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Multivalued Stochastic Partial Differential-Integral Equations Via Backward Doubly Stochastic Differential Equations Driven by a Lévy Process*

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Abstract

In this paper, we deal with a class of backward doubly stochastic differential equations (BDSDEs, in short) involving subdifferential operator of a convex function and driven by Teugels martingales associated with a Lévy process. We show the existence and uniqueness result by means of Yosida approximation. As an application, we give the existence of stochastic viscosity solution for a class of multivalued stochastic partial differential-integral equations (MSPIDEs, in short).

AMS Subject Classification: 60H10; 60H30.

Keywords: Backward doubly stochastic differential equation, subdifferential operator, Lévy process, Teugels martingale, multivalued stochastic partial differential-integral equation.

1 Introduction

Backward stochastic differential equations (BSDEs, in short) related to a multivalued maximal monotone operator defined by the subdifferential of a convex function have first been introduced by Gegout-Petit and Pardoux [14]. Further, Pardoux and Răşcanu [27] proved the existence and uniqueness of the solution of BSDEs, on a random (possibly infinite) time

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interval, involving a subdifferential operator in order to give a probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Following, Ouknine [24], N'zi and Ouknine [19, 20], Bahlali et al. [3, 4] discussed this type of BSDEs driven by a Brownian motion or the combination of a Brownian motion and an independent Poisson point process under the conditions of Lipschitz, locally Lipschitz or some monotone conditions on the coefficients.

Recently, a new class of BSDEs, named backward doubly stochastic differential equations (BDSDEs, in short) involving a standard forward stochastic integral and a backward stochastic integral has been introduced by Pardoux and Peng [26] in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs, in short). Following it, Matoussi and Scheutzow [18], Bally and Matoussi [5], Zhang and Zhao [32], Aman and Mrhardy[1] and Boufoussi et al. [6, 7] studied this kind of BDSDEs from different aspects.

The main tool in the theory of BSDEs is the martingale representation theorem for a martingale which is adapted to the filtration of a Brownian motion or a Poisson point process (Pardoux and Peng [25], Tang and Li [31]). Recently, Nualart and Schoutens [21] gave a martingale representation theorem associated with a Lévy process. This class of Lévy processes includes Brownian motion, Poisson process, Gamma process, negative binomial process and Meixner process as special cases. Based on [21], they showed the existence and uniqueness of the solution for BSDEs driven by Teugels martingales associated with a Lévy process in [22]. These results were important from a pure mathematical point of view as well as from application point of view in the world of finance. Specifically, they could be used for the purpose of option pricing in a Lévy market and related partial differential equation which provided an analogue of the famous Black-Scholes formula. Motivated by [26] and [22], Ren et al. [30] considered a class of BDSDEs driven by Teugels martingales and an independent Brownian motion, obtained the existence and uniqueness of solutions to these equations, which allowed to give a probabilistic interpretation for the solution to a class of stochastic partial differential-integral equations (SPDIEs, in short). Very recently, Ren and Fan [29] derived the existence and uniqueness of the solution for BSDEs driven by a Lévy process involving a subdifferential operator and gave a probabilistic interpretation for the solutions of a class of partial differential-integral inclusions (PDIIs, in short).

Motivated by the above works, the first aim of this paper is to derive existence and uniqueness result to the following BDSDE involving subdifferential operator of a convex function and driven by Teugels martingales associated with a Lévy process: for each $t \in [0,T]$,

$$\begin{cases} \mathrm{d}Y_t + f(t,Y_t,Z_t)\,\mathrm{d}t + g(t,Y_t,Z_t)\,\mathrm{d}B_t \in \partial\varphi(Y_t)\,\mathrm{d}t + \sum_{i=1}^{\infty} Z_t^{(i)}\,\mathrm{d}H_t^{(i)}, \\ \\ Y_T = \xi, \end{cases} \tag{1.1}$$

where $\partial\varphi$ is a subdifferential operators. The integral with respect to $\{B_t\}$ is a backward Kunita-Itô integral (see Kunita [16]) and this one with respect to $\{H_t^{(i)}\}_{i\geq 1}$ is a standard forward Itô integral (see Gong [15]). Our method is based on the Yosida approximation.

On the other hand, since the pioneering paper due to Buckdahn and Ma [9],[10],[11], the notion of stochastic viscosity solution has been intensely studied in the last ten year. Among others, we can cite the work of Boufoussi et al. [6], [7], Aman and Mrhardy [1], Aman and Ren [2] and Ren et al. [28], etc. In all these different works, authors have set existence results to stochastic viscosity solution of several types of SPDE. The tool is entirely probabilistic and used the connection between these SPDE and associated BDSDEs. Following this way, the second goal in this paper is to give stochastic viscosity solution for multivalued stochastic partial differential-integral equations (MSPDIEs, in short): for each $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$\begin{cases} \left(\frac{\partial u}{\partial t}(t,x)+\mathcal{L}u(t,x)+f\left(t,x,u(t,x),(u_k^1(t,x))_{k=1}^{\infty}\right)+g(t,x,u(t,x))\dot{B}_t\right)\in\partial\varphi(x), \\ \\ u(T,x)=u_0(x), \end{cases} \tag{1.2}$$

where $\mathcal{L}$ is the second-order differential integral operator of the diffusion process given by

$$\begin{aligned} \mathcal{L}\phi(t,x) &= m_1\sum_{i=1}^{d}\sigma_i(x)\frac{\partial\phi}{\partial x_i}(t,x)+\frac{1}{2}\sum_{i=1}^{d}\sigma_i^2(x)\frac{\partial^2\phi}{\partial x_i^2}(t,x) \\ &\quad +\int_{\mathbb{R}}\left[\phi(t,x+\sigma(x)y)-\phi(t,x)-\langle\nabla\phi(t,x),\sigma(x)y\rangle\right]\nu(\mathrm{d}y), \end{aligned} \tag{1.3}$$

and

$$\phi_k^1(t,x)=\int_{\mathbb{R}}(\phi(t,x+\sigma(x)y)-\phi(t,x))p_k(y)\nu(\mathrm{d}y),$$

with σ a $\mathbb{R}^d$-valued function, which is the drift coefficient of SDE driven by the Lévy process $\{L_t:t\in[0,T]\}$: for each $t\in[0,T]$

$$X_t=x+\int_0^t\sigma(X_{s^-})dL_s. \tag{1.4}$$

The quantity m_1 is defined by $m_1=\mathbb{E}(L_1)$, and the definition of p_k will be given in Section 2. Notice that equation (1.4) is the stochastic version of the partial differential integral equation (PDIEs, in short) introduced by El Otmani in [23]. Our study is motivate by the fact that almost all deterministic problems in many applied fields have their stochastic counterparts. The method is also fully probabilistic and uses connection between MSPDIE (1.1) and BDSDE (1.1) in Markovian framework.

The paper is organized as follows. In Section 2, we introduce some preliminaries and notations. Section 3 is devoted to the existence and uniqueness result for BDSDEs involving subdifferential operator of a convex function and driven by a Lévy process. Finally, in section 4 we derive a probabilistic representation (in stochastic viscosity sense) for the solution of a class of MSPDIEs via BDSDEs proposed in Section 3.

2 Preliminaries and notations

Let $T>0$ be a fixed terminal time and $\{B_t:t\in[0,T]\}$ be a standard $\mathbb{R}$-valued Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let us also consider $\{L_t:t\in$

$[0,T]\}$, a $\mathbb{R}$-valued Lévy process corresponding to a standard Lévy measure ν defined on a complete probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ with the following characteristic function:

$$\mathbb{E}(\mathrm{e}^{iuL_t}) = \exp\left[iaut - \frac{1}{2}\kappa^2u^2t + t\int_{\mathbb{R}}(\mathrm{e}^{iux} - 1 - iux1_{\{|x|<1\}})\nu(\mathrm{d}x)\right],$$

where $a \in \mathbb{R}, \kappa \geq 0$. Moreover, the Lévy measure ν satisfies the following conditions:

1. $\int_{\mathbb{R}}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$,
2. $\int_{]-\varepsilon,\varepsilon[^c} \mathrm{e}^{\lambda|y|}\nu(\mathrm{d}y) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$,

which provides that L_t has moments of all orders, i.e. $\int_{-\infty}^{+\infty} |x|^i\nu(\mathrm{d}x) < \infty,\ \forall i \geq 2$.

We consider the product space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$, defined by

$$\bar{\Omega} = \Omega\times\Omega'; \quad \bar{\mathcal{F}} = \mathcal{F}\otimes\mathcal{F}' \ ; \quad \bar{\mathbb{P}} = \mathbb{P}\otimes\mathbb{P}'.$$

Further, random variables $\xi(\omega)$, $\omega \in \Omega$ and $\zeta(\omega')$, $\omega' \in \Omega'$ can be considered as random variables on $\bar{\Omega}$ via the following identifications:

$$\xi(\omega,\omega') = \xi(\omega); \quad \zeta(\omega,\omega') = \zeta(\omega').$$

In this fact, the processes B and L are assumed independent. Next, denoting by $\mathcal{N}$ the totality of $\bar{\mathbb{P}}$-null sets of $\bar{\mathcal{F}}$, and for each $t \in [0,T]$, we define

$$\mathcal{F}_t = \mathcal{F}^B_{t,T}\otimes\mathcal{F}^L_t \vee \mathcal{N},$$

where for any process $\{\eta_t\}, \mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s : s \leq r \leq t\}$ and $\mathcal{F}^\eta_t = \mathcal{F}^\eta_{0,t}$. Since $\{\mathcal{F}^B_{t,T}\}_{t\geq 0}$ is decreasing and $\{\mathcal{F}^L_t\}_{t\geq 0}$ is increasing, the objet $\{\mathcal{F}_t\}_{t\geq 0}$ is neither increasing nor decreasing. Thus it does not a filtration.

