African Diaspora Journal of Mathematics
Volume 11, Number 2, pp. 98–131 (2011)

ISSN 1539-854X
www.math-res-pub.org/adjm

THE POLAR DECOMPOSITION IN BANACH SPACES

TEPPER L. GILL*
Department of Mathematics and E&CE, Howard University,
Washington, DC 20059, USA

VERNISE STEADMAN†
Department of Mathematics, University of the District of Columbia,
Washington, DC 20059, USA

WOODFORD W. ZACHARY‡
Department of Mathematics and E&CE, Howard University,
Washington, DC 20059, USA

Abstract

In this paper we survey research progress related to the existence of an adjoint for linear operators on Banach spaces. We introduce a new pair separable Banach spaces which are required for the general theory. We then discuss a number ways one can explicitly construct an adjoint and then prove that one always exists for bounded linear operators. However, this is not true for the class of closed densely defined linear operators. In this case, we can only show that one exists for operators of Baire class one. The existence of an adjoint allows us to construct the polar decomposition. As applications, we extend the Poincaré inequality and the Stone-von Neumann version of the spectral theorem to all operators of Baire class one on a separable Banach space. Our results even show that the spectral theorem is natural for Hilbert spaces (in a certain well-defined sense). As a final application, we provide the natural Banach space version of the Schatten class of compact operators.

AMS Subject Classification: 46B03; 47D03; 47H06; 47F05.

Keywords: Poincaré inequality spectral theorem, semigroups, vector measures, vector-valued functions, Schatten-class.

Introduction

The motivation for this work is the desire to extend advances in the Feynman operator calculus (and path integrals) to Banach spaces (see [GZ] and [GZ4]). The direct approach means that more of the known operator theory on Hilbert spaces is required for Banach spaces. This paper represents our progress to date.

*E-mail address: tgill@access4less.net
†E-mail address: vsteadma@udc.edu
‡deceased

Summary

The first section is devoted to the construction of two separable Hilbert spaces $KS^2[\mathbb{R}^n]$ and $GS^2[\mathbb{R}^n]$. These spaces have the special property that, for each classical Banach space $\mathcal{B}$, we have:

$$GS^2[\mathbb{R}^n] \hookrightarrow \mathcal{B} \hookrightarrow KS^2[\mathbb{R}^n] \ \text{(as continuous dense embeddings)}. \tag{0.1}$$

Remark 0.1. *This statement may be disconcerting on first blush, since, for example, $L^\infty[\mathbb{R}^n]$ is not separable. Actually, our results show explicitly that the separable or (non-separable) property of a space depends, not on the size of the space, but the nature of the topology imposed.*

We give examples to show that other (useful) spaces of this type may be constructed with similar or dissimilar properties.

In the second section, we construct the adjoint for operators of Baire class one (i.e., strong limits of continuous linear operators). In the third section, we extend the standard Poincaré inequality to Banach spaces. In the fourth section we extend the Stone-von Neumann spectral theorem and show that, with a slight generalization of the notion of a spectral measure, the theorem holds for all closed densely defined linear operators on a Hilbert space. In the fourth section we construct the natural version of the Schatten classes for Banach spaces.

1 Preliminaries

In this section, we want to construct two new Hilbert spaces $KS^2[\mathbb{R}^n]$ and $GS^2[\mathbb{R}^n]$ which satisfy equation (0.1) for the following classical Banach spaces:

1. the bounded continuous functions on $\mathbb{R}^n$, $\mathbb{C}_0[\mathbb{R}^n]$, which vanish at infinity;
2. the bounded uniformly continuous functions on $\mathbb{R}^n$, $\mathbb{C}_u[\mathbb{R}^n]$;
3. the bounded continuous functions on $\mathbb{R}^n$, $\mathbb{C}_b[\mathbb{R}^n]$; and,
4. the Lebesgue spaces $\mathbf{L}^p[\mathbb{R}^n]$, for $1 \le p \le \infty$.

1.1 The Hilbert Space KS^2

In order to construct our first Hilbert space, recall that Alexiewicz [AL] has shown that the class $D(\mathbb{R})$, of Denjoy integrable functions (restricted and wide sense), can be normed in the following manner: for $f \in D(\mathbb{R})$, define $\|f\|_D$ by

$$\|f\|_D = \sup_s \left| \int_{-\infty}^{s} f(r)dr \right|. \tag{1.1}$$

It is clear that this is a norm, and it is known that $D(\mathbb{R})$ is not complete. The restricted Denjoy integral is equivalent to the Henstock-Kurzweil integral (see [HS] and [KW]).

Replacing $\mathbb{R}$ by $\mathbb{R}^n$ in (1.1), for $f \in D(\mathbb{R}^n)$, we also obtain a norm on $D(\mathbb{R}^n)$:

$$\|f\|_D = \sup_{r>0}\left|\int_{\mathbf{B}_r} f(\mathbf{x})d\mathbf{x}\right| = \sup_{r>0}\left|\int_{\mathbf{R}^n} \mathcal{E}_{\mathbf{B}_r}(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right| < \infty, \tag{1.2}$$

where $\mathbf{B}_r$ is any closed cube of diagonal r centered at the origin in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and $\mathcal{E}_{\mathbf{B}_r}(\mathbf{x})$ is the indicator function of $\mathbf{B}_r$.

To construct our space on $\mathbb{R}^n$, let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed cube centered at $\mathbf{x}_i$, with sides parallel to the coordinate axes and diagonal $r_l = 2^{-l}, l \in \mathbb{N}$. Now choose the natural order which maps $\mathbb{N}\times\mathbb{N}$ bijectively to $\mathbb{N}$:

$$\{(1,1),\ (2,1),\ (1,2),\ (1,3),\ (2,2),\ (3,1),\ (3,2),\ (2,3),\ \cdots\}.$$

Let $\{\mathbf{B}_k,\ k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_l(\mathbf{x}_i)\ |(l,i) \in \mathbb{N}\times\mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$, and let $\mathcal{E}_k(\mathbf{x})$ be the indicator function of $\mathbf{B}_k$, so that $\mathcal{E}_k(\mathbf{x})$ is in $\mathbf{L}^p[\mathbb{R}^n] \cap \mathbf{L}^\infty[\mathbb{R}^n]$ for $1 \le p < \infty$. Define $F_k(\,\cdot\,)$ on $\mathbf{L}^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}. \tag{1.3}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $\mathbf{L}^p[\mathbb{R}^n]$ for each k, $\|F_k\|_\infty \le 1$ and, if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $\mathbf{L}^p[\mathbb{R}^n]$ for $1 \le p \le \infty$.

Let $t_k = 2^{-k}$ so that $\sum_{k=1}^\infty t_k = 1$ and define a measure $d\mathbf{P}(\mathbf{x},\mathbf{y})$ on $\mathbb{R}^n \times \mathbb{R}^n$ by:

$$d\mathbf{P}(\mathbf{x},\mathbf{y}) = \left[\textstyle\sum_{k=1}^\infty t_k \mathcal{E}_k(\mathbf{x})\mathcal{E}_k(\mathbf{y})\right] d\mathbf{x}d\mathbf{y}.$$

We now define an inner product $(\,\cdot\,)_2$ on $L^1[\mathbb{R}^n]$ by

$$\begin{aligned}(f,g)_2 &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\mathbf{x})g(\mathbf{y})^* d\mathbf{P}(\mathbf{x},\mathbf{y}) \\ &= \textstyle\sum_{k=1}^\infty t_k \left[\displaystyle\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right]\left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y})g(\mathbf{y})d\mathbf{y}\right]^*.\end{aligned} \tag{1.4}$$

We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space, $KS^2[\mathbb{R}^n]$. This space was first constructed in [ST]. Here, one was interested in showing that $L^1[\mathbb{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the Denjoy-integrable functions. To see that this is the case, let $f \in D[\mathbb{R}^n]$, then:

$$\|f\|_{KS^2}^2 = \textstyle\sum_{k=1}^\infty t_k \left|\displaystyle\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \sup_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \|f\|_D^2,$$

so $f \in KS^2[\mathbb{R}^n]$. (This space is related to one constructed by Kuelbs [KB] for other purposes.) It is shown in [GZ2] that this space also provides a complete mathematical foundation for the path integral and allows us to construct it in the manner originally suggested by Feynman (see [FH]).

It is clear that $\mathbb{C}_0[\mathbb{R}^n]$, $\mathbb{C}_b[\mathbb{R}^n]$ and $\mathbb{C}_u[\mathbb{R}^n]$ are contained as continuous dense embeddings in $KS^2[\mathbb{R}^n]$.

Theorem 1.1. *The space $KS^2[\mathbb{R}^n]$ contains $L^p[\mathbb{R}^n]$ (for each p, $1 \leqslant p \leqslant \infty$) as continuous, compact, dense embeddings.*

Proof. The proof of the first part is easy, if we notice that $L^1[\mathbb{R}^n] \cap L^p[\mathbb{R}^n]$ is dense for $1 \leq p < \infty$. If $f \in L^\infty[\mathbb{R}^n]$, then $\left|\int_{B_k} f(\mathbf{x})d\mathbf{x}\right|^2 \leqslant \|f\|_{L^\infty}^2$ for all k, so that $\|f\|_{KS^2} \leqslant \|f\|_{L^\infty}$. The proof of compactness follows from the fact that, if $\{f_n\}$ is any weakly convergent sequence in $L^p[\mathbb{R}^n]$ with limit f, then $\mathcal{E}_k(\mathbf{x}) \in L^q[\mathbb{R}^n]$, $1 < q \leq \infty$, so that

$$\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x})\,[f_n(\mathbf{x}) - f(\mathbf{x})]\,d\mathbf{x} \to 0$$

for each k. Thus, $\{f_n\}$ converges strongly to f in $KS^2[\mathbb{R}^n]$. Finally, note that $d\mu_k = \mathcal{E}_k(\mathbf{x})d\mathbf{x}$ defines a measure in $\mathfrak{M}[\mathbb{R}^n]$, the dual space of $L^\infty[\mathbb{R}^n]$, and that $KS^2[\mathbb{R}^n] \supset L^1[\mathbb{R}^n]^{**} = \mathfrak{M}[\mathbb{R}^n]$. □

The fact that $L^\infty[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$, while $KS^2[\mathbb{R}^n]$ is separable, makes it clear in a very forceful manner that separability is not an inherited property.

It is of particular interest that $KS^2[\mathbb{R}^n] \supset \mathfrak{M}[\mathbb{R}^n]$, the space of bounded finitely additive set functions defined on the Borel sets $\mathfrak{B}[\mathbb{R}]^n$. (Recall that $\mathfrak{M}[\mathbb{R}^n]$ also contains the Dirac delta measure.)

Remark 1.2. *There is quite a lot of flexibility in the choice of the family of positive numbers $\{t_k\}$, $\sum_{k=1}^{\infty} t_k = 1$. This is somewhat akin to the standard metric used for $\mathbb{R}^\infty$. Recall that, for any two points X, $Y \in \mathbb{R}^\infty$, $d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{|X-Y|}{1+|X-Y|}$. The family of numbers $\{\frac{1}{2^n}\}$ can be replaced by any other sequence of positive numbers whose sum is one, without affecting the topology. In [GZ] and [GZ1], we have used physical analysis to choose the family $\{t_k\}$ so they are interpreted as probabilities for the occurrence of a particular discrete path.*

There is also some flexibility with the order for $\mathbb{N} \times \mathbb{N}$. The important fact is that the properties of $KS^2[\mathbb{R}^n]$ are invariant for any choices.

Remark 1.3. *We could replace the family $\{\mathcal{E}_k,\ k \in \mathbb{N}\}$ by the Hermite functions for R^n, obtaining similar results. We favor the present approach because the construction methodology is simple, does not depend on a basis and is independent of the particular Banach space. For examples, of other constructions, see Section 1.3.*

1.2 The Hilbert Space GS^2

Let $\mathcal{B}$ be a dense continuous embedding in a separable Hilbert space $\mathcal{H}$, so there is an $M > 0$ such that $\|x\|_{\mathcal{H}} \leqslant M\|x\|_{\mathcal{B}}$, for all $x \in \mathcal{B}$. In what follows, we assume that $M = 1$. In order to construct our second Hilbert space, we need the following result by Lax [L].

