-\chi_i t+\frac{1}{2F_1}\frac{d_2F_1}{dX_i}\right]\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\phi$$

$$+\left[\frac{d(G+D)}{dX_i}\ln(1-X_i)-\frac{G+D}{1-X_i}\right]\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\phi\Big\}\Big]=0 \tag{3.17}$$

and

$$\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}+\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\right]\phi$$

$$+\left[\frac{d(E+F)}{dX_0}\ln(1-X_0)-\frac{E+F-1}{1-X_0}-\frac{1}{t}\psi(t(1-X_0))-\chi_0 t+\frac{1}{2F_1}\frac{d_2F_1}{dX_0}\right]\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\phi$$

$$-\left[\frac{d(G+D)}{dX_0}\ln(1-X_0)-\frac{G+D}{1-X_0}\right]\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_0}\phi-\Big[\sum_{i=1}^{3}\Big\{\left[\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}+\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\right]\phi$$

$$-\left[\frac{d(E+F)}{dX_i}\ln(1-X_i)-\frac{E+F-1}{1-X_i}-\frac{1}{t}\psi(t(1-X_i))-\chi_i t+\frac{1}{2F_1}\frac{d_2F_1}{dX_i}\right]\underline{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\phi$$

$$-\left[\frac{d(G+D)}{dX_i}\ln(1-X_i)-\frac{G+D}{1-X_i}\right]\bar{D}^{(\alpha,\beta);+}_{-(\gamma);X_i}\phi\Big\}\Big]=0. \tag{3.18}$$

The previous equations are interesting from phenomenological points of view, as they also admit complexified action integrals which could have considerable consequences in quantum field theory as they may guide to a novel complexified dynamics which may well differs entirely from the classical mechanics cardinally [73–77]. Work in this direction is under progress. The results obtained here declare further generalization, recommend additional study and comparison with other models found in literature [78–80].

# 4 CONCLUSIONS AND PERSPECTIVES

Fractional calculus is a rapidly growing subject of interest for physicists. The reason for this is that problems may be discussed in a much more stringent and elegant way than using traditional methods. This work represents the first attempt to extend the FALVA based on SKHGFD to the framework of field theory. To the best of our knowledge, the question of obtaining conditions on the Lagrangian assuring the existence of stationary trajectories is an entirely open question in the fractional scenery. One important additional feature of the new formalism is the hierarchy of differential equations which can be obtained from the generalized fractional Euler-Lagrange equations in both the classical and the quantum aspects. Furthermore, the complexified dynamics explored here may have interesting consequences in both the classical and quantum aspects. Nevertheless, it is noteworthy that a complexified action is mandated by chiral theories, where the Lagrangian contains complex coupling constants to account for charge-parity symmetry violations in particle physics interactions. The complexified fractional Klein-Gordon equation obtained here may be useful to explore some novel properties of fractal Brownian motion within the context of nonlinear quantum field theories. Future research efforts may be directed towards formulating predictions that can be tracked tested numerically. Contemporaneous research efforts are required to validate or falsify, develop or disprove the fractional dynamics discussed here including our preliminary findings.

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