We denote by $(H^{(i)})_{i\geq 1}$ the linear combination of so-called Teugels martingale $Y^{(i)}_t$ associated with the Lévy process $\{L_t : t \in [0,T]\}$ defined by

$$H^{(i)}_t = c_{i,i}Y^{(i)}_t + c_{i,i-1}Y^{(i-1)}_t + \cdots + c_{i,1}Y^{(1)}_t,$$

where for all $i \geq 1$, $Y^{(i)}_t = L^{(i)}_t - E[L^{(i)}_t] = L^{(i)}_t - tE[L^{(i)}_1]$ for each $t \in [0,T]$. More precisely, denoting $\Delta L_s = L_s - L_{s^-}$, the processes $L^{(i)}_t$ is defined as follows: $L^{(1)}_t = L_t$ and $L^{(i)}_t = \sum_{0<s\leq t}(\Delta L_s)^i$ for $i \geq 2$. It was shown in Nualart and Schoutens [21] that $L^{(i)}_t$ is a power-jump processes and the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}$ with respect to the measure $\mu(\mathrm{d}x) = x^2\nu(\mathrm{d}x) + \kappa^2\delta_0(\mathrm{d}x)$:

$$\int_{\mathbb{R}} q_n(x)q_m(x)\mu(dx) = 0 \text{ if } n \neq m \text{ and } \int_{\mathbb{R}} q^2_n(x)\mu(dx) = 1.$$

We set

$$p_k(x) = xq_{k-1}(x).$$

The martingales $(H^{(i)})_{i\geq 1}$ can be chosen to be pairwise strongly orthonormal martingales, i.e. $[H^{(i)},H^{(j)}] = 0, i \neq j$, and $\{[H^{(i)},H^{(i)}]_t - t\}_{t\geq 0}$ are uniformly integrable martingales with initial value 0 and $\left\langle H^{(i)},H^{(j)}\right\rangle_t = \delta_{ij}t$.

Remark 2.1. The case of $\nu = 0$ corresponds to the classic Brownian case and all non-zero degree polynomials $q_i(x)$ will vanish, giving $H_t^{(i)} = 0, i = 2, 3, \cdots$, i.e. all power jump processes of order strictly greater than one will be equal to zero. If ν only has mass at 1, we have the Poisson case; here also $H_t^{(i)} = 0, i = 2, 3, \cdots$, i.e. all power jumps processes will be the same, and equal to the original Poisson process. Both cases are degenerate in this Lévy framework.

Let us introduce the following appropriate spaces:

- $\ell^2 = \left\{ x = (x_n)_{n\geq 1};\ \|x\|_{\ell^2} = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} < \infty \right\}$.
- $\mathcal{H}^2$ the subspace of the $\mathcal{F}_t$-measurable and $\mathbb{R}$-valued processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|^2_{\mathcal{H}^2} = \mathbb{E}\int_0^T |Y_t|^2\,\mathrm{d}t < +\infty.$$

- S^2 the subspace of the $\mathbb{R}$-valued, $\mathcal{F}_t$-measurable, right continuous left limited (rcll, in short) processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|^2_{S^2} = \mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2\right) < +\infty.$$

- $\mathcal{P}^2(l^2)$ the space of jointly predictable processes $(Z)_{t\in[0,T]}$ taking values in ℓ^2 such that

$$\|Z\|^2_{\mathcal{P}^2(l^2)} = \mathbb{E}\int_0^T \|Z_s\|^2_{\ell^2}\,\mathrm{d}s = \sum_{i=1}^{\infty}\mathbb{E}\int_0^T |Z_s^{(i)}|^2\,\mathrm{d}s < \infty.$$

Now, we make the following assumptions:

(H1) The coefficients $f : [0,T]\times\Omega\times\mathbb{R}\times\ell^2 \to \mathbb{R}$ and $g : [0,T]\times\Omega\times\mathbb{R}\times\ell^2 \to \mathbb{R}$ satisfy, for all $t \in [0,T]$, $y \in \mathbb{R}$ and $z \in \ell^2$,

(i) $f(t,\cdot,y,z)$ and $g(t,\cdot,y,z)$ are $\mathcal{F}_t$-measurable,

(ii) $f(\cdot,0,0)$, $g(\cdot,0,0) \in \mathcal{H}^2$;

(H2) There exist some constants $C > 0$ and $0 < \alpha < 1$ such that for every $(t,\omega) \in [0,T]\times\Omega$, (y_1,z_1), $(y_2,z_2) \in \mathbb{R}\times\ell^2$

$$|f(t,\omega,y_1,z_1) - f(t,\omega,y_2,z_2)|^2 \leq C\left(|y_1 - y_2|^2 + \|z_1 - z_2\|^2_{\ell^2}\right),$$

$$|g(t,\omega,y_1,z_1) - g(t,\omega,y_2,z_2)|^2 \leq C|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2_{\ell^2};$$

(H3) Let $\varphi : \mathbb{R} \to (-\infty,+\infty]$ be a proper lower semi continuous convex function satisfying $\varphi(y) \geq \varphi(0) = 0$;

(H4) The terminal value $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ satisfies

$$\mathbb{E}\left(|\xi|^2 + \varphi(\xi)\right) < \infty.$$

Define:

$$\mathrm{Dom}(\varphi) = \{u \in \mathbb{R} : \varphi(u) < +\infty\},$$

$$\partial\varphi(u) = \{u^* \in \mathbb{R} : \langle u^*, v-u\rangle + \varphi(u) \le \varphi(v), \text{ for all } v \in \mathbb{R}\},$$

$$\mathrm{Dom}(\partial\varphi) = \{u \in \mathbb{R} : \partial\varphi(u) \neq \emptyset\},$$

$$\mathrm{Gr}(\partial\varphi) = \{(u,u^*) \in \mathbb{R}^2 : u \in \mathrm{Dom}(\partial\varphi), u^* \in \partial\varphi(u)\}.$$

Notice that the subdifferential $\partial\varphi$ is often identified with its graph $\mathrm{Gr}(\partial\varphi)$.

Now, we introduce a multi-valued maximal monotone operator on $\mathbb{R}$ defined by the subdifferential of the above function φ.

For all $x \in \mathbb{R}$, define

$$\varphi_\varepsilon(x) = \min_y \left(\frac{1}{2}|x-y|^2 + \varepsilon\varphi(y)\right) = \frac{1}{2}|x - J_\varepsilon(x)|^2 + \varepsilon\varphi(J_\varepsilon(x)),$$

where $J_\varepsilon(x) = (I + \varepsilon\partial\varphi)^{-1}(x)$ is called the resolvent of the monotone operator $A = \partial\varphi$. Then, we have the following proposition which appeared in Brezis [8].

Proposition 2.2. (1) *The function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a convex with Lipschitz continuous derivatives;*

(2) *for all $x \in \mathbb{R}$,*

$$\frac{1}{\varepsilon}D\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\partial\varphi_\varepsilon(x) = \frac{1}{\varepsilon}(x - J_\varepsilon(x)) \in \partial\varphi(J_\varepsilon(x));$$

(3) *for all $x, y \in \mathbb{R}$,*

$$|J_\varepsilon(x) - J_\varepsilon(y)| \le |x - y|;$$

(4) *for all $x \in \mathbb{R}$,*

$$0 \le \varphi_\varepsilon(x) \le \langle D\varphi_\varepsilon(x), x\rangle;$$

(5) *for all $x, y \in \mathbb{R}$ and $\varepsilon, \delta > 0$,*

$$\left\langle \frac{1}{\varepsilon}D\varphi_\varepsilon(x) - \frac{1}{\delta}D\varphi_\delta(y), x-y \right\rangle \ge -\left(\frac{1}{\varepsilon} + \frac{1}{\delta}\right)|D\varphi_\varepsilon(x)||D\varphi_\delta(y)|.$$

We first give the definition of BDSDEs involving subdifferential operator of a convex function and driven by Lévy process.

Definition 2.3. By definition a solution to BDSDE (ξ, f, g, φ) is a triple of (Y, U, Z) of jointly measurable processes such that

1. $(Y, Z) \in S^2 \times \mathcal{P}^2(l^2),\ U \in \mathcal{H}^2$;
2. $(Y_t, U_t) \in \partial\varphi,\ \mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-a.e. on $[0, T]$;
3. for all $t \in [0, T]$

$$Y_t + \int_t^T U_s\,\mathrm{d}s = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}B_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}\,\mathrm{d}H_s^{(i)}.$$

3 Existence and uniqueness result for BDSDE driven by Lévy process

The first result of the paper is the following theorem:

Theorem 3.1. *Assume that the assumptions* (H1)–(H4) *hold. Then the BDSDE* (ξ, f, g, φ) *has a unique solution.*

For the prove of this theorem, let us consider the following BDSDEs:

$$
\begin{aligned}
Y_t^\varepsilon + \frac{1}{\varepsilon}\int_t^T D\varphi_\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s &= \xi + \int_t^T f(s, Y_s^\varepsilon, Z_s^\varepsilon)\,\mathrm{d}s + \int_t^T g(s, Y_s^\varepsilon, Z_s^\varepsilon)\,\mathrm{d}B_s \\
&\quad - \sum_{i=1}^\infty \int_t^T Z_s^{\varepsilon,(i)}\,\mathrm{d}H_s^{(i)},\ 0 \le t \le T,
\end{aligned} \tag{3.1}
$$

where φ_ε is the Yosida approximation of the operator $A = \partial\varphi$. Since $\frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon)$ is Lipschitz continuous, it is known from a recent result of Ren et al. [30], that Eq. (3.1) has a unique solution $(Y^\varepsilon, Z^\varepsilon) \in S^2 \times \mathcal{P}^2(l^2)$.