Theorem 1.4 (Lax)**.** *Let $A \in L[\mathcal{B}]$. If A is selfadjoint on $\mathcal{H}$ (i.e., $(Ax,y)_{\mathcal{H}} = (x,Ay)_{\mathcal{H}}, \forall x,y \in \mathcal{B}$), then A is bounded on $\mathcal{H}$ and $\|A\|_{\mathcal{H}} \leqslant k\,\|A\|_{\mathcal{B}}$ for some positive constant k.*

Proof. Let $x \in \mathcal{B}$ and, without loss, we can assume that $k = 1$ and $\|x\|_{\mathcal{H}} = 1$. Since A is selfadjoint,

$$\|Ax\|_{\mathcal{H}}^2 = (Ax,Ax) = \left(x,A^2x\right) \leqslant \|x\|_{\mathcal{H}}\left\|A^2x\right\|_{\mathcal{H}} = \left\|A^2x\right\|_{\mathcal{H}}.$$

Thus, we have $\|Ax\|_{\mathcal{H}}^4 \leqslant \left\|A^4x\right\|_{\mathcal{H}}$, so it is easy to see that $\|Ax\|_{\mathcal{H}}^{2n} \leqslant \left\|A^{2n}x\right\|_{\mathcal{H}}$ for all n. It follows that:

$$\begin{aligned}\|Ax\|_{\mathcal{H}} &\leqslant (\left\|A^{2n}x\right\|_{\mathcal{H}})^{1/2n} \leqslant (\left\|A^{2n}x\right\|_{\mathcal{B}})^{1/2n}\\ &\leqslant (\left\|A^{2n}\right\|_{\mathcal{B}})^{1/2n}(\|x\|_{\mathcal{B}})^{1/2n} \leqslant \|A\|_{\mathcal{B}}\,(\|x\|_{\mathcal{B}})^{1/2n}.\end{aligned}$$

Letting $n \to \infty$, we get that $\|Ax\|_{\mathcal{H}} \leqslant \|A\|_{\mathcal{B}}$ for x in a dense set of the unit ball of $\mathcal{H}$. We are done, since the norm is attained on a dense set of the unit ball. □

For our second Hilbert space, fix $\mathcal{B}$ and define $GS^2_{\mathcal{B}}$ by:

$$GS^2_{\mathcal{B}} = \left\{ u \in \mathcal{B} \,\middle|\, \sum\nolimits_{n=1}^{\infty} t_n^{-1} \left|(u, \mathcal{E}_n)_2\right|^2 < \infty \right\}, \quad \text{with}$$
$$(u,v)_1 = \sum\nolimits_{n=1}^{\infty} t_n^{-1} \,(u, \mathcal{E}_n)_2\,(\mathcal{E}_n, v)_2\,.$$

For convenience, let $\mathcal{H}_1 = GS^2_{\mathcal{B}}$ and $KS^2 = \mathcal{H}_2$. for $u \in \mathcal{B}$, let $T_{12}u$ be defined by $T_{12}u = \sum_{n=1}^{\infty} t_n \,(u, \mathcal{E}_n)_2\, \mathcal{E}_n$.

Theorem 1.5. *The operator* $\mathbf{T}_{12}$ *is a positive trace class operator on* $\mathcal{B}$ *with a bounded extension to* $\mathcal{H}_2$*. In addition,* $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ *(as continuous dense embeddings),* $\left(T_{12}^{1/2}u,\ T_{12}^{1/2}v\right)_1 = (u,\ v)_2$ *and* $\left(T_{12}^{-1/2}u,\ T_{12}^{-1/2}v\right)_2 = (u,\ v)_1$.

Proof. First, since terms of the form $\{u_N = \sum_{k=1}^{N} t_n^{-1}\,(u, \mathcal{E}_k)_2\,\mathcal{E}_k :\ u \in \mathcal{B}\}$ are dense in $\mathcal{B}$, we see that $\mathcal{H}_1$ is dense in $\mathcal{B}$. It follows that $\mathcal{H}_1$ is also dense in $\mathcal{H}_2$.

For the operator T_{12}, we see that $\mathcal{B} \subset \mathcal{H}_2 \Rightarrow (u, \mathcal{E}_n)_2$ is defined for all $u \in \mathcal{B}$, so that $\mathbf{T}_{12}$ maps $\mathcal{B} \to \mathcal{B}$ and:

$$\|T_{12}u\|^2_{\mathcal{B}} \le \left[\sum\nolimits_{n=1}^{\infty} t_n^2 \,\|\mathcal{E}_n\|^2_{\mathcal{B}}\right] \left[\sum\nolimits_{n=1}^{\infty} \left|(u, \mathcal{E}_n)_2\right|^2\right] = M\,\|u\|_2^2 \le M\,\|u\|^2_{\mathcal{B}}\,.$$

Thus, T_{12} is a bounded operator on $\mathcal{B}$. It is clearly trace class and, since $(T_{12}u,\ u)_2 = \sum_{n=1}^{\infty} t_n \left|(u, \mathcal{E}_n)_2\right|^2 > 0$, it is positive. From here, it's easy to see that T_{12} is selfadjoint on $\mathcal{H}_2$ so, by Theorem 1.4, it has a bounded extension to $\mathcal{H}_2$.

An easy calculation now shows that $\left(T_{12}^{1/2}u,\ T_{12}^{1/2}v\right)_1 = (u,\ v)_2$ and $\left(T_{12}^{-1/2}u,\ T_{12}^{-1/2}v\right)_2 = (u,\ v)_1$. □

We call $GS^2_{\mathcal{B}}$ the Gross-Steadman space for $\mathcal{B}$. Historically, Gross [G] first proved that every real separable Banach space contains a separable Hilbert space as a dense embedding, and that this space is the support of a Gaussian measure.

1.3 The Uniqueness Problem

The purpose of this section is to take a look at other spaces with many of the same and/or different properties compared to KS^2 and GS^2.

In this first example we show that the Banach space $\mathbb{C}[0,1]$ has (at least) two pair of Hilbert spaces satisfying $\mathcal{H}_1 \subset \mathbb{C}[0,1] \subset \mathcal{H}_2$, as dense continuous embeddings.

Example 1.6. *The first pair is $H_0^1[0,1] \subset \mathbb{C}[0,1] \subset L^2[0,1]$, where $\mathbb{C}[0,1]$ is the set of continuous functions with the sup norm and $H_0^1[0,1]$ is the completion in the L^2 norm of the functions $u(x) \in \mathbb{C}[0,1]$, with $u'(x)$ continuous and $u(0) = u(1) = 0$. In this case, the norms for our respective Hilbert spaces are generated by the following inner products:*

$$(u,v)_2 = \int_0^1 u(x)v(x)dx \text{ and } (u,v)_1 = \int_0^1 u'(x)v'(x)dx.$$

We can take $\mathbf{J}_2 = \mathbf{I}_2$. However, from

$$\langle u, \mathbf{J}_1 v\rangle = \int_0^1 u'(x)v'(x)dx,$$

we must take $\mathbf{J}_1 = \left[-\frac{d^2}{dx^2}\right]$, with Dirichlet boundary conditions (see Barbu [B], pg., 4). It follows that the natural operator relating the spaces must be $\mathbf{T}_{12} = \left[-\frac{d^2}{dx^2}\right]^{-1}$.

With additional effort, one can show that $\mathbf{T}_{12}$ is a bounded compact operator on $\mathbb{C}[0,1]$, with a bounded extension to $L^2[0,1]$. Furthermore, one has $(u,v)_1 = (\mathbf{T}_{12}^{-1/2}u, \mathbf{T}_{12}^{-1/2}v)_2$ and $(u,v)_2 = (\mathbf{T}_{12}^{1/2}u, \mathbf{T}_{12}^{1/2}v)_1$.

The second pair of spaces will be used in the next section, so we provide additional detail. To construct them, let $\{e_n(x),\ n \in \mathbb{N}\}$, be the orthonormal basis for $L^2[0,1]$ generated by the polynomials $\{1, x, x^2, \dots\} \subset \mathbb{C}[0,1]$. If we set

$$F_n(u) = \int_0^1 e_n(x)u(x)dx,$$

we see that the set of vectors $\{F_n,\ n \in \mathbb{N}\}$ is fundamental on $\mathbb{C}[0,1]$ (and also $L^1[0,1]$). If $t_n = \frac{1}{2^n}$, define an inner product on $\mathbb{C}[0,1]$ by:

$$(u,v)_2 = \textstyle\sum_{n=1}^{\infty} t_n F_n(u)\bar{F}_n(v) = \sum_{n=1}^{\infty} t_n \displaystyle\int_0^1 \int_0^1 e_n(x)\bar{e}_n(y)u(x)\bar{v}(y)dxdy.$$

If $\mathcal{H}_2[0,1]$ is the completion of $\mathbb{C}[0,1]$ in the norm generated by the inner product, we obtain a Hilbert space.

Since

$$\|u\|_2 = \left[\textstyle\sum_{n=1}^{\infty} t_n \left|\displaystyle\int_0^1 e_n(x)u(x)dx\right|^2\right]^{1/2},$$

Just as with KS^2, it is easy to see that $\mathcal{H}_2$ contains all L^p spaces, $1 \le p \le \infty$ as continuous dense compact embeddings.

Now, define the operator $\mathbf{T}_{12}$ on $\mathbb{C}[0,1]$ by:

$$\mathbf{T}_{12}u = \sum_{n=1}^{\infty} t_n \,(u, e_n)_2\, e_n.$$

Since $\mathbb{C}[0,1] \subset \mathcal{H}_2$, $(u,e_n)_2$ is defined for all $u \in \mathbb{C}[0,1]$. Thus, $\mathbf{T}_{12}$ maps $\mathbb{C}[0,1] \to \mathbb{C}[0,1]$ and:

$$\|\mathbf{T}_{12}u\|_0^2 \le \left[\sum_{n=1}^{\infty} t_n^2\right] \left[\textstyle\sum_{n=1}^{\infty} |(u,e_n)_2|^2\right] = M\,\|u\|_2^2 \le M\,\|u\|_0^2.$$

Thus, $\mathbf{T}_{12}$ *is a bounded operator on* $\mathbb{C}[0,1]$. *Define* $\mathcal{H}_1$ *by:*

$$\mathcal{H}_1 = \left\{ u \in \mathbb{C}[0,1] \left| \sum_{n=1}^{\infty} \tfrac{1}{t_n} \left|(u,e_n)_2\right|^2 < \infty \right. \right\}, \ (u,v)_1 = \sum_{n=1}^{\infty} \tfrac{1}{t_n} (u,e_n)_2 (e_n,v)_2 .$$

With the above inner product, $\mathcal{H}_1$ *is a Hilbert space and, since terms of the form* $\{u_N = \sum_{k=1}^{N} \frac{1}{t_n} (u,e_k)_2 e_k : u \in \mathbb{C}[0,1]\}$ *are dense in* $\mathbb{C}[0,1]$, *we see that* $\mathcal{H}_1$ *is dense in* $\mathbb{C}[0,1]$. *It follows that* $\mathcal{H}_1$ *is also dense in* $\mathcal{H}_2$. *It is easy to see that* $\mathbf{T}_{12}$ *is a positive selfadjoint trace class operator with respect to the* $\mathcal{H}_2$ *inner product so, by Theorem 1.4 of Lax,* $\mathbf{T}_{12}$ *has a bounded extension to* $\mathcal{H}_2$ *and* $\|\mathbf{T}_{12}\|_2 \le \|\mathbf{T}_{12}\|_0$. *Finally, it is easy to see that, for* $u,v \in \mathcal{H}_1$, $(u,v)_1 = (\mathbf{T}_{12}^{-1/2}u, \mathbf{T}_{12}^{-1/2}v)_2$ *and* $(u,v)_2 = (\mathbf{T}_{12}^{1/2}u, \mathbf{T}_{12}^{1/2}v)_1$. *It follows that* $\mathcal{H}_1$ *is continuously embedded in* $\mathcal{H}_2$, *hence also in* $\mathbb{C}[0,1]$.

1.3.1 The Strong Distribution Pair

In this example, we construct a related pair of Hilbert spaces $SD^2[\mathbb{R}^n]$ and $WD^2[\mathbb{R}^n]$ that we call strong distribution spaces. It will be shown that $SD^2[\mathbb{R}^n]$ contains $W^{k,p}(\mathbb{R}^n)$, for each k,p, in addition to other interesting properties.

In order to construct $SD^2[\mathbb{R}^n]$, we return to our construction of $KS^2[\mathbb{R}^n]$ and replace $\mathcal{E}_k(\mathbf{x})$ by $\mathcal{G}_k(\mathbf{x})$, where

$$\mathcal{G}_k(\mathbf{x}) = e_0^{\mathbf{x}} \mathcal{E}_k(\mathbf{x}), \quad e_0^{\mathbf{x}} = \frac{1}{n} \sum_{j=1}^{n} \frac{e^{x_j}}{3^{[|\mathbf{x}^i|+1]}}.$$

It is easy to see that $|e_0^{\mathbf{x}}| < 1$. Following the same steps as in the construction of KS^p, $1 \le p \le \infty$, we obtain the strong distribution spaces SD^p:

$$SD^p[\mathbb{R}^n] = \left\{ u(\mathbf{x}) \left| \sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{G}_k(\mathbf{x}) u(\mathbf{x}) d\mathbf{x} \right|^p < \infty \right. \right\}.$$

The proof of the next theorem is now easy.

Theorem 1.7. *For* $1 \le p \le \infty$ *and each* q, $1 \leqslant q \leqslant \infty$, $SD^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ *as a continuous, compact, dense embedding.*

As before the spaces are reflexive for $1 < p < \infty$, so that that $SD^p[\mathbb{R}^n] \supset L^1[\mathbb{R}^n]^{**} = \mathfrak{M}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$.

If D denotes the standard partial differential operator, let

$$D^{\alpha} = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_k}.$$

Theorem 1.8. *If* $u \in SD^2[\mathbb{R}^n]$ *and* $D^{\alpha}u = v_{\alpha}$ *in the weak distributional sense, then* $v_{\alpha} \in SD^2[\mathbb{R}^n]$.