Setting $U_t^\varepsilon = \frac{1}{\varepsilon}D\varphi_\varepsilon(Y_t^\varepsilon)$, $0 \le t \le T$, our aim is to prove that the net $(Y^\varepsilon, U^\varepsilon, Z^\varepsilon)$ converges to a process (Y, U, Z) which is the desired solution of the BDSDEs.

In the sequel, $C > 0$ is a constant which can change its value from line to line. Firstly, we give a priori estimates on the solution.

Lemma 3.2. *Assume that assumptions* (H1)–(H4) *hold. Then there exists a constant* $C_1 > 0$ *such that for all* $\varepsilon > 0$

$$
\mathbb{E}\left(\sup_{0\le t\le T}|Y_t^\varepsilon|^2 + \int_0^T \|Z_s^\varepsilon\|_{\ell^2}^2\,\mathrm{d}s\right) \le C_1.
$$

Proof. Applying the Itô formula to $|Y_t^\varepsilon|^2$ yields that

$$
\begin{aligned}
|Y_t^\varepsilon|^2 + \frac{2}{\varepsilon}\int_t^T Y_s^\varepsilon D\varphi_\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s &= |\xi|^2 + 2\int_t^T Y_{s-}^\varepsilon f(s, Y_s^\varepsilon, Z_s^\varepsilon)\,\mathrm{d}s + 2\int_t^T Y_{s-}^\varepsilon g(s, Y_s^\varepsilon, Z_s^\varepsilon)\,\mathrm{d}B_s \\
&\quad + \int_t^T |g(s, Y_s^\varepsilon, Z_s^\varepsilon)|^2\,\mathrm{d}s - \sum_{i=1}^\infty \int_t^T |Z_s^{\varepsilon,(i)}|^2\,\mathrm{d}[\mathrm{H}^{(\mathrm{i})}, \mathrm{H}^{(\mathrm{i})}]_s \\
&\quad - 2\sum_{i=1}^\infty \int_t^T Y_{s-}^\varepsilon Z_s^{\varepsilon,(i)}\,\mathrm{d}H_s^{(i)}.
\end{aligned} \tag{3.2}
$$

Noting that the fact $Y_s^\varepsilon D\varphi_\varepsilon(Y_s^\varepsilon) \ge 0$ and taking expectation on the both sides, we obtain

$$
\begin{aligned}
\mathbb{E}|Y_t^\varepsilon|^2 + \mathbb{E}\int_t^T \|Z_s^\varepsilon\|_{\ell^2}^2\,\mathrm{d}s &\le \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_t^T Y_{s-}^\varepsilon f(s, Y_s^\varepsilon, Z_s^\varepsilon)\,\mathrm{d}s \\
&\quad + \mathbb{E}\int_t^T |g(s, Y_s^\varepsilon, Z_s^\varepsilon)|^2\,\mathrm{d}s.
\end{aligned} \tag{3.3}
$$

Using the elementary inequality $2ab \le \beta^2 a^2 + \frac{b^2}{\beta^2}$ for all $a,b \ge 0, \beta > 0$, and (H2), we obtain for all $M > 0$

$$\begin{aligned} 2yf(s,y,z) &= 2y(f(s,y,z) - f(s,0,0)) + 2yf(s,0,0) \\ &\le \frac{1}{M}|y|^2 + MC|y|^2 + MC\|z\|^2_{\ell^2} + |y|^2 + |f(s,0,0)|^2 \\ &\le \left(1 + \frac{1}{M} + MC\right)|y|^2 + |f(s,0,0)|^2 + MC\|z\|^2_{\ell^2} \end{aligned}$$

and

$$\begin{aligned} |g(s,y,z)|^2 &= |g(s,y,z) - g(s,0,0) + g(s,0,0)|^2 \\ &\le \left(1 + \frac{1}{\beta}\right)|g(s,y,z) - g(s,0,0)|^2 + (1+\beta)|g(s,0,0)|^2 \\ &\le \left(1 + \frac{1}{\beta}\right)C|y|^2 + (1+\beta)|g(s,0,0)|^2 + \alpha\left(1 + \frac{1}{\beta}\right)\|z\|^2_{\ell^2}. \end{aligned}$$

Choosing $M = \frac{1-\alpha}{2C}, \beta = \frac{3\alpha}{1-\alpha}$, where $0 < \alpha < 1$ is a constant appearing in (H2), it follows from (3.3) that

$$\begin{aligned} &\mathbb{E}|Y^\varepsilon_t|^2 + \frac{1-\alpha}{6}\mathbb{E}\int_t^T \|Z^\varepsilon_s\|^2_{\ell^2}\,\mathrm{d}s \\ \le\ & C\mathbb{E}\left(|\xi|^2 + \int_t^T |Y^\varepsilon_s|^2\,\mathrm{d}s + \int_0^T |f(s,0,0)|^2\,\mathrm{d}s + \int_0^T |g(s,0,0)|^2\,\mathrm{d}s\right). \end{aligned}$$

Gronwall inequality and Bulkholder-Davis-Gundy inequality show the desired result. □

Lemma 3.3. *Assume that the assumptions* (H1)–(H4) *hold. Then there exists a constant* $C_2 > 0$ *independent of* ε *such that*

(i) $\mathbb{E}\displaystyle\int_0^T \left(\frac{1}{\varepsilon}\left|D\varphi_\varepsilon(Y^\varepsilon_s)\right|\right)^2 \mathrm{d}s \le C_2$;

(ii) $\mathbb{E}\varphi(J_\varepsilon(Y^\varepsilon_t)) \le C_2$;

(iii) $\mathbb{E}|Y^\varepsilon_t - J_\varepsilon(Y^\varepsilon_t)|^2 \le \varepsilon^2 C_2$.

Proof. (i) For each $t \in [0,T]$, given an equidistant partition of interval $[t,T]$ such that $t = t_0 < t_1 < t_2 < \cdots < t_n = T$ and $t_{i+1} - t_i = \frac{1}{n}$, the subdifferential inequality shows

$$\varphi_\varepsilon(Y^\varepsilon_{t_{i+1}}) \ge \varphi_\varepsilon(Y^\varepsilon_{t_i}) + (Y^\varepsilon_{t_{i+1}} - Y^\varepsilon_{t_i})D\varphi_\varepsilon(Y^\varepsilon_{t_i}).$$

From (3.1), we obtain

$$\begin{aligned} \varphi_\varepsilon(Y^\varepsilon_{t_i}) + \frac{1}{\varepsilon}\int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y^\varepsilon_{t_i})D\varphi_\varepsilon(Y^\varepsilon_s)\,\mathrm{d}s \ &\le\ \varphi_\varepsilon(Y^\varepsilon_{t_{i+1}}) + \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y^\varepsilon_{t_i})f(s,Y^\varepsilon_s,Z^\varepsilon_s)\,\mathrm{d}s \\ &\quad + \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y^\varepsilon_{t_i})g(s,Y^\varepsilon_s,Z^\varepsilon_s)\,\mathrm{d}B_s \\ &\quad -2\sum_{j=1}^\infty \int_{t_i}^{t_{i+1}} D\varphi_\varepsilon(Y^\varepsilon_{t_i})(Z^\varepsilon_s)^{(j)}\,\mathrm{d}H^{(j)}_s. \end{aligned}$$

Summing up the above formula over i and letting $n \to \infty$, we obtain

$$\begin{aligned}
&\varphi_\varepsilon(Y^\varepsilon_{t^-}) + \frac{1}{\varepsilon}\int_t^T |D\varphi_\varepsilon(Y^\varepsilon_{s^-})|^2\,\mathrm{d}s \\
\leq\ & \varphi_\varepsilon(\xi) + \int_t^T D\varphi_\varepsilon(Y^\varepsilon_{s^-})f(s,Y^\varepsilon_s,Z^\varepsilon_s)\,\mathrm{d}s + \int_t^T D\varphi_\varepsilon(Y^\varepsilon_{s^-})g(s,Y^\varepsilon_s,Z^\varepsilon_s)\,\mathrm{d}B_s \\
& -2\sum_{j=1}^\infty \int_t^T D\varphi_\varepsilon(Y^\varepsilon_{s^-})(Z^\varepsilon_s)^{(j)}\,\mathrm{d}H^{(j)}_s.
\end{aligned}$$

Taking expectation on the both sides, we get

$$\mathbb{E}\varphi_\varepsilon(Y^\varepsilon_t) + \frac{1}{\varepsilon}\mathbb{E}\int_t^T |D\varphi_\varepsilon(Y^\varepsilon_s)|^2\,\mathrm{d}s \leq \mathbb{E}\varphi_\varepsilon(\xi) + \mathbb{E}\int_0^T D\varphi_\varepsilon(Y^\varepsilon_s)f(s,Y^\varepsilon_s,Z^\varepsilon_s)\,\mathrm{d}s. \quad (3.4)$$