Proof. Since each $\mathcal{G}_k \in \mathbb{C}_c^{\infty}[\mathbb{R}^n]$, we have

$$\int_{\mathbb{R}^n} \mathcal{G}_k(\mathbf{x}) D^{\alpha} u(\mathbf{x}) d\mathbf{x} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} D^{\alpha} \mathcal{G}_k(\mathbf{x}) v_{\alpha}(\mathbf{x}) d\mathbf{x}.$$

An easy calculation shows that, for any i, $\partial_{x_i} \mathcal{G}_k(\mathbf{x}) = \mathcal{G}_k(\mathbf{x})$, so that

$$\int_{\mathbb{R}^n} \mathcal{G}_k(\mathbf{x}) D^\alpha u(\mathbf{x}) d\mathbf{x} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \mathcal{G}_k(\mathbf{x}) v_\alpha(\mathbf{x}) d\mathbf{x}.$$

It now follows that, for any $w \in SD^2[\mathbb{R}^n]$, $(D^\alpha u, w)_{SD^2} = (-1)^{|\alpha|}(v_\alpha, w)_{SD^2}$, so that $v_\alpha \in SD^2[\mathbb{R}^n]$. □

The next result explains our use of the term strong distribution in describing $SD^2[\mathbb{R}^n]$.

Corollary 1.9. *If u is in the domain of D^α, then for any $w \in SD^2[\mathbb{R}^n]$, $(D^\alpha u, w)_{\mathbf{SD}^2} = (-1)^{|\alpha|}(u, w)_{SD^2}$ so that, in particular, $\|D^\alpha u\|_{SD^2} = \|u\|_{SD^2}$.*

Corollary 1.10. *For any p, $1 \le p \le \infty$ and all $k \in \mathbb{N}$, $W^{k,p}[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n]$.*

A look at the definition of $SD^p[\mathbb{R}^n]$, for any p makes it clear that the last three results hold for these spaces also.

We can define $\mathbf{T}_{12}$ by

$$\mathbf{T}_{12} u(\mathbf{x}) = \textstyle\sum_{k=1}^\infty t_k (u, \mathcal{G}_k)_{SD} \mathcal{G}_k(\mathbf{x})$$

and

$$WD^2[\mathbb{R}^n] = \left\{ u \left| \textstyle\sum_{k=1}^\infty t_k^{-1} \left|(u, \mathcal{G}_k)_{SD}\right|^2 < \infty \right. \right\} \text{ with}$$
$$(u, v)_{WD} = \textstyle\sum_{k=1}^\infty t_k^{-1} (u, \mathcal{G}_k)_{SD} (\mathcal{G}_k, v)_{SD}$$

It is clear that the pairs of Hilbert spaces rigging $L^1[\mathbb{R}^n]$ are not unique.

Definition 1.11. *For the classical Banach spaces, the pair of Hilbert spaces $KS^2[\mathbb{R}^n]$ and $GS^2[\mathbb{R}^n]$ will be called the adjoint canonical pair.*

2 Adjoint Theory on Banach Spaces

In this section, $\mathcal{H}_2$ will always represent $KS^2[\mathbb{R}^n]$, $\mathcal{B}$ will be one of the classical Banach spaces and $\mathcal{H}_1$ will represent $GS^2[\mathbb{R}^n]$.

2.1 Preliminaries

Let $L[\mathcal{B}]$ denote the bounded linear operators on $\mathcal{B}$. By a duality map $J : \mathcal{B} \mapsto \mathcal{B}'$, we mean the set

$$J(u) = \left\{ f_u \in \mathcal{B}' \left| \langle u, f_u \rangle = \|u\|^2 = \|f_u\|^2 \right. \right\}, \ \forall u \in \mathcal{B}.$$

For fixed u define a seminorm $p_u(\,\cdot\,)$ on $\mathcal{B}$ by $p_u(x) = \|u\|_{\mathcal{B}} \|x\|_{\mathcal{B}}$, and define $\hat{f}_u^s(\cdot)$ by:

$$\hat{f}_u^s(x) = \frac{\|u\|_{\mathcal{B}}^2}{\|u\|_2^2} (x, u)_2 .$$

On the closed subspace $M = \langle u \rangle$, $\left|\hat{f}_u^s(x)\right| = \|u\|_B \|x\|_B \leqslant p_u(x)$. By the Hahn-Banach Theorem, $\hat{f}_u^s(\cdot)$ has an extension, $f_u^s(\cdot)$, to $\mathcal{B}$ such that $|f_u^s(x)| \leqslant p_u(x) = \|u\|_B \|x\|_B$ for all $x \in \mathcal{B}$

(see Rudin [RU], Theorem 3.3, page 57). From here, we see that $\|f_u^s\|_{\mathcal{B}'} \leq \|u\|_{\mathcal{B}}$. On the other hand, we have $\|u\|_{\mathcal{B}}^2 = f_u^s(u) \leqslant \|u\|_{\mathcal{B}} \|f_u^s\|_{\mathcal{B}'}$, so that $f_u^s(\cdot)$ is a duality mapping for u. We call $f_u^s(\cdot)$ the *Steadman duality map* on $\mathcal{B}$ associated with $\mathcal{H}_2$.

Recall that a densely defined operator A is called accretive if $\operatorname{Re}\langle Au, f_u\rangle \geq 0$ for $u \in D(A)$ and any duallity map f_u. It is called m-accretive if, in addition, it is closed and $Ran(I+A) = \mathcal{B}$. (Its called m-dissipative if $-A$ is m-accretive.)

2.2 Faces of The Adjoint

Before stating the main results of this section, we begin with a few examples associated with the question of adjoints for Banach spaces that are not Hilbert spaces.

If $\mathcal{B} \subset \mathcal{H} \subset \mathcal{B}^*$, with $\mathcal{B}$ a continuous dense embedding in $\mathcal{H}$, It is reasonable to identify $\mathcal{H}$ with $\mathcal{H}^*$ use this relationship to construct A^*. The following can happen in this case:

1. It is not always possible to define an A^* on $\mathcal{B}$.

2. The restriction of an operator on $\mathcal{H}$ to $\mathcal{B}$ need not exist.

3. If A extends from $\mathcal{B}$ to a bounded linear operator on $\mathcal{H}$, so that A^* exists, then A^* need not restrict to $\mathcal{B}$.

Example 2.1. *Let $\mathcal{B} = \ell_1$, $\mathcal{H} = \ell_2$ and $\mathcal{B}^* = \ell_\infty$, so that $\mathcal{B} \subset \mathcal{H} \subset \mathcal{B}^*$. Let $A : \ell_1 \to \ell_1$ be defined by*

$$A\mathbf{x} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \cdots\cdots & \\ 0 & 0 & 0 & \cdots\cdots & \cdots & & & \\ 0 & 0 & 0 & \cdots\cdots & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^\infty \frac{1}{n} x_n \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}.$$

Since $\|A\mathbf{x}\|_1 = \left|\sum_{n=1}^\infty \frac{1}{n} x_n\right| \leqslant \sum_{n=1}^\infty |x_n| = \|x\|_1$, it is clear that A is bounded. We also have that

$$A^*\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots\cdots & & & \\ \frac{1}{2} & 0 & 0 & \cdots & \cdots\cdots & & & \\ \frac{1}{3} & 0 & 0 & \cdots & \cdots\cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots\cdots & & \\ \frac{1}{n} & 0 & 0 & \cdots & \cdots\cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \vdots \\ \frac{1}{n} \\ \vdots \end{bmatrix},$$

so that

$$\|A^*\mathbf{x}\|_2 = |x_1| \left[\sum_{n=1}^\infty \frac{1}{n^2}\right]^{1/2} \leqslant \left[\sum_{n=1}^\infty |x_n|^2\right]^{1/2} \left[\sum_{n=1}^\infty \frac{1}{n^2}\right]^{1/2} < \infty$$

and $\|A^*\mathbf{x}\|_1 = |x_1| \sum_{n=1}^\infty \frac{1}{n} = \infty$. Thus, A^* is not bounded on ℓ_1 (not even densely defined). The fact that it is bounded on ℓ_2 also shows that A has a bounded extension to ℓ_2.

It is well-known that we cannot always identify $\mathcal{H}$ with $\mathcal{H}^*$ without consequence. Our next example shows how one might be able to give an explicit definition of the adjoint without a Hilbert space. There are two possibilities a reflexive or non-reflexive Banach space. We first consider the reflexive case. (We assume the field is $\mathbb{R}$.)

Example 2.2. *Let $\mathcal{B} = L^p[0,1]$ and let $A : \mathcal{B} \to \mathcal{B}$, where A is defined by*

$$Au(x) = \int_0^1 K(x,y)u(y)dy,\ u \in L^p[0,1].$$

In this case, we assume that K is Lebesgue integrable on $[0,1]\times[0,1] = I \times I$ and that $\frac{1}{p}+\frac{1}{q} = 1$. (Weaker conditions are possible.)

If $p > 2$, then $L^p[0,1] \subset L^q[0,1]$ and for $1 < p < 2$, $L^q[0,1] \subset L^p[0,1]$.

In the first case, A^* is already defined on $L^p[0,1]$ and we need to know if $A^*u \in L^p[0,1]$, for all $u \in L^p[0,1]$.

$$\|A^*u\|_p^p = \int_0^1 |A^*u(x)|^p dx = \int_0^1 \left|\int_0^1 K(x,y)u(y)dy\right|^p dx.$$

Lemma 2.3. *If $\int_0^1 \|K(x,\cdot)\|_q^p\, dx < \infty$, there exists a constant m such that*

$$\|A^*u\|_p \leqslant m\|u\|_p \text{ for } u \in L^p[0,1].$$

Proof. As a function of y, $K(x,y) \in L^q[0,1]$, so that $K(x,y)u(y) \in L^1[0,1]$. By Hölder's inequality, we have:

$$\|A^*u\|_p^p \leqslant \int_0^1 \left|\int_0^1 K(x,y)u(y)dy\right|^p dx \leqslant \int_0^1 \left[\|K(x,\cdot)\|_q \|u\|_p\right]^p dx$$
$$= \|u\|_p^p \int_0^1 \|K(x,\cdot)\|_q^p\, dx.$$

Since the last integral is finite, we are done. □

In the second case, when $1 < p < 2$ and $L^q[0,1] \subset L^p[0,1]$, we assume that K is a continuous function on $I \times I$. It now follows that $D(A^*) = L^q[0,1]$ is dense in $L^p[0,1]$.

Lemma 2.4. *The operator A^* has a unique extension to a bounded linear operator on $L^p[0,1]$.*

Proof. First note that, since K is continuous on a compact set, it is uniformly bounded, so that

$$\sup_{I\times I} |K(x,y)| \leqslant m < \infty.$$

Now,

$$\|A^*u\|_p^p = \int_0^1 \left|\int_0^1 K(x,y)u(y)dy\right|^p dx \leqslant \int_0^1 \left[\int_0^1 |K(x,y)|\,|u(y)|\,dy\right]^p dx.$$

In this case, we also have that $u \in L^p[0,1]$ implies that $u \in L^1[0,1]$, so that $\|A^*u\|_p^p \leqslant m^p \|u\|_1^p$. □

In our next example shows that even in the non-reflexive case it is still possible to define an adjoint without using a Hilbert space.

Example 2.5. *In this example, we take $\mathcal{B} = L^1[0,1]$, let $K \in L^1[I \times I] \cap L^\infty[I \times I]$ and define $A : L^1[0,1] \to L^1[0,1]$ by*

$$Au(x) = \int_0^1 K(x,y)u(y)dy.$$

Since the adjoint $A^* : L^\infty[0,1] \to L^\infty[0,1]$, if $u \in L^1[0,1]$ and $v^* \in L^\infty[0,1]$, from Fubini's Theorem, we have:

$$\langle Au, v^* \rangle = \int_0^1 \left[\int_0^1 K(x,y)u(y)dy \right] v^*(x)dx$$
$$= \int_0^1 \left[\int_0^1 K(x,y)v^*(x)dx \right] u(y)dy = \langle u, A^*v^* \rangle .$$

Since

$$\|A^*v^*\|_\infty \leqslant es\sup_{I\times I} |K(x,y)| \, \|v^*\|_\infty < \infty,$$

we see that A^* is the standard adjoint mapping of $L^\infty[0,1] \to L^\infty[0,1]$.

Since $L^1[0,1] \cap L^\infty[0,1]$ is dense, $A^*u \in L^1[0,1]$ for all $u \in L^1[0,1] \cap L^\infty[0,1]$. The extension of A^* to a bounded linear operator on $L^1[0,1]$ follows from

$$|A^*u(x)| \leqslant es\sup_{I\times I} |K(x,y)| \int_0^1 |u(y)|\, dy.$$

Remark 2.6. *It should be noted that, in the examples no symmetry properties were imposed on the kernel, so in each case the adjoint is distinct.*

2.3 Banach Space Adjoint

The purpose of this section is to prove that, all bounded linear operators on one of the classical Banach spaces has an adjoint. The first result appeared in Gill et al [GBZS] and generalizes the well-known result of von Neumann [VN] for bounded operators on Hilbert spaces. For convenience, we provide a proof. (We delay the proof of (1) and (3) until after Theorem 2.9.)