From the inequalities

$$\begin{aligned}
D\varphi_\varepsilon(y)f(s,y,z) &\leq \frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2 + \frac{\varepsilon}{2}|f(s,y,z)|^2 \\
&\leq \frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2 + \varepsilon(|f(s,y,z)-f(s,0,0)|^2 + |f(s,0,0)|^2) \\
&\leq \frac{1}{2\varepsilon}|D\varphi_\varepsilon(y)|^2 + \varepsilon C|y|^2 + \varepsilon C\|z\|^2_{\ell^2} + \varepsilon|f(s,0,0)|^2,
\end{aligned}$$

from the facts that $\varphi_\varepsilon(Y^\varepsilon_t) \geq 0$ and $\varphi_\varepsilon(\xi) \leq \varepsilon\varphi(\xi)$ and by employing (3.4), we obtain

$$\frac{1}{2\varepsilon}\mathbb{E}\int_0^T |D\varphi_\varepsilon(Y^\varepsilon_s)|^2\,\mathrm{d}s \leq C\mathbb{E}\left(\varphi(\xi) + \int_0^T |f(s,0,0)|^2\,\mathrm{d}s + T\sup_{0\leq t\leq T}|Y^\varepsilon_t|^2 + \int_0^T \|Z^\varepsilon_t\|^2_{\ell^2}\,\mathrm{d}t\right).$$

Lemma 3.2 shows the desired result.
(ii) From (3.4) we obtain

$$\mathbb{E}\varphi_\varepsilon(Y^\varepsilon_t) \leq \varepsilon\mathbb{E}\varphi(\xi) + \frac{1}{2\varepsilon}\mathbb{E}\int_t^T |D\varphi_\varepsilon(Y^\varepsilon_s)|^2\,\mathrm{d}s + \varepsilon\mathbb{E}\int_t^T |f(s,Y^\varepsilon_s,Z^\varepsilon_s)|^2\,\mathrm{d}s.$$

Using $\varphi(J_\varepsilon(Y^\varepsilon_t)) \leq \frac{1}{\varepsilon}\varphi_\varepsilon(Y^\varepsilon_t)$ and (*i*), we obtain (*ii*).
The last part of the Lemma simply follows from the fact that

$$|x - J_\varepsilon(x)| = 2\varphi_\varepsilon(x) - 2\varepsilon\varphi(J_\varepsilon(x)).$$

This completes the proof of Lemma 3.3. □

Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of strictly positif real numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$. In what follows, we aim to show that $(Y^{\varepsilon_n}, Z^{\varepsilon_n})_{n\geq 0}$ is a Cauchy sequence in $S^2 \times \mathcal{P}^2(l^2)$.

Lemma 3.4. *Assume that the assumptions* (H1)–(H4) *hold. Then there exists a constant* C_3 *independent of* ε, $\delta > 0$ *such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y^\varepsilon_t - Y^\delta_t|^2 + \int_0^T \|Z^\varepsilon_t - Z^\delta_t\|^2_{\ell^2}\,\mathrm{d}t\right) \leq C_3(\varepsilon + \delta).$$

Proof. Applying the Itô formula to $|Y_t^\varepsilon - Y_t^\delta|^2$ yields the equality

$$
\begin{aligned}
|Y_t^\varepsilon - Y_t^\delta|^2 &= -2\int_t^T (Y_s^\varepsilon - Y_s^\delta)\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s - \frac{1}{\delta}D\varphi_\delta(Y_s^\delta)\right)\mathrm{d}s \\
&\quad +2\int_t^T (Y_s^\varepsilon - Y_s^\delta)(f(s,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,Y_s^\delta,Z_s^\delta))\,\mathrm{d}s \\
&\quad +2\int_t^T (Y_s^\varepsilon - Y_s^\delta)(g(s,Y_s^\varepsilon,Z_s^\varepsilon) - g(s,Y_s^\delta,Z_s^\delta))\,\mathrm{d}B_s \\
&\quad +\int_t^T |g(s,Y_s^\varepsilon,Z_s^\varepsilon) - g(s,Y_s^\delta,Z_s^\delta)|^2\,\mathrm{d}s - \int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2\,\mathrm{d}s \\
&\quad -2\sum_{i=1}^\infty \int_t^T (Y_s^\varepsilon - Y_s^\delta)(Z_s^{\varepsilon,(i)} - Z_s^{\delta,(i)})\,\mathrm{d}H_s^{(i)}.
\end{aligned} \tag{3.5}
$$

Taking expectations, we obtain

$$
\begin{aligned}
&\mathbb{E}|Y_t^\varepsilon - Y_t^\delta|^2 + \mathbb{E}\int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2\,\mathrm{d}s \\
&\quad = -2\mathbb{E}\int_t^T (Y_s^\varepsilon - Y_s^\delta)\left(\frac{1}{\varepsilon}D\varphi_\varepsilon(Y_s^\varepsilon)\,\mathrm{d}s - \frac{1}{\delta}D\varphi_\delta(Y_s^\delta)\right)\mathrm{d}s \\
&\qquad +2\mathbb{E}\int_t^T (Y_s^\varepsilon - Y_s^\delta)(f(s,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,Y_s^\delta,Z_s^\delta))\,\mathrm{d}s \\
&\qquad +\mathbb{E}\int_t^T |g(s,Y_s^\varepsilon,Z_s^\varepsilon) - g(s,Y_s^\delta,Z_s^\delta)|^2\,\mathrm{d}s.
\end{aligned} \tag{3.6}
$$

Using the elementary inequality $2ab \le \beta^2 a^2 + \frac{b^2}{\beta^2}$ for all $a,b \ge 0$ and (H2), we get as in the proof of Lemma 3.2

$$
\begin{aligned}
&(Y_s^\varepsilon - Y_s^\delta)(f(s,Y_s^\varepsilon,Z_s^\varepsilon) - f(s,Y_s^\delta,Z_s^\delta)) \\
&\quad \le \frac{2C}{1-\alpha}|Y_s^\varepsilon - Y_s^\delta|^2 + \frac{1-\alpha}{2}|Y_s^\varepsilon - Y_s^\delta|^2 + \frac{1-\alpha}{2}\|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2
\end{aligned}
$$

and

$$|g(s,Y_s^\varepsilon,Z_s^\varepsilon) - g(s,Y_s^\delta,Z_s^\delta)|^2 \le C|Y_s^\varepsilon - Y_s^\delta|^2 + \alpha\|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2.$$

Noting (5) of Proposition 2.2, we obtain

$$
\begin{aligned}
&\mathbb{E}|Y_t^\varepsilon - Y_t^\delta|^2 + \frac{1-\alpha}{2}\mathbb{E}\int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2\,\mathrm{d}s \\
&\quad \le C\varepsilon\mathbb{E}\int_t^T |Y_s^\varepsilon - Y_s^\delta|^2\,\mathrm{d}s \\
&\qquad +2\left(\frac{1}{\varepsilon} + \frac{1}{\delta}\right)\mathbb{E}\int_t^T |D\varphi_\varepsilon(Y_s^\varepsilon)||D\varphi_\delta(Y_s^\delta))|\,\mathrm{d}s.
\end{aligned} \tag{3.7}
$$

Lemma 3.3 shows that

$$2\left(\frac{1}{\varepsilon} + \frac{1}{\delta}\right)\mathbb{E}\int_t^T |D\varphi_\varepsilon(Y_s^\varepsilon)||D\varphi_\delta(Y_s^\delta))|\,\mathrm{d}s \le (\varepsilon+\delta)C.$$

So, we can obtain

$$\mathbb{E}|Y_t^\varepsilon - Y_t^\delta|^2 + \mathbb{E}\int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 \,\mathrm{d}s \le C\mathbb{E}\int_t^T |Y_s^\varepsilon - Y_s^\delta|^2 \,\mathrm{d}s + C(\varepsilon+\delta).$$

The Gronwall inequality shows that

$$\sup_{0\le t\le T} \mathbb{E}|Y_t^\varepsilon - Y_t^\delta|^2 + \mathbb{E}\int_t^T \|Z_s^\varepsilon - Z_s^\delta\|_{\ell^2}^2 \,\mathrm{d}s \le C(\varepsilon+\delta).$$

The Bulkholder-Davis-Gundy inequality yields the desired result. □

Proof of Theorem 3.1

Existence. Lemma 3.4 shows that $(Y^{\varepsilon_n}, Z^{\varepsilon_n})$ is a Cauchy sequence in $S^2 \times \mathcal{P}^2(l^2)$, whenever $\lim_{n\to\infty} \varepsilon_n = 0$. As a consequence, the $\lim_{\varepsilon\downarrow 0}(Y^\varepsilon, Z^\varepsilon)$ exist in the same space. Denote this limit by (Y,Z). From the same lemma it follows that (Y,Z) belongs to $S^2 \times \mathcal{P}^2(l^2)$. For each $\varepsilon \ge 0$, let us define

$$U_t^\varepsilon = \frac{1}{\varepsilon} D\varphi_\varepsilon(Y_t^\varepsilon)$$

and

$$\bar{U}_t^\varepsilon = \int_0^t U_s^\varepsilon \,\mathrm{d}s.$$

Then, for all ε, $\delta > 0$, it follows from (3.1) that

$$\mathbb{E}\left(\sup_{0\le t\le T} |\bar{U}_t^\varepsilon - \bar{U}_t^\delta|^2\right) \le C\mathbb{E}\left(\sup_{0\le t\le T} |Y_t^\varepsilon - Y_t^\delta|^2 + \int_0^T \|Z_t^\varepsilon - Z_t^\delta\|_{\ell^2}^2 \,\mathrm{d}t\right),$$

which, together with Lemma 3.4 shows that $(\bar{U}^\varepsilon)$ is a Cauchy net. Hence, there exists a measurable process $\bar{U}_t$ such that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left(\sup_{0\le t\le T} |\bar{U}_t^\varepsilon - \bar{U}_t|^2\right) = 0.$$

Furthermore, Lemma 3.3 (i) implies that

$$\sup_\varepsilon \mathbb{E}\int_0^T |U_t^\varepsilon|^2 \,\mathrm{d}t = \sup_\varepsilon \mathbb{E}\int_0^T \left(\frac{1}{\varepsilon}|D\varphi_\varepsilon(Y_t^\varepsilon)|\right)^2 \mathrm{d}t < \infty,$$

which shows that $(\bar{U}^\varepsilon)_\varepsilon$ is bounded in the space $L^2(\Omega, H^1[0,T])$, and converges weakly to $\bar{U}$ in the same space. In particular $\bar{U}$ is absolutely continuous so that there exists a measurable process $(U_t)_{0\le t\le T} \in \mathcal{H}^2$ such that $\bar{U}_t = \int_0^t U_s \,\mathrm{d}s$.