Theorem 2.7. *If A is a bounded linear operator on $\mathcal{B}$, then A has a well-defined adjoint A^* defined on $\mathcal{B}$ such that:*

1. *the operator $A^*A \geq 0$ (m-accretive),*
2. *$(A^*A)^* = A^*A$ (selfadjoint), and*
3. *$I + A^*A$ has a bounded inverse.*

Proof. If $\mathbf{J} : \mathcal{H} \longrightarrow \mathcal{H}'$ is the standard conjugate isomorphism between a HIlbert space and its dual, so that $< u, \mathbf{J}(u) > = (u,u)_2 = \|u\|_2^2$, let $\mathbf{J}_i : \mathcal{H}_i \to \mathcal{H}_i'$, $(i = 1,2)$. Then $A_1 = A_{|\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_2$, and $A'_1 : \mathcal{H}_2' \to \mathcal{H}_1'$.

It follows that ${A'}_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}_1'$ and $\mathbf{J}_1^{-1}{A'}_1\mathbf{J}_2 : \mathcal{H}_2 \to \mathcal{H}_1 \subset \mathcal{B}$ so that, if we define $A^* = [\mathbf{J}_1^{-1}{A'}_1\mathbf{J}_2]_{\mathcal{B}}$, then $A^* : \mathcal{B} \to \mathcal{B}$ (i.e., $A^* \in L[\mathcal{B}]$).

To prove (2), we have that for $x \in \mathcal{H}_1$,

$$\begin{aligned}(A^*A)^*x &= (\{\mathbf{J}_1^{-1}[\{[\mathbf{J}_1^{-1}{A'}_1\mathbf{J}_2]|_{\mathcal{B}}A\}_1]'\mathbf{J}_2\}|_{\mathcal{B}})x \\ &= (\{\mathbf{J}_1^{-1}[\{{A'}_1[\mathbf{J}_2A_1\mathbf{J}_1^{-1}]|_{\mathcal{B}}\}]\mathbf{J}_2\}|_{\mathcal{B}})x \\ &= A^*Ax.\end{aligned}$$

It follows that the same result holds on $\mathcal{B}$. □

The operator A^*A is selfadjoint on $\mathcal{B}$. By the Theorem of Lax [L], it is natural to expect that the same is true on $\mathcal{H}_2$. However, in general, this need not be the case. To obtain a simple counterexample, recall that, in standard notation, the simplest class of bounded linear operators on $\mathcal{B}$ is $\mathcal{B} \otimes \mathcal{B}'$, in the sense that:

$$\mathcal{B} \otimes \mathcal{B}' : \mathcal{B} \to \mathcal{B}, \text{ by } Au = (b \otimes l_{b'}(\cdot))u = \langle b', u \rangle b.$$

Thus, if $l_{b'}(\cdot) \in \mathcal{B}' \backslash \mathcal{H}_2'$, then $J_2\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(u)\}$ is not in $\mathcal{H}_2'$, so that A^*A is not defined as an operator on all of $\mathcal{H}_2$ and thus cannot have a bounded extension.

The following example shows explicitly that an extension is not possible for one of the standard Banach-Hilbert space couples.

Example 2.8. *Let $\ell_1 \to \ell_2$ be the natural embedding, and let e_n be the natural unit basis. Put $T(e_1) = e_1$ and $T(e_n) = e_1 + e_n$ for $n > 1$. This operator has a natural extension to a bounded linear operator in ℓ_1. Put $x_n = n^{-1}(e_1 + \cdots + e_n)$. Then $\|x_n\|_2 \to 0$, $\|T(x_n) - e_1\|_2 \to 0$ but $T(0) \neq e_1$. Thus, T cannot be extended to a closed operator on ℓ_2. It follows that ℓ_2 is not the correct Hilbert space for the extension of bounded linear operators or for the construction of adjoints for bounded linear operators on ℓ_1.*

We now recall that $\mathcal{B}' \subset \mathcal{H}_2$ for each of our classical Banach spaces so that the above discussion and example does not apply for $\mathcal{H}_2$.

Theorem 2.9. *Let A be a bounded linear operator on $\mathcal{B}$, then A has a bounded extension to $L[\mathcal{H}_2]$, with $\|A\|_{\mathcal{H}_2} \le k\|A\|_{\mathcal{B}}$ (for some positive k).*

Proof. If $T = A^*A$, then $\langle Tx, \mathbf{J}_2(y)\rangle = (Tx,y)_{\mathcal{H}_2}$ is well defined for all x, $y \in \mathcal{B}$, and $(Tx,y)_{\mathcal{H}_2} = (x,Ty)_{\mathcal{H}_2}$. Thus, we can now apply Lax's Theorem to see that $\|T\|_{\mathcal{H}_2} = \|A\|_{\mathcal{H}_2}^2 \le k^2\|A\|_{\mathcal{B}}^2$. □

We can now finish our proof of Theorem 2.7.

To prove (1), let $x \in \mathcal{B}$, then $(A^*Ax, x)_{\mathcal{H}_2} \ge 0$ for all $x \in \mathcal{B}$. Hence $\langle A^*Ax, f_x^s\rangle \ge 0$, so that A^*A is accretive. (Since its bounded, its m-accretive.) The proof of (3), that $I + A^*A$ is invertible, follows the same lines as in von Neumann's theorem.

Remark 2.10. *Theorem 2.9 tells us that $L[\mathcal{B}] \subset L[\mathcal{H}_2]$ as a continuous embedding. (It can be shown that, if $\mathcal{B}$ has the approximation property, the embedding is dense.)*

The algebra $L[\mathcal{B}]$ also has a $*$-operation that makes it much closer to $L[\mathcal{H}_2]$ then expected. However, in general $\|A^*A\|_{\mathcal{B}} \neq \|A\|_{\mathcal{B}}^2$. Furthermore, if $A \neq B, B^*$ then, unless

$$\left(B\big|_{\mathcal{H}_1}\right)'\left(A\big|_{\mathcal{H}_1}\right)' = \left(AB\big|_{\mathcal{H}_1}\right)', \text{ we have } (AB)^* \neq B^*A^*.$$

Thus, $L[\mathcal{B}]$ is a not a $*$-algebra in the traditional sense.

3 Closed Linear Operators on $\mathcal{B}$

We now consider the case that $A \in \mathcal{C}[\mathcal{B}]$, the closed densely defined linear operators on $\mathcal{B}$. By definition, A is of Baire class one if it can be approximated by a sequence, $\{A_n\}$, of bounded linear operators. In this case, it is natural to define $A^* = s - \lim A_n^*$ (see below). In case $\mathcal{B}$ is a Hilbert space, every $A \in \mathcal{C}[\mathcal{B}]$ is of Baire class one. However, it turns out that, if $\mathcal{B}$ is not a Hilbert space, there may be operators $A \in \mathcal{C}[\mathcal{B}]$ that are not of Baire class one, so that it is not reasonable to expect Theorem 2.7 to hold for all of $\mathcal{C}[\mathcal{B}]$. In order to understand the problem, we need the following:

Definition 3.1. *A Banach space $\mathcal{B}$ is said to be:*

1. *quasi-reflexive if* $\dim\left\{\mathcal{B}''/\mathcal{B}\right\} < \infty$, *and*
2. *nonquasi-reflexive if* $\dim\left\{\mathcal{B}''/\mathcal{B}\right\} = \infty$.

An important result by Vinokurov, Petunin and Pliczko [VPP] shows that, for every nonquasi-reflexive Banach space $\mathcal{B}$ (for example, $C[0;1]$ or $L^1[\mathbb{R}^n]$, $n \in \mathbb{N}$), there is a closed densely defined linear operator A, which is not of Baire class one. It can even be arranged so that A^{-1} is a bounded linear injective operator (with a dense range). This means, in particular, that there does not exist a sequence of bounded linear operators $A_n \in L[\mathcal{B}]$ such that, for $x \in D(A)$, $A_n x \to Ax$, as $n \to \infty$ for each $A \in \mathcal{C}[\mathcal{B}]$.

Recall that an m-dissipative linear operator is the generator of a C_0-contraction semigroup and $Ran(\lambda I - A) = \mathcal{B}$ for every $\lambda > 0$ (see Pazy [PZ]). Furthermore, the Yosida approximator [YS], $A_\lambda = \lambda A R(\lambda, A)$, is a bounded linear operator which converges strongly to A on $D(A)$. The following result shows that every operator of Baire class one has an adjoint.

Theorem 3.2. *Let $A \in \mathcal{C}[\mathcal{B}]$. The operator A is in the first Baire class if and only if it has an adjoint A^*.*

Proof. Let $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ and suppose that A has an adjoint $A^* \in \mathcal{C}[\mathcal{B}]$. Let $T = [A^*A]^{1/2}$, $\bar{T} = [AA^*]^{1/2}$ (the negatives of each generate C_0-contraction semigroups). Since T is nonnegative, it follows that $I + \alpha T$ has a bounded inverse $S(\alpha) = (I + \alpha T)^{-1}$, for $\alpha > 0$. It is also easy to see that $AS(\alpha)$ is bounded and, on $D(A)$, $AS(\alpha) = \bar{S}(\alpha)A = (I + \alpha\bar{T})^{-1}A$ (see Kato [K], pages 335 and 481). Using this result, we have:

$$\lim_{\alpha \to 0^+} AS(\alpha)x = \lim_{\alpha \to 0^+} \bar{S}(\alpha)Ax = Ax, \text{ for } x \in D(A).$$

It follows that A is in the first Baire class.

To prove the converse, suppose that A is in the first Baire class. Thus, there is a sequence of bounded linear operators $\{A_n\}$ such that, for $x \in D(A)$, $A_n x \to Ax$ as $n \to \infty$. Since each A_n is bounded, by Theorem 2.1, each A_n has an adjoint A_n^* and both can be extended to bounded linear operators $\bar{A}_n$, $\bar{A}_n^*$ on $\mathcal{H}_2$ (by Theorem 2.3). Furthermore, we have $\left\|\bar{A}_n\right\|_{\mathcal{H}_2} \leq k\left\|A_n\right\|_{\mathcal{B}}$ and $\left\|\bar{A}_n^*\right\|_{\mathcal{H}_2} \leq k\left\|A_n^*\right\|_{\mathcal{B}}$. It follows that the sequence $\{\bar{A}_n x\}$ converges for each $x \in D(A)$. If we define $\bar{A}$ as the closure in $\mathcal{H}_2$ of $lim_{n\to\infty}\bar{A}_n x$ for $x \in D(A)$, then $\bar{A} \in \mathcal{C}[\mathcal{H}_2]$.

Since $\bar{A}$ is a closed densely defined linear operator, its $\mathcal{H}_2$ adjoint, $\bar{A}^*$, is densely defined and $\bar{A} = \bar{A}^{**}$ (see Rudin [RU], Theorem 13.12, page 335). From this, we see that $\bar{A}^*$ is a closed densely defined linear operator on $\mathcal{H}_2$. Since $\bar{A}$ restricted to $\mathcal{B}$ is A, $\bar{A}^*$ restricted to $\mathcal{B}$ defines A^*. □

As noted above, if $\mathcal{B}$ is a quasi-reflexive separable Banach space, there is at least one closed densely defined linear operator that is not of the first Baire class. However, to our knowledge, it has not been shown that every operator $A \in \mathcal{C}[\mathcal{B}]$ is of the first Baire class even when $\mathcal{B}$ is reflexive. From a theoretical point of view, the following theorems hold for all operators of first Baire class. However, unless an operator is the generator of a semigroup, there is no known way to find the particular sequence of bounded linear operators that may be used to approximate it. Since the operators of interest for applications are (for the most part) generators of C_0-contraction semigroups, in what follows, we restrict our consideration to C_0 generators.

Theorem 3.3. *If A generates a C_0-contraction semigroup and $\mathcal{B}$ is one of the classical Banach spaces, then:*

1. *A has a closed densely defined extension $\bar{A}$ to $\mathcal{H}_2$, which is also the generator of a C_0-contraction semigroup.*
2. $\rho(\bar{A}) = \rho(A)$ *and* $\sigma(\bar{A}) = \sigma(A)$.
3. *The adjoint of $\bar{A}$, $\bar{A}^*$, restricted to $\mathcal{B}$, is the adjoint A^* of A, that is:*
   - *the operator $A^*A \geqslant 0$,*
   - $(A^*A)^* = A^*A$ *and*
   - *$I + A^*A$ has a bounded inverse.*

Proof. **Part I**
Let $T(t)$ be the semigroup generated by A. By Theorem 2.9, as a bounded linear operator, $T(t)$ has a bounded extension $\bar{T}(t)$ to $\mathcal{H}_2$.

We prove that $\bar{T}(t)$ is a C_0-semigroup. (The fact that it is a contraction semigroup will follow later.) It is clear that $\bar{T}(t)$ has the semigroup property. To prove that it is strongly continuous, use the fact that $\mathcal{B}$ is dense in $\mathcal{H}_2$ so that, for each $u \in \mathcal{H}_2$, there is a sequence $\{u_n\}$ in $\mathcal{B}$ converging to u. We then have:

$$\begin{aligned}
&\lim_{t\to 0}\|\bar{T}(t)u - u\|_2 \leqslant \lim_{t\to 0}\{\|\bar{T}(t)u - \bar{T}(t)u_n\|_2 + \|\bar{T}(t)u_n - u_n\|_2\} + \|u_n - u\|_2 \\
&\leqslant k\,\|u - u_n\|_2 + \lim_{t\to 0}\|\bar{T}(t)u_n - u_n\|_2 + \|u_n - u\|_2 \\
&= (k+1)\,\|u - u_n\|_2 + \lim_{t\to 0}\|T(t)u_n - u_n\|_2 = (k+1)\,\|u - u_n\|_2\,,
\end{aligned}$$

where we have used the fact that $\bar{T}(t)u_n = T(t)u_n$ for $u_n \in \mathcal{B}$, and k is the constant in Theorem 2.9. It is clear that we can make the last term on the right as small as we like by choosing n large enough, so that $\bar{T}(t)$ is a C_0-semigroup.

To prove (1), note that, if $\bar{A}$ is the extension of A, and $\lambda I - \bar{A}$ has an inverse, then $\lambda I - A$ also has one, so $\rho(\bar{A}) \subset \rho(A)$ and $Ran(\lambda I - A)_{\mathcal{B}} \subset Ran(\lambda I - \bar{A})_{\mathcal{H}_2} \subset \overline{Ran(\lambda I - A)}_{\mathcal{H}_2}$ for any $\lambda \in \mathbb{C}$. For the other direction, note that, since A generates a C_0-contraction semigroup, $\rho(A) \neq \emptyset$. Thus, if $\lambda \in \rho(A)$, then $(\lambda I - A)^{-1}$ is a continuous mapping from $Ran(\lambda I - A)$

onto $D(A)$ and $Ran(\lambda I - A)$ is dense in $\mathcal{B}$. Let $u \in D(\bar{A})$, so that $(u, \bar{A}u) \in \hat{G}(A)$, the closure of the graph of A in $\mathcal{H}_2$. Thus, there exists a sequence $\{u_n\} \subset D(A)$ such that $\|u - u_n\|_G = \|u - u_n\|_{\mathcal{H}_2} + \|\bar{A}u - \bar{A}u_n\|_{\mathcal{H}_2} \to 0$ as $n \to \infty$. Since $\bar{A}u_n = Au_n$, it follows that $(\lambda I - \bar{A})u = \lim_{n\to\infty}(\lambda I - A)u_n$. However, by the boundedness of $(\lambda I - A)^{-1}$ on $Ran(\lambda I - A)$, we have that, for some $\delta > 0$,

$$\left\|(\lambda I - \bar{A})u\right\|_{\mathcal{H}_2} = \lim_{n\to\infty} \|(\lambda I - A)u_n\|_{\mathcal{H}_2} \geq \lim_{n\to\infty} \delta \|u_n\|_{\mathcal{H}_2} = \delta \|u\|_{\mathcal{H}_2}.$$

It follows that $\lambda I - \bar{A}$ has a bounded inverse and, since $D(A) \subset D(\bar{A})$ implies that $Ran(\lambda I - A) \subset Ran(\lambda I - \bar{A})$, we see that $Ran(\lambda I - \bar{A})$ is dense in $\mathcal{H}_2$ so that $\lambda \in \rho(\bar{A})$ and hence $\rho(A) \subset \rho(\bar{A})$. It follows that $\rho(A) = \rho(\bar{A})$ and necessarily, $\sigma(A) = \sigma(\bar{A})$.

Since A generates a C_0-contraction semigroup, it is m-dissipative. From the Lumer-Phillips Theorem (see Pazy [PZ]), we have that $Ran(\lambda I - A) = \mathcal{B}$ for $\lambda > 0$. It follows that $\bar{A}$ is m-dissipative and $Ran(\lambda I - \bar{A}) = \mathcal{H}_2$. Thus, $\bar{T}(t)$ is a C_0-contraction semigroup.

We now observe that the same proof applies to $\bar{T}^*(t)$, so that $\bar{A}^*$ is also the generator of a C_0-contraction semigroup on $\mathcal{H}_2$.

Clearly $\bar{A}^*$ is the adjoint of $\bar{A}$ so that, from von Neumann's Theorem, $\bar{A}^*\bar{A}$ has the expected properties. By a result of Kato [K] (see page 276), $\bar{\mathbf{D}} = D(\bar{A}^*\bar{A})$ is a core for $\bar{A}$ (i.e., the set of elements $\{u, \bar{A}u\}$ is dense in the graph, $G[\bar{A}]$, of $\bar{A}$ for $u \in \bar{\mathbf{D}}$). From here, we see that the restriction A^* of $\bar{A}^*$ to $\mathcal{B}$ is the generator of a C_0-contraction semigroup and $\mathbf{D} = D(A^*A)$ is a core for A. The proof of (3) for A^*A now follows. □

Theorem 3.4. *Let $A \in \mathcal{C}[\mathcal{B}]$ be the generator of a C_0-contraction semigroup. Then there exist an m-accretive operator T and a partial isometry W such that $A = WT$ and $D(A) = D(T)$.*

Proof. The fact that $\mathcal{B}' \subset \mathcal{H}_2$ ensures that A^*A is a closed selfadjoint operator on $\mathcal{B}$ by Theorem 3.3. Furthermore, both A and A^* have closed densely defined extensions $\bar{A}$ and $\bar{A}^*$ to $\mathcal{H}_2$. Thus, the operator $\hat{T} = [\bar{A}^*\bar{A}]^{1/2}$ is a well-defined m-accretive selfadjoint linear operator on $\mathcal{H}_2, \bar{A} = \bar{W}\bar{T}$ for some partial isometry $\bar{W}$ defined on $\mathcal{H}_2$, and $D(\bar{A}) = D(\bar{T})$. Our proof is complete when we notice that the restriction of $\bar{A}$ to $\mathcal{B}$ is A and $\bar{T}^2$ restricted to $\mathcal{B}$ is A^*A, so that the restriction of $\bar{W}$ to $\mathcal{B}$ is well-defined and must be a partial isometry. The equality of the domains is obvious. □

3.0.1 The Adjoint is not Unique

In this section, we show that, for a given operator A defined on a fixed Banach space $\mathcal{B}$, two different Hilbert space riggings can produce two different adjoints for A.

Let us return to the two pair of Hibert spaces $H_0^1[0,1] \subset \mathbb{C}[0,1] \subset L^2[0,1]$ and $\mathcal{H}_1[0,1] \subset \mathbb{C}[0,1] \subset \mathcal{H}_2[0,1]$ of Example 1.6.

Let $A = \left[-\frac{d^2}{dx^2}\right]$ be defined on $\mathbb{C}[0,1]$, with domain

$$D_c(A) = \left\{u'' \in \mathbb{C}[0,1] \,|u(0) = u(1) = 0\right\}.$$

It is easy to see that A extends to a selfadjoint operator on $L^2[0,1]$, with domain

$$D_2(A) = \left\{u'' \in L^2[0,1] \left|u(0) = u(1) = 0 \text{ and, } u' \text{ is absolutely continuous}\right.\right\}.$$

To begin, we first compute the adjoint A^*, of A directly as an operator on $\mathbb{C}[0,1]$. Since $\mathbb{C}^*[0,1] = BV[0,1]$, the functions of bounded variation on $[0,1]$, it follows from

$$\langle Au, v\rangle = -\int_0^1 u''(x)v(x)dx,$$

that

$$\langle u, A^*v\rangle = -\int_0^1 u(x)v''(x)dx$$

and

$$D_c(A^*) = \left\{u'' \in BV[0,1]\, | u(0) = u(1) = 0\right\}.$$

Note that $D_c(A) \subset D_c(A^*)$ (proper). Thus, if we restrict A^* to $D_c(A)$, it becomes a selfadjoint operator on $\mathbb{C}[0,1]$, without the rigging.

We now investigate the adjoint obtained from use of the first rigging, $H_0^1[0,1] \subset \mathbb{C}[0,1] \subset L^2[0,1]$ (see Barbu [B], pg. 4). In this case, $\mathbf{J}_1 = \left[-\frac{d^2}{dx^2}\right]$ and $\mathbf{J}_2 = \mathbf{I}_2$, the identity operator on $L^2[0,1]$, so that

$$A_1^* = \mathbf{J}_1^{-1}A_1'\mathbf{J}_2, = \mathbf{I}_2.$$

In the second rigging, $\mathcal{H}_1[0,1] \subset \mathbb{C}[0,1] \subset \mathcal{H}_2[0,1]$, constructed in Example 1.6, we have

$$A_2^* = \mathbf{J}_1^{-1}A_1'\mathbf{J}_2.$$

In this case,

$$\mathbf{J}_1(v) = \sum\nolimits_{n=1}^{\infty} t_n^{-1}(e_n, v)_2(\,\cdot\,, e_n)_2, \quad \mathbf{J}_2(v) = \sum\nolimits_{n=1}^{\infty} t_n\bar{F}_n(v)F_n(\,\cdot\,)$$

and

$$(e_n, v)_2 = \sum\nolimits_{k=1}^{\infty} t_k\bar{F}_k(v)F_k(e_n) = t_n\bar{F}_n(v),$$

so that $\mathbf{J}_1(v) = \sum_{n=1}^{\infty} \bar{F}_n(v)(\,\cdot\,, e_n)_2$. However,

$$(\,\cdot\,, e_n)_2 = \sum\nolimits_{k=1}^{\infty} t_k\bar{F}_k(e_n)F_k(\,\cdot\,) = t_nF_n(\,\cdot\,), \text{ so that } \mathbf{J}_1 = \mathbf{J}_2.$$

It follows that $\mathbf{J}_2(A_2^*u) = \mathbf{J}_2(Au)$, so that $A_2^* = A = \left[-\frac{d^2}{dx^2}\right]$, with the same domains.

It follows that the natural adjoint obtained on $\mathbb{C}[0,1]$ coincides with the canonical adjoint for this space. On the other hand, we also see that different riggings give distinct adjoints. (It is clear that the requirements of von Neumann's Theorem are satisfied by both adjoints.)

Definition 3.5. *The operator A^* constructed using the adjoint canonical pair is called the canonical adjoint.*

With the adjoint canonical pair fixed, we see that the canonical adjoint is unique.

Remark 3.6. *We note that the non-uniqueness result is always compatible with a selfadjoint linear operator, even on a Hilbert space. However, on a Hilbert space, it will never appear as a possible adjoint because of the restrictive definition.*

3.1 Closed Operators on $L^p[\mathbb{R}^n]$

In this section, we present a few examples of well-known linear operators on $L^p[\mathbb{R}^n]$, $1 < n < \infty$. In each case, the adjoint can be computed without actually using the canonical Hilbert space pair.

3.1.1 The Hilbert and Riesz Transforms

In one dimension, the Hilbert transform can be defined on $L^2[\mathbb{R}]$ via its Fourier transform:

$$\widehat{H(f)} = -i\,\mathrm{sgn}\,x\,\hat{f}.$$

It can also be defined directly as principal-value integral:

$$(Hf)(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|\geqslant\varepsilon}\frac{f(y)}{x-y}dy.$$

For a proof of the following results see Grafakos [GRA], chapter 4.

Theorem 3.7. *The Hilbert transform on $L^2[\mathbb{R}]$ satisfies:*

1. *H is an isometry, $\|H(f)\|_2 = \|f\|_2$ and $H^* = -H$.*
2. *For $f \in L^p[\mathbb{R}]$, $1 < p < \infty$, there exists a constant $C_p > 0$ such that,*

$$\|H(f)\|_p \leq C_p\|f\|_p. \tag{3.1}$$

The next result is technically obvious, but conceptually non-trivial.

Corollary 3.8. *The adjoint of H, H^* defines a bounded linear operator on $L^p[\mathbb{R}]$ for $1 < p < \infty$, and H^* satisfies equation (3.1) for the same constant C_p.*

The Riesz transform, **R**, is the n-dimensional analogue of the Hilbert transform and its j^{th} component is defined for $f \in L^p[\mathbb{R}^n]$, $1 < p < \infty$, by:

$$R_j(f) = c_n \lim_{\varepsilon\to 0}\int_{|\mathbf{y}-\mathbf{x}|\geqslant\varepsilon}\frac{y_j - x_j}{|\mathbf{y}-\mathbf{x}|^{n+1}}f(\mathbf{y})d\mathbf{y}, \quad c_n = \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{(n+1)/2}}.$$

Definition 3.9. *Let Ω be defined on the unit sphere S^{n-1} in R^n.*

1. *The function $\Omega(x)$ is said to be homogeneous of degree n if $\Omega(tx) = t^n\Omega(x)$.*
2. *The function $\Omega(x)$ is said to have the cancellation property if*

$$\int_{S^{n-1}}\Omega(\mathbf{y})d\sigma(\mathbf{y}) = 0, \text{ where } d\sigma \text{ is the induced Euclidean measure on } S^{n-1}.$$

3. *The function $\Omega(x)$ is said to have the Dini-type condition if*

$$\sup_{\substack{|\mathbf{x}-\mathbf{y}|\leqslant\delta \\ |\mathbf{x}|=|\mathbf{y}|=1}} |\Omega(\mathbf{x}) - \Omega(\mathbf{y})| \leqslant \omega(\delta) \Rightarrow \quad \int_0^1 \frac{\omega(\delta)d\delta}{\delta} < \infty.$$

A proof of the following theorem can be found in Stein [STE] (see pg., 39).