Next, we will show that $(Y_t, U_t) \in \partial\varphi$, $\mathrm{d}\mathbb{P}\otimes \mathrm{d}t$-a.e. on $[0,T]$. A consequence of assertion (2) in Proposition 2.2 is that U_t^ε belongs to the subdifferential $\partial\varphi(J_\varepsilon(Y_t^\varepsilon))$, and hence

$$\int_a^b U_t^\varepsilon(V_t - J_\varepsilon(Y_t^\varepsilon))dt + \int_a^b \varphi(J_\varepsilon(Y_t^\varepsilon))dt \le \int_a^b \varphi(V_t)dt \tag{3.8}$$

for all $V=(V_t)_{t\in[a,b]}\in\mathcal{H}([a,b])$. Moreover, for all $0\le a<b\le T$ and all processes $V\in\mathcal{H}^2([a,b])$, Lemma 5.8 in [14] provides us the convergence

$$\int_a^b U_t^\varepsilon(V_t-Y_t^\varepsilon)\,\mathrm{d}t\to\int_a^b U_t(V_t-Y_t)\,\mathrm{d}t,\ \text{ in probability as } \varepsilon\downarrow 0. \tag{3.9}$$

From (*i*) of Lemma 3.3, i.e. from $\sup_{\varepsilon>0}\mathbb{E}\int_0^T|U_t^\varepsilon|^2dt<\infty$ together with assertion (*iii*) in Lemma 3.3 we see

$$\int_a^b U_t^\varepsilon(J_\varepsilon(Y_t^\varepsilon)-Y_t^\varepsilon)\,\mathrm{d}t\to 0,\ \text{as } \varepsilon\downarrow 0. \tag{3.10}$$

which together with Proposition 2.2 provides that $U_t^\varepsilon\in\partial\varphi(J_\varepsilon(Y_t^\varepsilon))$ and

$$\int_a^b U_t^\varepsilon(V_t-J_\varepsilon(Y_t^\varepsilon)\,\mathrm{d}t+\int_a^b\varphi(J_\varepsilon(Y_t^\varepsilon))\,\mathrm{d}t\le\int_a^b\varphi(V_t)\,\mathrm{d}t.$$

In virtue of the inequality in (3.8) we obtain:

$$\begin{aligned}
&\int_a^b U_t(V_t-Y_t)dt+\int_a^b\varphi(Y_t)dt\\
=\ &\int_a^b U_t(V_t-Y_t)dt-\int_a^b U_t^\varepsilon(V_t-Y_t^\varepsilon)dt\\
&+\int_a^b U_t^\varepsilon(V_t-J_\varepsilon(Y_t^\varepsilon))dt+\int_a^b\varphi(J_\varepsilon(Y_t^\varepsilon))dt+\int_a^b U_t^\varepsilon(J_\varepsilon(Y_t^\varepsilon)-Y_t^\varepsilon)dt\\
\le\ &\int_a^b U_t(V_t-Y_t)dt-\int_a^b U_t^\varepsilon(V_t-Y_t^\varepsilon)dt+\int_a^b\varphi(V_t)dt\\
&+\int_a^b U_t^\varepsilon(J_\varepsilon(Y_t^\varepsilon)-Y_t^\varepsilon)dt.
\end{aligned}\tag{3.11}$$

Taking the liminf (in probability) in the inequality (3.11) as $\varepsilon\downarrow 0$ and involving (3.9) and (3.10), we obtain

$$\int_a^b U_t(V_t-Y_t)\,\mathrm{d}t+\int_a^b\varphi(Y_t)\,\mathrm{d}t\le\int_a^b\varphi(V_t)\,\mathrm{d}t. \tag{3.12}$$

Since in (3.12) we may choose any pair $(a,b)\in\mathbb{R}\times\mathbb{R}$ for which $0\le a\le b\le T$, and the process $V\in\mathcal{H}([a,b])$ is arbitrary, we infer

$$U_t(V_t-Y_t)+\varphi(Y_t)\le\varphi(V_t),\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.e.}. \tag{3.13}$$

From (3.13) it follows that $(Y_t,U_t)\in\partial\varphi$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$-a.e. on $[0,T]$. Taking limits on both sides of (3.1), we obtain the existence of the solution.

Uniqueness. Let $(Y_t,U_t,Z_t)_{0\le t\le T}$ and $(Y'_t,U'_t,Z'_t)_{0\le t\le T}$ be two solutions of BDSDE(ξ,f,g,φ). Define

$$(\Delta Y_t,\Delta U_t,\Delta Z_t)_{0\le t\le T}=(Y_t-Y'_t,U_t-U'_t,Z_t-Z'_t)_{0\le t\le T}.$$

Applying the Itô formula to $|\Delta Y_t|^2$ shows that

$$
\begin{aligned}
\mathbb{E}|\Delta Y_t|^2 + 2\mathbb{E}\int_t^T \Delta U_s \Delta Y_s \,\mathrm{d}s + \mathbb{E}\int_t^T \|\Delta Z_t\|^2_{\ell^2}\,\mathrm{d}s& \\
&= 2\mathbb{E}\int_t^T \Delta Y_s[f(s,Y_s,Z_s) - f(s,Y'_s,Z'_s)]\,\mathrm{d}s \\
&\quad +\mathbb{E}\int_t^T |g(s,Y_s,Z_s) - g(s,Y'_s,Z'_s)|^2\,\mathrm{d}s. \qquad (3.14)
\end{aligned}
$$

Since $\partial\varphi$ is monotone, and $U_t \in \partial\varphi(Y_t)$, for all $t \in [0,T]$, we have

$$\Delta U_s \Delta Y_s \geq 0, \ \ \mathrm{d}\mathbb{P}\otimes \mathrm{d}t\text{-a.e.}$$

Furthermore, by applying the same procedure as in the proof of Lemma 3.4, we obtain

$$\mathbb{E}|\Delta Y_t|^2 + \mathbb{E}\int_t^T \|\Delta Z_t\|^2_{\ell^2}\,\mathrm{d}s \leq C\mathbb{E}\int_t^T |\Delta Y_s|^2\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_t^T \|\Delta Z_s\|^2_{\ell^2}\,\mathrm{d}s.$$

The Gronwall inequality entails $\Delta Y_t = 0$ $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$-a.e., and so the uniqueness of the solution follows. □

4 Stochastic viscosity solutions of multivalued SPDIEs

In this section, we derive the existence of the stochastic viscosity solution of a class of multivalued SPDIE (1.2) via BDSDE with subdifferential operator and driven by Lvy process studied in the previous section.

4.1 Notion of stochastic viscosity solution of multivalued SPDIEs

Recall that $\mathbf{F}^B = \{\mathcal{F}^B_{t,T}\}_{0\leq t\leq T}$ stands for the filtration generated by Brownian motion $B = \{B_t\}_{0\leq t\leq T}$. The objet $\mathcal{M}^B_{0,T}$ denotes all the $\mathbf{F}^B$-stopping times τ such $0 \leq \tau \leq T$, a.s. and $\mathcal{M}^B_\infty$ is the set of all almost surely finite $\mathbf{F}^B$-stopping times. For generic Euclidean spaces E and E_1, we introduce the following spaces:

1. The symbol $C^{k,n}([0,T]\times E;E_1)$ stands for the space of all E_1-valued functions defined on $[0,T]\times E$ which are k-times continuously differentiable in t and n-times continuously differentiable in x, and $C^{k,n}_b([0,T]\times E;E_1)$ denotes the subspace of $C^{k,n}([0,T]\times E;E_1)$ in which all functions have uniformly bounded partial derivatives.
2. For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}^B_T$, $C^{k,n}(\mathcal{G},[0,T]\times E;E_1)$ (resp. $C^{k,n}_b(\mathcal{G},[0,T]\times E;E_1)$) denotes the space of all $C^{k,n}([0,T]\times E;E_1)$ (resp. $C^{k,n}_b([0,T]\times E;E_1)$-valued random variable that are $\mathcal{G}\otimes\mathcal{B}([0,T]\times E)$-measurable;
3. $C^{k,n}(\mathbf{F}^B,[0,T]\times E;E_1)$ (resp.$C^{k,n}_b(\mathbf{F}^B,[0,T]\times E;E_1)$) is the space of all random fields $\varphi \in C^{k,n}(\mathcal{F}_T,[0,T]\times E;E_1$ (resp. $C^{k,n}(\mathcal{F}_T,[0,T]\times E;E_1)$, such that for fixed $x\in E$ and $t\in[0,T]$, the mapping $\omega \to \alpha(t,\omega,x)$ is $\mathbf{F}^B$-progressively measurable.
4. For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}^B$ and a real number $p \geq 0$, $L^p(\mathcal{G};E)$ denotes the set of all E-valued, $\mathcal{G}$-measurable random variable ξ such that $\mathbb{E}|\xi|^p < \infty$.