Theorem 3.10. *Suppose that Ω is homogeneous of degree* 0*, satisfying both the cancellation property and the Dini-type condition. If $f \in L^p[\mathbb{R}^n]$, $1 < p < \infty$ and*

$$T_\varepsilon(f)(\mathbf{x}) = \int_{|\mathbf{y}-\mathbf{x}|\geqslant\varepsilon} \frac{\Omega(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^n} f(\mathbf{y})d\mathbf{y}.$$

Then

1. *There exists a constant A_p, independent of both f and ε such that*

$$\|T_\varepsilon(f)\|_p \leqslant A_p\|f\|_p.$$

2. *Furthermore, $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ exists in the L^p norm and*

$$\|T(f)\|_p \leqslant A_p\|f\|_p. \tag{3.2}$$

Treating $T(f)$ as a special case of the Henstock-Kurzweil integral, we can write it as

$$T(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Omega(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^n} f(\mathbf{y})d\mathbf{y}.$$

For $g \in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, we have $\langle T(f), g\rangle = \langle f, T^*(g)\rangle$. Using Fubini's Theorem for the Henstock-Kurzweil integral (see [HS]), we have that

Corollary 3.11. *The adjoint of T, $T^* = -T$, is defined on L^p and satisfies equation (3.2)*

It is easy to see that the Riesz transform is a special case of the above Theorem and Corollary.

Another closely related integral operator is the Riesz potential, $I_\alpha(f)(\mathbf{x}) = (-\Delta)^{-\alpha/2}f(\mathbf{x})$, $0 < \alpha < n$, is defined on $L^p[\mathbb{R}^n]$, $1 < p < \infty$, by (see Stein [STE], pg., 117):

$$I_\alpha(f)(\mathbf{x}) = \gamma^{-1}(\alpha)\int_{\mathbb{R}^n} \frac{f(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}}, \text{ and } \gamma(\alpha) = 2^\alpha\pi^{\frac{n}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

Since the kernel is symmetric, application of Fubini's Theorem shows that the adjoint $I_\alpha^* = I_\alpha$, is also defined on $L^p[\mathbb{R}^n]$. Since $(-\Delta)^{-1}$ is not bounded, we cannot obtain L^p bounds for $I_\alpha(f)(\mathbf{x})$. However, if $1/q = 1/p - \alpha/n$, we have the following (see Stein [STE], pg., 119)

Theorem 3.12. *If $f \in L^p[\mathbb{R}^n]$ and $0 < \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$, then the integral defining $I_\alpha(f)$ converges absolutely for almost all* **x***. Furthermore, there is a constant $A_{p,q}$, such that*

$$\|I_\alpha(f)\|_q \leqslant A_{p,q}\|f\|_p. \tag{3.3}$$

Remark 3.13. *In physics, the interesting operator is not $(-\Delta)^{-\alpha/2}$, but $(-\Delta)^{1/2}$. In quantum mechanics, it appears as the absolute value operator of the physical momentum,* $\mathbf{p}$, $|\mathbf{p}| = \hbar\sqrt{-\Delta}$. (*It also appears as the photon propagator in quantum field theory* (*see Schweber [SCH]*).)

In [STE2], Stein shows that, if $\psi \in \mathbb{C}_0^\infty[\mathbb{R}^3]$, then:

$$\sqrt{-\Delta}\phi(\mathbf{x}) = \lim_{\varepsilon\to 0}\int_{|\mathbf{y}|\geqslant\varepsilon} \frac{\phi(\mathbf{x}+\mathbf{y})-\phi(\mathbf{x})}{|\mathbf{y}|^4}d\mathbf{y}. \tag{3.4}$$

If we look closely at the power of the term in the denominator of equation (3.4), we see that, at the point of singularity, it appears to (effectively) diverge like the standard Coulomb potential (see Schiff [SC], page 80). However, this is physical guessing (and could miss the mark). In [GMZ], the following representation was discovered (for $\psi \in \mathbb{C}_0^\infty[\mathbb{R}^3]$):

$$\sqrt{-\Delta}\psi(\mathbf{x}) = \frac{-\pi}{\pi^3+2}\int_{\mathbf{R}^3}\frac{\psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - 4\pi\delta(\mathbf{x}-\mathbf{y})\right]d\mathbf{y}. \tag{3.5}$$

3.1.2 Differential Operators

We begin with the following useful result (see Kato [K], pg. 168).

Theorem 3.14. *Let T be a densely defined linear operator on a reflexive Banach space $\mathcal{B}$. Then, the following holds:*

1. *The adjoint of T, T^* is a closed linear operator.*
2. *The operator T has a closed extension if and only if $D(T^*)$ is dense in $\mathcal{B}^*$. In this case, the closure $\bar{T} = T^{**}$.*
3. *If T is closable, then $(\bar{T})^* = T^*$.*

From the above result, we see that, for any closed densely defined linear operator A defined on $L^p[\mathbb{R}^n]$, $1 < p < \infty$, for which the domain of A^*, $D(A^*) \subset L^q[\mathbb{R}^n], 1/p + 1/q = 1$, is dense in $L^p[\mathbb{R}^n]$, also has a closed densely defined extension to $L^p[\mathbb{R}^n]$.

Example 3.15. *Let A be a second order differential operator on $L^p[\mathbb{R}^n]$, of the form*

$$A = \sum_{i,j=1}^n a_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^n b_{ij}(\mathbf{x})x_j\frac{\partial}{\partial x_i},$$

where $\mathbf{a}(\mathbf{x}) = [\![a_{ij}(\mathbf{x})]\!]$ and $\mathbf{b}(\mathbf{x}) = [\![b_{ij}(\mathbf{x})]\!]$ are matrix-valued functions in $\mathbb{C}_c^\infty[\mathbb{R}^n\times\mathbb{R}^n]$ (infinitely differentiable functions with compact support). We also assume that, for all $\mathbf{x}\in\mathbb{R}^n$ det $[\![a_{ij}(\mathbf{x})]\!] > \varepsilon$ and the imaginary part of the eigenvalues of $\mathbf{b}(\mathbf{x})$ are bounded above by $-\varepsilon$, for some $\varepsilon > 0$. Note, since we don't require $\mathbf{a}$ or $\mathbf{b}$ to be symmetric, $A \neq A^*$.

It is well-known that $\mathbb{C}_c^\infty[\mathbb{R}^n]$ is dense in $L^p[\mathbb{R}^n]\cap L^q[\mathbb{R}^n]$ for all $p,q\in[1,\infty)\cap\mathbb{N}$. Furthermore, since A^* is invariant on $\mathbb{C}_c^\infty[\mathbb{R}^n]$, $A^* : L^p[\mathbb{R}^n] \to L^p[\mathbb{R}^n]$. It now follows from Theorem 3.14, that A^* has a closed densely defined extension to $L^p[\mathbb{R}^n]$.

We see from this example, that both $[(-\Delta)^{-\alpha/2}]^*$ and $[(-\Delta)^{1/2}]^*$ have closed densely defined extensions to $L^p[\mathbb{R}^n]$.

3.2 Operators on $\mathcal{B}$

In this section, we look at the general theory of operators on $\mathcal{B}$ and relate it to other studies.

Definition 3.16. *Let S be bounded, let A be closed and densely defined, and let $\mathcal{U}$, $\mathcal{V}$ be subspaces of $\mathcal{B}$:*

1. *A is said to be naturally self-adjoint if $D(A) = D(A^*)$ and $A = A^*$.*
2. *A is said to be normal if $AA^* = A^*A$.*
3. *S is unitary if $SS^* = S^*S = I$.*
4. *The subspace $\mathcal{U}$ is $\perp$ to $\mathcal{V}$ if, for each $v \in \mathcal{V}$ and $\forall u \in \mathcal{U}$, $\langle v, f_u^s \rangle = 0$ and, for each $u \in \mathcal{U}$ and $\forall v \in \mathcal{V}$, $\langle u, f_v^s \rangle = 0$.*

The last definition is transparent since, for example,

$$\langle v, f_u^s \rangle = 0 \Leftrightarrow \langle v, J_2(u) \rangle = (v,u)_2 = 0 \ \ \forall v \in \mathcal{V}.$$

For later reference, we note that orthogonal subspaces in $\mathcal{H}_2$ induce orthogonal subspaces in $\mathcal{B}$.

With respect to our definition of natural selfadjointness, the following related definition is due to Palmer [PL], where the operator is called symmetric. This is essentially the same as a Hermitian operator as defined by Lumer [LU].

Definition 3.17. *A closed densely defined linear operator A on $\mathcal{B}$ is called self-conjugate if both iA and $-iA$ are dissipative.*

Theorem 3.18. *(Vidav-Palmer) A linear operator A, defined on $\mathcal{B}$, is self-conjugate if and only if iA and $-iA$ are generators of isometric semigroups.*

Theorem 3.19. *The operator A, defined on $\mathcal{B}$, is self-conjugate if and only if it is naturally self-adjoint.*

Proof. Let $\bar{A}$ and $\bar{A}^*$ be the closed densely defined extensions of A and A^* to $\mathcal{H}_2$. On $\mathcal{H}_2$, $\bar{A}$ is naturally self-adjoint if and only if $i\bar{A}$ generates a unitary group, if and only if it is self-conjugate. Thus, both definitions coincide on $\mathcal{H}_2$. It follows that the restrictions coincide on $\mathcal{B}$. □

Theorem 3.20. *(Gram-Schmidt) If $\mathcal{B}$ has a basis $\{\varphi_i,\ 1 \leqslant i < \infty\}$, then there is an orthonormal basis $\{\psi_i,\ 1 \leqslant i < \infty\}$ for $\mathcal{B}$ with a corresponding set of orthonormal duality maps $\{f_i^s,\ 1 \leqslant i < \infty\}$ (i.e., $\langle \psi_i, f_i^s \rangle = \delta_{ij}$).*

Proof. Since each φ_i is in $\mathcal{H}_2$, we can construct an orthogonal set of vectors $\{\phi_i,\ 1 \leqslant i < \infty\}$ in $\mathcal{H}_2$ by the standard Gram-Schmidt process. Set $\psi_i = \phi_i / \|\phi_i\|_{\mathcal{B}}$ and $\hat{f}_i^s = J(\psi_i)/\|\psi_i\|_{\mathcal{H}}^2$ on the subspace $M_i =< \psi_i >$. For each i, let $M_i^{\perp}$ be the subspace spanned by $\{\psi_j,\ i \neq j\}$. Now use the Hahn-Banach Theorem to extend $\hat{f}_i^s$ to f_i^s, defined on all of $\mathcal{B}$, with $f_i^s = 0$ on $M_i^{\perp}$ (see [RS], pg. 77 Corollary 3). From here, it is easy to check that $\{\{\psi_i\}, \{f_i^s\},\ 1 \leqslant i < \infty\}$ is a biorthonormal basis for $\mathcal{B}$. □

We close this section with the following observation about the use of KS^2. Let A be any closed densely defined positive linear operator on $\mathcal{B}$ with a discrete positive spectrum $\{\lambda_i\}$. In this case, $-A$ generates a C_0-contraction semigroup, so that it can be extended to $\mathcal{H}_2$ with the same properties. If we compute the ratio $\frac{\langle A\psi, f_\psi^s\rangle}{\langle \psi, f_\psi^s\rangle}$ in $\mathcal{B}$, it will be "close" to the value of $\frac{(\bar{A}\psi,\psi)_{\mathcal{H}_2}}{(\psi,\psi)_{\mathcal{H}_2}}$ in $\mathcal{H}_2$. On the other hand, note that we can use the min-max theorem on $\mathcal{H}_2$ to compute the eigenvalues and eigenfunctions of A via $\bar{A}$ exactly on $\mathcal{H}_2$. Thus, in this sense, the min-max theorem holds on $\mathcal{B}$.

4 Extension of the Poincaré inequality

4.1 Introduction

While studying the eigenvalue problem for Laplace's equation, Poincaré [PO] derived the following inequality

$$\int_U |f(\mathbf{x})|^2 \, d\mathbf{x} \leqslant M \int_U |Df(\mathbf{x})|^2 \, d\mathbf{x}, \text{ for all } f \in \mathbb{C}_0^\infty(U), \tag{4.1}$$

where U is a bounded domain in $\mathbb{R}^n$. Friedrichs [FR] extended this inequality and used it in the development of existence theory for general linear elliptic differential equations.

This theorem also arises in probability theory, in the following form:

Theorem 4.1. *Let U be a domain of $\mathbb{R}^n$ with a $\mathbb{C}^1$ boundary ∂U and, for each $f \in L^p[\mathbb{R}^n]$, let $\bar{f}_U$ denote the average of f over U, $\bar{f}_U = \bar{\int}_U f(\mathbf{y})d\mathbf{y}$. If $W^{1,p}(U)$ is the set of all locally integrable functions mapping U into $\mathbb{R}$ with a first order weak derivative, $Df \in L^p[U]$, then there is a constant M, depending only on n, p and U, such that*

$$\left\| f - \bar{f}_U \right\|_{L^p[U]} \leqslant M \left\| Df \right\|_{L^p[U]}.$$

A proof of Theorem 3.1 can be found in Evans [EV].

Let A be any closed densely defined linear operator of Baire class one on a separable Banach space $\mathcal{B}$, or any closed densely defined linear operator in case $\mathcal{B}$ is a Hilbert space. Let $T = -[A^*A]^{1/2}$ and recall that T is a (closed selfadjoint) m-dissipative generator of an analytic contraction semigroup $S(t)$ and $A = -WT$, where W is a partial isometry (see Kato [K], Yosida [YS], Pazy [PZ] or Vrabie [VR]).

The following result shows, under what conditions, the Poincaré inequality has an extension.

Theorem 4.2. *If there exists an $\alpha = \alpha(t) > 0$ such that, for all*

$$u \in D(A), \; \|S(t)u - u\|_{\mathcal{B}} \geq \alpha(t) \|u\|_{\mathcal{B}}.$$

Then there exists a $\lambda > 0$ such that

$$\lambda \|u\|_{\mathcal{B}} \leqslant \|Au\|_{\mathcal{H}}, \text{ for all } u \in D(A).$$

Proof. First observe that

$$\int_0^t TS(s)u\,dt = \int_0^t \frac{d}{ds}S(s)u\,ds = S(t)u - u.$$

Now choose $\alpha(t)$ so that $\|S(t)u-u\|_{\mathcal{B}} \geq \alpha(t)\,\|u\|_{\mathcal{B}}$. It follows that

$$\alpha(t)\,\|u\|_{\mathcal{B}} \leq \int_0^t \|S(t)Tu\|_{\mathcal{B}}\,dt \leqslant t\,\|Tu\|_{\mathcal{B}} = t\,\|Au\|_{\mathcal{B}}.$$

Thus, we have that $\frac{\alpha(t)}{t}\,\|u\|_{\mathcal{B}} \leqslant \|Au\|_{\mathcal{B}}$. It follows that $\frac{\alpha(t)}{t}$ has a least upper bound, λ. □

The following example shows that the theorem is not true if we relax the condition on $\alpha(t)$.

Example 4.3. *Let $\mathcal{B} = L^2[0,1]$ and let $Au(x) = xu(x)$ for all $u(x) \in \mathcal{B}$. In this case, if λ exists then*

$$\lambda^2 \int_0^1 u^2(x)dx \leqslant \int_0^1 x^2u^2(x)dx.$$

Let $E = (0,\varepsilon)$ and let $u(x) = I_E(x)$, where $I_E(x)$ is the indicator function for E. We then have

$$\lambda^2 \int_0^1 u^2(x)dx = \lambda^2\varepsilon, \text{ and } \quad \int_0^1 x^2u^2(x)dx = \frac{\varepsilon^3}{3}.$$

This leads to a contradiction for fixed λ, if ε is small enough.

Before discussing operational conditions that ensure the existence of a least upper bound λ, we need a definition.

Definition 4.4. *A strongly continuous semigroup T(t) is said to be*

1. *uniformly strongly stable if there is an $\omega_0 > 0$ such that*

$$\lim_{t\to\infty} e^{t\omega_0}\,\|T(t)u\|_{\mathcal{B}} = 0, \text{ for all } u \in \mathcal{B},$$

uniformly exponentially stable if there are strictly positive constants M, ω such that

$$\|T(t)\|_{\mathcal{B}} \leq Me^{-t\omega}$$

and uniformly stable if

$$\lim_{t\to\infty} \|T(t)\|_{\mathcal{B}} = 0.$$

Proofs of the next two theorems can be found in Engel and Nagel ([EN], pg. 296) and Pazy ([PZ], pg. 118) respectively.

Theorem 4.5. *For a strongly continuous semigroup $T(t)$, the following are equivalent.*

1. *$T(t)$ is uniformly strongly stable.*
2. *$T(t)$ is uniformly stable.*
3. *$T(t)$ is uniformly exponentially stable.*

Theorem 4.6. *Let A be the generator of an analytic semigroup $T(t)$ and let $\sigma(A)$ be the spectrum of A. If*

$$\sigma = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 0,$$

then $T(t)$ is uniformly exponentially stable.

Pazy ([PZ], pg. 117), gives an instructive example to show that we cannot drop the analytic requirement on A.

Theorem 4.7. *Let A be a closed densely defined linear operator on $\mathcal{B}$. Then the following conditions are sufficient for the existence of a $\lambda > 0$ such that $\lambda \|u\|_{\mathcal{B}} \leq \|Au\|_{\mathcal{B}}$ for all $u \in D(A)$.*

1. *There is a $r > 0,\ r \neq 1$ such that $\|S(r)u\|_{\mathcal{B}} \leqslant r\|u\|_{\mathcal{B}}$.*
2. *The operator $T = [A^*A]^{1/2}$ is strongly accretive, $\delta\|u\|^2 \leqslant (Tu, f_u)$, for some $\delta > 0$ and f_u is a duallity map for u.*
3. *If $\mathcal{B}$ is a Hilbert space and A is strongly accretive, $\delta\|u\|_{\mathcal{B}}^2 \leqslant (Au, u)_{\mathcal{B}}$, for $\delta > 0$.*

Proof. To prove (1), note that

$$|\|u\|_{\mathcal{B}} - \|S(r)u\||_{\mathcal{B}} \leqslant \|S(r)u - u\|_{\mathcal{B}} \leqslant r\|Tu\|_{\mathcal{B}} = r\|Au\|_{\mathcal{B}}.$$

Thus, if $0 < r < 1$, we have

$$(1-r)\|u\|_{\mathcal{B}} \leqslant r\|Au\|_{\mathcal{B}}, \Rightarrow \lambda = \frac{1-r}{r}$$

and, in the contrary case, we have

$$(r-1)\|u\|_{\mathcal{B}} \leqslant r\|Au\|_{\mathcal{B}}, \Rightarrow \lambda = \frac{r-1}{r}.$$

The proof of (2) is easy since

$$\delta\|u\|_{\mathcal{B}}^2 = (Tu, f_u)_{\mathcal{B}} \leq \|Tu\|_{\mathcal{B}}\|u\|_{\mathcal{B}} = \|Au\|_{\mathcal{B}}\|u\|_{\mathcal{B}}.$$

For the proof of (3), we note that

$$\delta\|u\|_{\mathcal{B}}^2 \leqslant (Au, u)_{\mathcal{B}} \leqslant \|Au\|_{\mathcal{B}}\|u\|_{\mathcal{B}}, \Rightarrow \delta\|u\|_{\mathcal{B}} \leqslant \|Au\|_{\mathcal{B}},$$

so we set $\lambda = \delta$.

□

Remark 4.8. *The first condition (1) is imposed directly on the semigroup, and is satisfied for all uniformly exponentially stable semigroups. In this case, note that A need not be a generator. The second condition (2) is weaker. It only implies that $-[A^*A]^{1/2} = -T$ is the generator of an analytic semigroup, so that A need not be a generator.*

It can be shown that condition (3) implies that $-A$ is the generator of an analytic semigroup.

5 Extension Of The Spectral Theorem

5.1 Introduction

For any selfadjoint operator in $\mathcal{C}[\mathcal{H}]$, the following theorem is well-known. A proof can be found in [DS], pages 1192-99 (see also Reed and Simon [RS], page 263).

Theorem 5.1. *Let $A \in \mathcal{C}[\mathcal{H}]$ be a selfadjoint operator, with spectrum $\sigma(A) \subset \mathbb{R}$, then there exists a unique regular countably additive projection-valued (= spectral) measure $\mathbf{E}(\Omega)$ mapping the Borel sets, $\mathfrak{B}[\mathbb{R}]$, over $\mathbb{R}$ into $\mathcal{H}$ such that, for each $x \in D(A)$, we have:*

1. *$D(A)$ also satisfies*

$$D(A) = \left\{ x \in \mathcal{H} \mid \int_{\sigma(A)} \lambda^2 \left(\mathbf{E}(d\lambda)x, x\right)_{\mathcal{H}} < \infty \right\}$$

and

2. $$Ax = \lim_{n\to\infty} \int_{-n}^{n} \lambda \mathbf{E}(d\lambda)x, \text{ for } x \in D(A).$$

3. *If $g(\cdot)$ is a complex-valued Borel function defined (a.e) on $\mathbb{R}$, then $g(A) \in \mathcal{C}[\mathcal{H}]$ and, for $x \in D(g(A)) = D_g(A)$,*

$$g(A)x = \lim_{n\to\infty} \int_{-n}^{n} g(\lambda)\mathbf{E}(d\lambda)x,$$

where

$$D_g(A) = \left\{ x \in \mathcal{H} \mid \int_{\sigma(A)} |g(\lambda)|^2 \left(\mathbf{E}(d\lambda)x, x\right)_{\mathcal{H}} < \infty \right\}$$

and $g(A^*) = \bar{g}(A)$.

It is an exercise to show that $\mathbf{E}(\Omega)x$ is of bounded variation. (For $\Omega = (-\infty, \lambda]$, $\mathbf{E}(\lambda)x$ is called a spectral function and $\{\mathbf{E}(\lambda)\}$ is called a spectral family.)

Theorem 4.1 initiated the general study of operators that have a spectral representation (or functional calculus). This research has moved in many directions. The Rellich-Titchmarsh-Kato line is concerned with applications to problems in physics and applied mathematics. In this direction, one is interested in concrete detailed information about the spectrum of various specific operators subject to different constraints (see Rellich [RL], Titchmarsh [TI] and Kato [K]). Another line of study follows more closely the approach developed by Stone and von Neumann (independently extending the bounded case by HIlbert). In this direction one seeks to extend Theorem 4.1 to a larger class of operators via operator theory and functional analysis (see Dunford and Schwartz [DS] and Yosida [YS]). The notes starting on page 2089 (in [DS]) are especially helpful in understanding the history (and the many other approaches).

5.2 Background

Dunford and Schwartz define a spectral operator as one that has a spectral family similar to that defined in Theorem 4.1 for selfadjoint operators. (A spectral operator is an operator with countably additive spectral measure on the Borel sets of the complex plane.) Strauss and Trunk [STT] define a bounded linear operator A, on a Hilbert space $\mathcal{H}$, to be spectralizable if there exists a non-constant polynomial p such that the operator $p(A)$ is a scalar spectral operator (has a representation as in Theorem 4.1 (2)). Another interesting line of attack is represented in the book of Colojoară and Foiaş [CF], where they study the class of generalized spectral operators. Here, one is not opposed to allowing the spectral resolution to exist in a generalized sense, so as to include operators with spectral singularities.

The following theorem was proven by Helffer and Sjöstrand [HSJ] (see Proposition 7.2):

Theorem 5.2. *Let $g \in \mathcal{C}_0^\infty[\mathbb{R}]$ and let $\hat{g} \in \mathcal{C}_0^\infty[\mathbf{C}]$ be an extension of g, with $\frac{\partial \hat{g}}{\partial \bar{z}} = 0$ on $\mathbb{R}$. If A is a selfadjoint operator on $\mathcal{H}$, then*

$$g(A) = -\frac{1}{\pi}\iint\limits_{\mathbf{C}} \frac{\partial \hat{g}}{\partial \bar{z}}\,(z-A)^{-1}\,dxdy.$$

This defines a functional calculus. Davies [DA] showed that the above formula can be used to define a functional calculus on Banach spaces for a closed densely defined linear operator A, provided $\rho(A) \cap \mathbb{R} = \emptyset$. In this program the objective is to construct a functional calculus pre-supposing that the operator of concern has a reasonable resolvent.

5.3 Problem

The basic problem that causes additional difficulty is the fact that many bounded linear operators (on $\mathcal{H}_2$) are of the form $A = B + N$, where B is normal and N is nilpotent (i.e., there is a $k \in \mathbb{N}$, such that $N^{k+1} = 0,\ N^k \neq 0$). In this case, A does not have a representation with a standard spectral measure. On the other hand, $R = [N^*N]^{1/2}$ is a selfadjoint operator, and there is a unique partial isometry W such that $N = WR$. If $\mathbf{E}(\cdot)$ is the spectral measure associated with R, then $W\mathbf{E}(\Omega)x$ is not a spectral measure, but it is a measure of bounded variation. Thus, we just might be able to find an easier solution to the problem if we are willing to drop our requirement that the spectral representation be with respect to a spectral measure in the normal sense.

We begin by noting that, in either of the Strauss and Trunk [STT], Helffer and Sjöstrand [HSJ] or Davies [DA] cases, the operator A is in Baire class one. Thus, Theorem 3.3 shows that A has an adjoint and Theorem 3.4 shows that $A = WR$, where W is a partial isometry and R is a nonnegative selfadjoint linear operator. Before presenting our solution for the Hilbert space case, we need a few results about vector-valued functions of bounded variation.

Recall that a vector-valued function $\mathbf{e}(\lambda)$ defined on a subset of $\mathbb{R}$ to $\mathcal{H}$ is of bounded variation if

$$V(\mathbf{e},\mathbb{R}) = \sup_P \left\| \sum_{i=1}^{n} [\mathbf{e}(b_i) - \mathbf{e}(a_i)] \right\|,$$

where the supremum is over all partitions P of non-overlapping intervals (a_i, b_i) in $\mathbb{R}$ (see Hille and Phillips [HP] or Diestel and Uhl [DU]).

The next result is proved in Hille and Phillips [HP] (see page 63).