Furthermore, regardless of the dimension, we denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the inner product and norm in E and E_1, respectively. For $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$, we denote $D_x=\left(\frac{\partial}{\partial x_1},....,\frac{\partial}{\partial x_d}\right)$, $D_{xx}=(\partial^2_{x_ix_j})^d_{i,j=1}$, $D_y=\frac{\partial}{\partial y}$, $D_t=\frac{\partial}{\partial t}$. The meaning of D_{xy} and D_{yy} is then self-explanatory. The coefficients

$$\begin{array}{rcl} f &:& \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\ell^2\to\mathbb{R}\\ g &:& \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}\\ \sigma &:& \mathbb{R}^d\to\mathbb{R}^d\\ u_0 &:& \mathbb{R}^d\to\mathbb{R}, \end{array}$$

satisfying the assumptions:

$$\text{(H5)}\ \begin{cases} |f(t,x,y,z)|\le K(1+|x|+|y|+\|z\|),\\ \\ |u_0(x)|+|\varphi(u_0(x))|\le K(1+|x|). \end{cases}$$

$$\text{(H6)}\ \begin{cases} \|\sigma(x)-\sigma(x')\|\le K|x-x'|,\\ \\ |f(t,x,y,z)-f(t,x,y',z')|\le K(|y-y'|+\|z-z'\|_{\ell^2}). \end{cases}$$

(H7) The function $g\in C_b^{0,2,3}([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R})$.

The definition of stochastic viscosity solution to MSPDIE (1.1) uses the stochastic sub- and super-jets introduced by Buckdahn and Ma [9]. Let us recall the following relevant definitions.

Definition 4.1. Let $\tau\in\mathcal{M}^B_{0,T}$, and $\xi\in\mathcal{F}_\tau$. We say that a sequence of random variables (τ_k,ξ_k) is a (τ,ξ)-approximating sequence if for all k, $(\tau_k,\xi_k)\in\mathcal{M}^B_\infty\times L^2(\mathcal{F}_\tau,\mathbb{R}^d)$ such that

(i) $\xi_k\to\xi$ in probability;

(ii) either $\tau_k\uparrow\tau$ a.s., and $\tau_k<\tau$ on the set $\{\tau>0\}$; or $\tau_k\downarrow\tau$ a.s., and $\tau_k>\tau$ on the set $\{\tau<T\}$.

The symbol $\mathcal{S}(n)$, in the next definition, stands for the set of all symmetric $n\times n$ matrices.

Definition 4.2. Let $(\tau,\xi)\in\mathcal{M}^B_{0,T}\times L^2\left(\mathcal{F}^B_\tau;\mathbb{R}^d\right)$ and $u\in C\left(\mathbf{F}^B,[0,T]\times\mathbb{R}^d\right)$. We denote by $\mathcal{J}^{1,2,+}_g u(\tau,\xi)$ the stochastic g-superjet of u at (τ,ξ) the set of $\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}(n)$-valued and $\mathcal{F}^B_\tau$-measurable random vector (a,p,X) which is such that for all (τ,ξ)-approximating sequence (τ_k,ξ_k), we have

$$\begin{aligned} u(\tau_k,\xi_k) \;\le\;& u(\tau,\xi)+a(\tau_k-\tau)+b(B_{\tau_k}-B_\tau)+\frac{c}{2}(B_{\tau_k}-B_\tau)^2+\langle p,\xi_k-\xi\rangle\\ &+\langle q,\xi_k-\xi\rangle(B_{\tau_k}-B_\tau)+\frac{1}{2}\langle X(\xi_k-\xi),\xi_k-\xi\rangle\\ &+o(|\tau_k-\tau|)+o(|\xi_k-\xi|^2). \end{aligned} \tag{4.1}$$

The $\mathcal{F}_\tau^B$-measurable random vector (b,c,q) taking values in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ is defined by

$$\begin{cases} b = g(\tau,\xi,u(\tau,\xi)), \quad c = (g\partial_u g)(\tau,\xi,u(\tau,\xi)) \\ \\ q = \partial_x g(\tau,\xi,u(\tau,\xi)) + \partial_u g(\tau,\xi,u(\tau,\xi))p. \end{cases}$$

Similarly, $\mathcal{J}_g^{1,2,-}u(\tau,\xi)$ denotes the set of all stochastic g-subjet of u at (τ,ξ) if the inequality in (4.1) is reversed.

Remark 4.3. Let us note that $\partial\varphi(y) = [\varphi_l'(y),\varphi_r'(y)]$, for every $y\in\mathrm{Dom}(\varphi)$, where $\varphi_l'(y)$ and $\varphi_r'(y)$ denote the left and right derivatives of φ.

In order to simplify the notation in the definition of stochastic viscosity solution of multivalued SPDIEs, we set

$$\begin{aligned} V_f(\tau,\xi,a,p,X) &= -a-\frac{1}{2}\mathrm{Trace}(\sigma\sigma^*(\xi)X) - m_1\langle p,\sigma(\xi)\rangle - \frac{1}{2}\int_{\mathbb{R}}\langle X\sigma(\xi),\sigma(\xi)\rangle y^2\nu(\mathrm{d}y) \\ &\quad -f\left(\tau,\xi,u(\tau,\xi),\int_{\mathbb{R}}\langle p,\sigma(\xi)y\rangle p_k(y)\nu(\mathrm{d}y)\right). \end{aligned}$$

Definition 4.4. (1) A random field $u\in C\left(\mathbf{F}^B,[0,T]\times\mathbb{R}^d\right)$ which satisfies $u(T,x) = u_0(x)$, for all $x\in\mathbb{R}^d$, is called a stochastic viscosity subsolution of MSPDIE (1.1) if

$$u(\tau,\xi) \in \mathrm{Dom}(\varphi), \quad \forall\,(\tau,\xi)\in\mathcal{M}_{0,T}^B\times L^2\left(\mathcal{F}_\tau^B;\mathbb{R}^d\right), \quad \mathbb{P}\text{-a.s.},$$

and at any $(\tau,\xi)\in\mathcal{M}_{0,T}^B\times L^2\left(\mathcal{F}_\tau^B;\mathbb{R}^d\right)$, for any $(a,p,X)\in\mathcal{J}_g^{1,2,+}u(\tau,\xi)$, the following inequality holds $\mathbb{P}$-a.s.

$$V_f(\tau,\xi,a,p,X)+\varphi_l'(u(\tau,\xi)-\frac{1}{2}(g\partial_u g)(\tau,\xi,u(\tau,\xi))\le 0; \tag{4.2}$$

(2) A random field $u\in C\left(\mathbf{F}^B,[0,T]\times\mathbb{R}^d\right)$ which satisfies that $u(T,x) = u_0(x)$, for all $x\in\mathbb{R}^d$, is called a stochastic viscosity supersolution of MSPDIE (1.2) if

$$u(\tau,\xi) \in \mathrm{Dom}(\varphi), \quad \forall\,(\tau,\xi)\in\mathcal{M}_{0,T}^B\times L^2\left(\mathcal{F}_\tau^B;\mathbb{R}^d\right), \quad \mathbb{P}\text{-a.s.},$$

and at any $(\tau,\xi)\in\mathcal{M}_{0,T}^B\times L^2\left(\mathcal{F}_\tau^B;\mathbb{R}^d\right)$, for any $(a,p,X)\in\mathcal{J}_g^{1,2,-}u(\tau,\xi)$, the following inequality holds $\mathbb{P}$-a.s.

$$V(\tau,\xi,a,p,X)+\varphi_r'(u(\tau,\xi)-\frac{1}{2}(g\partial_u g)(\tau,\xi,u(\tau,\xi))\ge 0; \tag{4.3}$$

(3) A random field $u\in C\left(\mathbf{F}^B,[0,T]\times\mathbb{R}^d\right)$ is called a stochastic viscosity solution of MSPDIE (1.1) if it is both a stochastic viscosity subsolution and a stochastic viscosity supersolution.

Remark 4.5. Observe that if f is deterministic and $g\equiv 0$, Definition 4.4 becomes the generalization of the definition of (deterministic) viscosity solution of MPDIE given by N'zi and Ouknine in [20].

To finish this section, we state the notion of random viscosity solution which will be a bridge linking the stochastic viscosity solution and its deterministic counterpart.

Definition 4.6. A random field $u\in C(\mathbf{F}^B,[0,T]\times\mathbb{R}^d)$ is called an ω-wise viscosity solution if for $\mathbb{P}$-almost all $\omega\in\Omega$, $u(\omega,\cdot,\cdot)$ is a (deterministic) viscosity solution of MSPDIE (1.2).

4.2 Doss-Sussmann transformation

In this section, we will introduce the stochastic flow $\eta \in C(\mathbf{F}^B, [0,T]\times\mathbb{R}^d\times\mathbb{R})$, unique solution of the following stochastic differential equation in the Stratonovich sense:

$$\eta(t,x,y) \quad = \quad y + \int_t^T \langle g(s,x,\eta(s,x,y)), \circ\mathrm{d}B_s\rangle, \tag{4.4}$$

where (4.4) should be viewed as going from T to t (i.e. y should be understood as the initial value). Under the assumption $(H7)$, the mapping $y \mapsto \eta(t,x,y)$ defines a diffeomorphism for all (t,x), $\mathbb{P}$-a.s. such that its y-inverse $\varepsilon(t,x,y)$ is the solution to the following first-order SPDE:

$$\varepsilon(t,x,y) = y - \int_t^T \langle D_y\varepsilon(s,x,y),\, g(s,x,\eta(s,x,y)) \circ \mathrm{d}B_s\rangle.$$

We refer the reader to [10] for a lucid discussion on this topic. We now give the following result which proof follows the same procedure as the proof of Lemma 4.8 in [7].

Proposition 4.7. *Assume that the assumptions* (H1)–(H7) *hold. If for* $(\tau,\xi) \in \mathcal{M}^B_{0,T} \times L^2\left(\mathcal{F}^B_\tau;\mathbb{R}^d\right)$, $u \in C\left(\mathbf{F}^B, [0,T]\times\mathbb{R}^d\right)$ *and* (a_u, X_u, p_u) *belongs to* $\mathcal{J}_g^{1,2,+}u(\tau,\xi)$, *then* (a_v, X_v, p_v) *belongs to* $\mathcal{J}_0^{1,2,+}v(\tau,\xi)$, *with* $v(\cdot,\cdot) = \varepsilon(\cdot,\cdot,u(\cdot,\cdot))$ *and*

$$\begin{cases} a_v = D_y\varepsilon(\tau,\xi,u(\tau,\xi))a_u \\ \\ p_v = D_y\varepsilon(\tau,\xi,u(\tau,\xi))p_u + D_x\varepsilon(\tau,\xi,u(\tau,\xi)) \\ \\ X_v = D_y\varepsilon(\tau,\xi,u(\tau,\xi))X_u + 2D_{xy}\varepsilon(\tau,\xi,u(\tau,\xi))p_u^* \\ \qquad + D_{xx}\varepsilon(\tau,\xi,u(\tau,\xi)) + D_{yy}\varepsilon(\tau,\xi,u(\tau,\xi))p_u p_u^*. \end{cases}$$

Conversely, if for $(\tau,\xi) \in \mathcal{M}^B_{0,T}\times L^2\left(\mathcal{F}^B_\tau;\mathbb{R}^d\right)$, $v \in C\left(\mathbf{F}^B, [0,T]\times\mathbb{R}^d\right)$ *and* $(a_v, X_v, p_v) \in \mathcal{J}_0^{1,2,+}v(\tau,\xi)$, *then* $(a_u, X_u, p_u) \in \mathcal{J}_g^{1,2,+}u(\tau,\xi)$ *with* $u(\cdot,\cdot) = \eta(\cdot,\cdot,v(\cdot,\cdot))$ *and*

$$\begin{cases} a_u = D_y\eta(\tau,\xi,v(\tau,\xi))a_v \\ \\ p_u = D_y\eta(\tau,\xi,v(\tau,\xi))p_v + D_x\eta(\tau,\xi,v(\tau,\xi)) \\ \\ X_u = D_y\eta(\tau,\xi,v(\tau,\xi))X_v + 2D_{xy}\eta(\tau,\xi,v(\tau,\xi))p_v^* \\ \qquad + D_{xx}\eta(\tau,\xi,v(\tau,\xi)) + D_{yy}\eta(\tau,\xi,v(\tau,\xi))p_v p_v^*. \end{cases}$$

One of the key ideas of Buckdahn and Ma is to use the Doss-Sussman transformation to convert a SPDE to a PDE with random coefficients, so that the stochastic viscosity solution can be studied ω-wisely. However, if we apply Doss-Sussman transformation to the MSPDIE (1.2) the resulting equation is not necessarily the multivalued PDIE studied by N'zi and Ouknine in [20], because of the presence of the subdifferential term. For this reason we will require the Doss-Sussman transformation in the following way:

Corollary 4.8. *Assume that the assumptions* (H1)–(H7) *hold. Let us define and consider* $(\tau,\xi) \in \mathcal{M}^B_{0,T}\times L^2\left(\mathcal{F}^B_\tau;\mathbb{R}^d\right)$, $u \in C\left(\mathbf{F}^B, [0,T]\times\mathbb{R}^d\right)$.

(1) *for* $(a_u, X_u, p_u) \in \mathcal{J}_g^{1,2,+}u(\tau,\xi)$, *u satisfies (4.2) if and only if* $v(\cdot,\cdot) = \varepsilon(\cdot,\cdot,u(\cdot,\cdot))$ *satisfies*

$$V_{\widetilde{f}}(\tau,\xi,a_v,p_v,X_v) + \frac{\varphi_l'(\eta(\tau,\xi,v(\tau,\xi))}{D_y\eta(\tau,\xi,v(\tau,\xi))} \leq 0; \tag{4.5}$$

(2) *for* $(a_u, X_u, p_u) \in \mathcal{J}_g^{1,2,-}u(\tau,\xi)$, *u satisfies (4.3) if and only if* $v(\cdot,\cdot) = \varepsilon(\cdot,\cdot,u(\cdot,\cdot))$ *satisfies*

$$V_{\widetilde{f}}(\tau,\xi,a_v,p_v,X_v) + \frac{\varphi_l'(\eta(\tau,\xi,v(\tau,\xi))}{D_y\eta(\tau,\xi,v(\tau,\xi))} \geq 0; \tag{4.6}$$

where (a_v, p_v, X_v) *is defined by Proposition 4.7 and*

$$\begin{aligned}\widetilde{f}(t,x,y,(\theta^k)_{k\geq 1}) \quad = \quad & \frac{1}{D_y\eta(t,x,y)}\Big[f\Big(t,x,\eta(t,x,y),D_y\eta(t,x,y)\theta^k + \eta_k^1(t,x,y)\Big)\\ & -\frac{1}{2}(g\partial_u g)(t,x,\eta(t,x,y)) + \mathcal{L}_x\eta(t,x,y) + \lambda\langle\sigma^*(x)D_{xy}\eta(t,x,y),\sigma(x)p_v\rangle\\ & +\frac{1}{2}\lambda D_{yy}\eta(t,x,y)|\sigma(x)p_v|^2\Big].\end{aligned}$$

with $\theta^k = \int_{\mathbb{R}}\langle p_v,\sigma(x)u\rangle p_k(u)\nu(du)$ *and* $\lambda = 1 + \int_{\mathbb{R}} u^2\nu(du)$.

Proof. Let $(\tau,\xi) \in \mathcal{M}_{0,T}^B \times L^2\left(\mathcal{F}_\tau^B;\mathbb{R}^d\right)$ be given and $(a_u, p_u, X_u) \in \mathcal{J}_g^{1,2,+}u(\tau,\xi)$. We assume that u is a stochastic subsolution of MSPDIE (1.2), i.e.

$$u(\tau,\xi) \quad \in \quad \mathrm{Dom}(\varphi), \qquad \forall\,(\tau,\xi) \in \mathcal{M}_{0,T}^B \times L^2\left(\mathcal{F}_\tau^B;\mathbb{R}^d\right), \quad \mathbb{P}\text{-a.s.},$$

such that

$$V_f(\tau,\xi,a,p,X) + \varphi_l'(u(\tau,\xi) - \frac{1}{2}(g\partial_u\, g)(\tau,\xi,u(\tau,\xi)) \leq 0, \ \mathbb{P}\text{-a.s.}$$

In view of Proposition 4.7 and since $D_y\eta(t,x,y) > 0$, for all (t,x,y), we obtain by little calculation

$$V_{\widetilde{f}}(\tau,\xi,a_v,p_v,X_v) + \frac{\varphi_l'(\eta(\tau,\xi,v(\tau,\xi)))}{D_y\eta(\tau,\xi,v(\tau,\xi))} \leq 0.$$

The converse part of (1) can be proved similarly. In the same manner one can show the second assertion (2). □

5 Probabilistic representation result for stochastic viscosity solution to MSPDIEs

In this section, we aim to show that the solution of multivalued BDSDE with jump gives the viscosity solution of a semi-linear MSPDIE in the Markovian case.

5.1 A class of reflected diffusion process

We now introduce a class of diffusion processes. Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ be a uniformly bounded function satisfying the uniform Lipschitz condition with some constant $C > 0$, for all $x, y \in \mathbb{R}^d$:

$$|\sigma(x) - \sigma(y)| \leq C|x - y|. \tag{5.1}$$

For each $(t,x) \in [0,T] \times \mathbb{R}^d$, from [17] and reference therein, let $\{X_s^{t,x}, s \in [t,T]\}$ be a unique pair of progressively measurable process, which is a solution to the SDE (1.4) in the Markovian framework:

$$X_s^{t,x} = x + \int_t^s \sigma(X_{r^-}^{t,x})\,\mathrm{d}L_r, \tag{5.2}$$

where, as above, $(L_t)_{0\leq t\leq T}$ is a given Lévy process. Furthermore, we have the following proposition.

Proposition 5.1. *There exists a constant* $C > 0$ *such that for all* $0 \leq t < t' \leq T$ *and* $x, x' \in \mathbb{R}^d$, *such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|X_s^{t,x} - X_s^{t',x'}\right|^4\right] \leq C\left(|t'-t|^2 + |x - x'|^4\right)$$

5.2 Existence of viscosity solution for MSPDIEs

Fix $T > 0$ and for all $(t,x) \in [0,T] \times \mathbb{R}^d$, let $X_s^{t,x}$, $s \in [t,T]$ denote the solution of the SDE (5.2). And we suppose now that the data (ξ, f, g) of the multi-valued BDSDE possibly with jumps take the form

$$\begin{aligned} \xi &= u_0(X_T^{t,x}), \\ f(s,y,z) &= f(s, X_s^{t,x}, y, z), \\ g(s,y) &= f(s, X_s^{t,x}, y). \end{aligned}$$

And we make the following assumptions:
We assume that $u_0 \in C(\mathbb{R}^d;\mathbb{R})$, $f \in C([0,T]\times\mathbb{R}^d\times\mathbb{R}\times\ell^2;\mathbb{R})$ and $g \in C([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R})$ such that the hypotheses (H1)–(H7) hold. It follows from the results of the Section 3 that, for all $(t,x) \in [0,T]\times\mathbb{R}^d$, there exists a unique triplet $(Y^{t,x}, Z^{t,x}, U^{t,x})$ for the solution of the following

(1) $(Y_s^{t,x}, U_s^{t,x}) \in \partial\varphi$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}s$-a.e. on $[t,T]$

(2) $$Y_s^{t,x} + \int_s^T U_r^{t,x}\,\mathrm{d}r = u_0(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,\mathrm{d}r + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x})\,\mathrm{d}B_r - \sum_{i=1}^{\infty}\int_s^T (Z^{t,x})_r^{(i)}\,\mathrm{d}H_r^{(i)},\ t \leq s \leq T.$$

We extend processes $Y^{t,x}$, $Z^{t,x}$, $U^{t,x}$ on $[0,T]$ by putting $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = 0$, $U_s^{t,x} = 0$, $s \in [0,t]$.

We have this result whose proof is similar to that of Theorem 2.1 which appeared in [26]

Proposition 5.2. *Let the ordered triplet* $(Y_s^{t,x}, U_s^{t,x}, Z_s^{t,x})$ *be the unique solution of the multivalued BDSDE* (5.2)*. Then, for* $(s,t,x) \in [0,T]\times[0,T]\times\mathbb{R}^d$*, the random field* $(s,t,x) \mapsto \mathbb{E}'(Y_s^{t,x})$ *is a.s. continuous (*$Y^{t,x}$ *has jumps), where* $\mathbb{E}'$ *is the expectation with respect to* $\mathbb{P}'$*, introduced at page 3.*

We are ready now to derive our main result in this section.

Theorem 5.3. *Suppose that the assumptions* (H1)–(H7) *are satisfied. Then, the function* $u(t,x)$ *defined by* $u(t,x) = Y_t^{t,x}$ *is a stochastic viscosity solution of MSPDIE* (1.2)

Proof. Since the variable $u(t,x) = Y_t^{t,x} = \mathbb{E}'(Y_t^{t,x})$, does not depend on ω', it follows from Proposition 5.2 that $u \in C(\mathcal{F}^B, [0,T]\times\mathbb{R}^d)$. Next, for all $(\tau,\xi) \in \mathcal{M}^B(0,T)\times L^2(\mathcal{F}^B,\mathbb{R}^d)$,

$$\varphi(u(\tau(\omega),\xi(\omega))) = \varphi\left(Y_{\tau(\omega)}^{\tau(\omega),\xi(\omega)}\right) < \infty,\ \mathbb{P}\text{-a.s.},$$

which implies that $u(\tau,\xi) \in \mathrm{Dom}(\varphi)$ $\mathbb{P}$-a.s.Thus it remains to show that u is the stochastic viscosity solution to MSPDIE (1.2). In other word, using Corollary 4.8, it suffices to prove that $v(t,x) = \varepsilon(t,x,u(t,x))$ satisfies (4.5) and (4.6). For this reason and, for each $(t,x) \in [0,T]\times\mathbb{R}^d$, $\delta > 0$, let $\{(Y_s^{t,x,\delta}, Z_s^{t,x,\delta}),\ 0 \le s \le T\}$ denote the solution of the following BDSDE:

$$\begin{aligned} Y_s^{t,x,\delta} + \frac{1}{\delta}\int_s^T D\varphi_\delta(Y_r^{t,x,\delta})\,\mathrm{d}r &= u_0(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,\delta}, Z_r^{t,x,\delta})\,\mathrm{d}r \\ &\quad + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x,\delta})\,\mathrm{d}B_r \\ &\quad - \sum_{i=1}^{\infty}\int_s^T (Z^{t,x,\delta})_r^{(i)}\,\mathrm{d}H_r^{(i)}. \end{aligned} \tag{5.3}$$

Setting $Y_t^{t,x,\delta} = u^\delta(t,x)$, it is shown by Theorem 3.6 in [2], that the function $v^\delta(t,x) = \varepsilon(t,x,u^\delta(t,x))$ is an ω-wise viscosity solution to this MSPDIE:

$$\begin{cases} (i)\ \left(\dfrac{\partial v^\delta}{\partial t}(t,x) - \left[\mathcal{L}v^\delta(t,x) + \widetilde{f_\delta}(t,x,v^\delta(t,x),\sigma^*(x)\nabla v^\delta(t,x))\right]\right) = 0, & (t,x) \in [0,T]\times\mathbb{R}^d, \\ (ii)\ v(T,x) = u_0(x), \quad x \in \mathbb{R}^d, & \end{cases} \tag{5.4}$$

where

$$\widetilde{f_\delta}(t,x,y,z) = \widetilde{f}(t,x,y,z) - \frac{\frac{1}{\delta}D\varphi_\delta(\eta(t,x,y))}{D_y\eta(t,x,y)}.$$

Moreover, an appeal to Lemma 3.5 shows that, along a subsequence,

$$|v^\delta(t,x) - v(t,x)| \to 0,\ \text{a.s., as } \delta \to 0, \tag{5.5}$$

for all $(t,x) \in [0,T]\times\mathbb{R}^d$.

On the other hand, for all $(\tau,\xi) \in \mathcal{M}^B(0,T)\times L^2(\mathcal{F}^B,\mathbb{R}^d)$ and $\omega \in \Omega$ be fixed, let us consider $(a_v, p_v, X_v) \in \mathcal{J}_0^{1,2,+}(v(\tau(\omega),\xi(\omega)))$. Thus, since v^δ is an ω-wise viscosity solution to the MSPDIE (5.4), and by Lemma 6.1 in Crandall- Ishii-Lions [12], there exist sequences

$$\begin{cases} \delta_n(\omega) \searrow 0, \\ (\tau_n(\omega),\xi_n(\omega)) \in [0,T]\times\mathbb{R}^d, \\ (a_v^n, p_v^n, X_v^n) \in \mathcal{J}_0^{1,2,+}(v^{\delta_n}(\tau_n(\omega),\xi_n(\omega))) \end{cases}$$

satisfying

$$\begin{aligned} (\tau_n(\omega),\xi_n(\omega),a_v^n,p_v^n,X_v^n,v^{\delta_n}(\tau_n(\omega),\xi_n(\omega))) &\to (\tau(\omega),\xi(\omega),a_v,p_v,X_v,v(\tau(\omega),\xi(\omega))) \\ n &\to \infty, \end{aligned}$$

such that for $(\tau_n(\omega),\xi_n(\omega)) \in [0,T]\times\mathbb{R}^d$,

$$V_{\widetilde{f(\omega)}}(\tau_n(\omega),\xi_n(\omega),a_v^n,X_v^n,p_v^n) + \frac{\frac{1}{\delta_n} D\varphi_{\delta_n}(\eta(\tau_n(\omega),\xi_n(\omega),v^{\delta_n}(\tau_n(\omega),\xi_n(\omega))))}{D_y\eta(\tau_n(\omega),\xi_n(\omega),v^{\delta_n}(\tau_n(\omega),\xi_n(\omega)))} \leq 0. \tag{5.6}$$

In order to simplify the notation, we remove the dependence of ω. Let $y \in \mathrm{Dom}(\varphi)$ be such that $y < u(\tau,\xi) = \eta(\tau,\xi,v(\tau,\xi))$. The uniform convergence on compact subsets in probability (ucp) of v^{δ_n} to v implies that there exists $n_0 > 0$ such that $\forall\, n \geq n_0,\;\; y < \eta(\tau_n,\xi_n,v^{\delta_n}(\tau_n,\xi_n))$. Therefore, inequality (5.6) yields

$$\begin{aligned} &\Big(\eta(\tau_n,\xi_n,v^{\delta_n}(\tau_n,\xi_n)) - y\Big) V_{\widetilde{f_{\delta_n}}}(\tau_n,\xi_n,a_v^n,X_v^n,p_v^n) \\ \leq\;& \Big[\varphi(y) - \varphi(J_{\delta_n}(\eta(\tau,\xi,v^{\delta_n}(\tau,\xi))))\Big] \frac{1}{D_y\eta(\tau_n,\xi_n,v^{\delta_n}(\tau_n,\xi_n))}. \end{aligned}$$

Taking the limit in this last inequality, we get for all $y < \eta(\tau,\xi,v(\tau,\xi))$

$$V_{\widetilde{f}}(\tau,\xi,a_v,X_v,p_v) \leq -\frac{\varphi(\eta(\tau,\xi,v(\tau,\xi))) - \varphi(y)}{\eta(\tau,\xi,v(\tau,\xi)) - y} \frac{1}{D_y\eta(\tau,\xi,v(\tau,\xi))},$$

which implies that

$$V_{\widetilde{f}}(\tau,\xi,a_v,X_v,p_v) + \frac{\varphi_l'(\eta(\tau,\xi,v(\tau,\xi)))}{D_y\eta(\tau,\xi,v(\tau,\xi))} \leq 0,$$

and we derive that v satisfies (4.5). Hence, according to Corollary 4.8, u is a stochastic viscosity subsolution of MSPDIE (1.2). By similar arguments, one can prove that u is a stochastic viscosity supersolution of MSPDIE (1.2). This completes the proof. □

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