Theorem 5.3. *Let $\mathbf{a}(\lambda)$ be a vector-valued function from $\mathbb{R}$ to $\mathcal{H}$ of bounded variation. If $h(\lambda)$ is a continuous complex-valued function on $(a,b) \subset \mathbb{R}$, then the following holds:*

1. *The integral $\int_a^b h(\lambda) d\mathbf{a}(\lambda)$ exists in the $\mathcal{H}$ norm.*
2. *If T is any operator in $L[\mathcal{H}]$, then $T\mathbf{a}(\lambda)$ is of bounded variation and*

$$T \int_a^b h(\lambda) d\mathbf{a}(\lambda) = \int_a^b h(\lambda) dT\mathbf{a}(\lambda).$$

5.4 Scalar case

Theorem 5.4. *If $A \in \mathcal{C}[\mathcal{B}]$ is an operator of Baire class one, then there exists a unique vector-valued function $\mathbf{e}_x(\lambda)$ of bounded variation such that, for each $x \in D(A)$, we have:*

1. *$D(A)$ also satisfies*

$$D(A) = \left\{ x \in \mathcal{B} \mid \int_{\sigma(A)} \lambda^2 \langle d\mathbf{e}_x(\lambda), f_x^s \rangle_{\mathcal{B}} < \infty \right\}$$

and

2. $$Ax = \lim_{n \to \infty} \int_{-n}^{n} \lambda d\mathbf{e}_x(\lambda), \text{ for all } x \in D(A).$$

3. *If $g(\cdot)$ is a complex-valued Borel function defined (a.e) on $\mathbb{R}$, then $g(A) \in \mathcal{C}[\mathcal{B}]$. Furthermore,*

$$D_g(A) = \left\{ x \in \mathcal{B} \mid \int_{\sigma(A)} |g(\lambda)|^2 \langle d\mathbf{e}_x(\lambda), f_x^s \rangle_{\mathcal{B}} < \infty \right\}$$

and

4. $$g(A)x = \lim_{n \to \infty} \int_{-n}^{n} g(\lambda) d\mathbf{e}_x(\lambda), \text{ for all } x \in D_g(A).$$

Proof. By Theorem 3.4, $A = WR$, where W is the unique partial isometry and $R = [A^*A]^{1/2}$. Let $\bar{R}$ be the extension of R to $\mathcal{H}_2$. From Theorem 5.1 (2), we see that there is a unique spectral measure $\bar{\mathbf{E}}(\Omega)$ such that for each $x \in D(\bar{R})$:

$$\bar{R}x = \lim_{n \to \infty} \int_0^n \lambda d\bar{\mathbf{E}}(d\lambda)x. \tag{5.1}$$

If we set $\bar{\mathbf{a}}_x(\lambda) = \bar{\mathbf{E}}(\lambda)x$, then $\bar{\mathbf{a}}_x(\lambda)$ is a vector-valued function of bounded variation. Furthermore, if $\bar{W}$ is the extension of W, $\bar{W}\bar{\mathbf{a}}_x(\lambda)$ is of bounded variation, with $Var(\bar{W}\bar{\mathbf{a}}_x, \mathbb{R}) \leq Var(\bar{\mathbf{a}}_x, \mathbb{R})$. If we set $\bar{\mathbf{e}}_x(\lambda) = \bar{W}\bar{\mathbf{a}}_x(\lambda)$, by Theorem 5.3, for each interval (a,b),

$$\left\{ \bar{W} \int_a^b \lambda d\bar{\mathbf{a}}_x(\lambda) \right\} = \int_a^b \lambda d\bar{\mathbf{e}}_x(\lambda).$$

Since $\bar{A}x = \bar{W}\bar{R}x$ and the restriction of $\bar{A}$ to $\mathcal{B}$ is A, we have, for all $x \in D(A)$,

$$Ax = \lim_{n\to\infty} \int_{-n}^{n} \lambda d\mathbf{e}_x(\lambda). \tag{5.2}$$

This proves (2). The proof of (1) follows from (1) in Theorem 5.1 and the definition of f_x^s. The proofs of (3) and (4) are direct adaptations of the Hilbert space case (see [RS]). □

5.5 General Case

In this section, we assume that, for each i, $1 \le i \le n$, $n \in \mathbb{N}$, $\mathcal{B}_i = \mathcal{B}$ is a fixed separable Banach space. We set $\mathfrak{B} = \times_{i=1}^{n} \mathcal{B}_i$, and represent a vector $\mathbf{x} \in \mathfrak{B}$ by $\mathbf{x}^t = [x_1, x_2, \cdots, x_n]$. An operator $\mathbf{A} = [A_{ij}] \in C[\mathfrak{B}]$ is defined whenever $A_{ij} : \mathcal{B} \to \mathcal{B}$, is in $\mathcal{C}[\mathcal{B}]$.

If $\mathcal{B}' \subset \mathcal{H}_2$ and A_{ij} is of Baire class one, then by Theorem 5.4, there exists a unique vector-valued function $\mathbf{e}_x^{ij}(\lambda)$ of bounded variation such that, for each $x \in D(A_{ij})$, we have:

1. $D(A_{ij})$ also satisfies

$$D(A_{ij}) = \left\{ x \in \mathcal{B} \mid \int_{\sigma(A_{ij})} \lambda^2 \left\langle d\mathbf{e}_x^{ij}(\lambda), f_x^s \right\rangle_{\mathcal{B}} < \infty \right\}$$

and

2.

$$A_{ij}x = \lim_{n\to\infty} \int_{-n}^{n} \lambda d\mathbf{e}_x^{ij}(\lambda), \text{ for all } x \in D(A_{ij}).$$

3. If $g(\cdot)$ is a complex-valued Borel function defined (a.e) on $\mathbb{R}$, then $g(A_{ij}) \in \mathcal{C}[\mathcal{B}]$. Furthermore,

$$D_g(A_{ij}) = \left\{ x \in \mathcal{B} \mid \int_{\sigma(A_{ij})} |g(\lambda)|^2 \left\langle d\mathbf{e}_x^{ij}(\lambda), f_x^s \right\rangle_{\mathcal{B}} < \infty \right\}$$

and

4.

$$g(A_{ij})x = \lim_{n\to\infty} \int_{-n}^{n} g(\lambda) d\mathbf{e}_x^{ij}(\lambda), \text{ for all } x \in D_g(A_{ij}).$$

If we let $d\mathcal{E}(\lambda) = [d\mathbf{e}^{ij}(\lambda)]$, then we can represent $\mathbf{A}$ and $g(\mathbf{A})$ by:

$$\mathbf{Ax} = \lim_{n\to\infty} \int_{-n}^{n} \lambda d\mathcal{E}(\lambda)\mathbf{x}, \text{ for all } \mathbf{x} \in D(\mathbf{A})$$

and

$$g(\mathbf{A})\mathbf{x} = \lim_{n\to\infty} \int_{-n}^{n} g(\lambda) d\mathcal{E}(\lambda)\mathbf{x}, \text{ for all } \mathbf{x} \in D(\mathbf{A}).$$

6 Schatten Classes

In this section, we show how our approach allows us to provide a natural definition for the Schatten class of operators on $\mathcal{B}$.

Let $\mathbb{K}(\mathcal{B})$ be the class of compact operators on $\mathcal{B}$ and let $\mathbb{F}(\mathcal{B})$ be the set of operators of finite rank. Recall that, for separable Banach spaces, $\mathbb{K}(\mathbf{B})$ is an ideal that need not be the maximal ideal in $L[\mathcal{B}]$. If $\mathbb{M}(\mathcal{B})$ is the set of weakly compact operators and $\mathbb{N}(\mathcal{B})$ is the set of operators that map weakly convergent sequences into strongly convergent sequences, it is known that both are closed two-sided ideals in the operator norm, and, in general, $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{M}(\mathcal{B})$ and $\mathbb{F}(\mathcal{B}) \subset \mathbb{K}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$ (see Dunford and Schwartz [DS], pg. 553). For reflexive Banach spaces $\mathbb{K}(\mathcal{B}) = \mathbb{N}(\mathcal{B})$ and $\mathbb{M}(\mathcal{B}) = L[\mathcal{B}]$. For the space of continuous functions $\mathbb{C}[\Omega]$, on a compact Hausdorff space Ω, Grothendieck [GR] has shown that $\mathbb{M}(\mathcal{B}) = \mathbb{N}(\mathcal{B})$. On the other hand, it is shown in Dunford and Schwartz [DS] that, for a positive measure space, (Ω, Σ, μ), on $L^1(\Omega, \Sigma, \mu)$, $\mathbb{M}(\mathcal{B}) \subset \mathbb{N}(\mathcal{B})$.

In this section, we assume that $\mathcal{B}$ has the approximation property (i.e., every compact operator can be approximated by operators of finite rank). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the canonical adjoint pair for $\mathcal{B}$, so that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$, as continuous dense embeddings. Let A be a compact operator on $\mathcal{B}$ and let $\bar{A}$ be its extension to $\mathcal{H}_2$. For each compact operator $\bar{A}$ on $\mathcal{H}_2$, there exists an orthonormal set of functions $\{\bar{\varphi}_n \,|n \geqslant 1\}$ such that

$$\bar{A} = \sum\nolimits_{n=1}^{\infty} \mu_n(\bar{A})\, (\cdot\, , \bar{\varphi}_n)_2\, \bar{U}\bar{\varphi}_n,$$

where the μ_n are the eigenvalues of $[\bar{A}^*\bar{A}]^{1/2} = \left|\bar{A}\right|$, counted by multiplicity and in decreasing order, and $\bar{U}$ is the partial isometry associated with the polar decomposition of $\bar{A} = \bar{U}\left|\bar{A}\right|$. Without loss, we can assume that the set of functions $\{\bar{\varphi}_n \,|n \geqslant 1\}$ is contained in $\mathcal{B}$ and $\{\varphi_n \,|n \geqslant 1\}$ is the normalized version in $\mathcal{B}$. If $\mathbb{S}_p[\mathcal{H}_2]$ is the Schatten Class of order p in $L[\mathcal{H}_2]$, it is well-known that, if $\bar{A} \in \mathbb{S}_p[\mathcal{H}_2]$, its norm can be represented as:

$$\left\|\bar{A}\right\|_p = \left\{\sum\nolimits_{n=1}^{\infty} \left(\bar{A}^*\bar{A}\bar{\varphi}_n, \bar{\varphi}_n\right)_2^{p/2}\right\}^{1/p} = \left\{\sum\nolimits_{n=1}^{\infty} \left|\mu_n(\bar{A})\right|^p\right\}^{1/p}.$$

Definition 6.1. *We define the Schatten Class of order p in $L[\mathcal{B}]$ by:*

$$\mathbb{S}_p[\mathcal{B}] = \mathbb{S}_p[\mathcal{H}_2] \cap L[\mathcal{B}]\,|_{\mathcal{B}}$$

Since $\bar{A}$ is the extension of $A \in \mathbb{S}_p[\mathcal{B}]$, we can represent A on $\mathcal{B}$ as

$$A = \sum\nolimits_{n=1}^{\infty} \mu_n(A)\, \langle \cdot\, , f_n^s \rangle\, U\varphi_n,$$

where f_n^s is the Steadman duality map obtained from the Hahn-Banach extension of $\mathbf{J}_2(\varphi_n) \big/ \|\varphi_n\|_2^2$ and U is the restriction of $\bar{U}$. The corresponding norm of A on $\mathbb{S}_p[\mathcal{B}]$ is defined by:

$$\|A\|_p = \left\{\sum\nolimits_{n=1}^{\infty} \langle A^*A\varphi_n, f_n^s\rangle^{p/2}\right\}^{1/p}.$$

Theorem 6.2. *Let $A \in \mathbb{S}_p[\mathcal{B}]$, then $\|A\|_p = \left\|\bar{A}\right\|_p$.*

Proof. It is clear that $\{\varphi_n \mid n \geqslant 1\}$ is a set of eigenfunctions for A^*A on $\mathcal{B}$. Furthermore, by Lax's Theorem A^*A is selfadjoint and the point spectrum of A^*A is unchanged by its extension to $\mathcal{H}_2$. It follows that $A^*A\varphi_n = \left|\mu_n(\bar{A})\right|^2 \varphi_n$,

$$\langle A^*A\varphi_n, f_n^s\rangle = \frac{|\mu_n|^2}{\|\varphi_n\|_2^2}\langle \varphi_n, f_n^s\rangle = \frac{|\mu_n|^2}{\|\varphi_n\|_2^2}(\varphi_n,\varphi_n)_2 = |\mu_n|^2$$

and

$$\|A\|_p = \left\{\sum\nolimits_{n=1}^{\infty}\langle A^*A\varphi_n, f_n^s\rangle^{p/2}\right\}^{1/p} = \left\{\sum\nolimits_{n=1}^{\infty}|\mu_n|^p\right\}^{1/p} = \left\|\bar{A}\right\|_p .$$

□

Lemma 6.3. *If $\mathcal{B}$ has the approximation property, the embedding of $L[\mathcal{B}]$ in $L[\mathcal{H}_2]$ is both continuous and dense.*

Proof. From Theorem 2.9, we see that the embedding is continuous. Since $\mathcal{B}$ has the approximation property, the finite rank operators $\mathbb{F}(\mathcal{B})$ on $\mathcal{B}$ are dense in the finite rank operators $\mathbb{F}(\mathcal{H}_2)$ on $\mathcal{H}_2$. It follows that, for each p, $\mathbb{S}_p[\mathcal{B}]$ is dense in $\mathbb{S}_p[\mathcal{H}_2]$. In particular, $\mathbb{S}_1[\mathcal{B}]$ is dense in $\mathbb{S}_1[\mathcal{H}_2]$ and, since $\mathbb{S}_1[\mathcal{H}_2]^* = L[\mathcal{H}_2]$, we see that $\mathbb{S}_1[\mathcal{B}]^* = L[\mathcal{B}]$ must be dense in $L[\mathcal{H}_2]$. □

It is clear that much of the theory of operator ideals on Hilbert spaces extend to separable Banach spaces in a straightforward way. We state a few of the more important results to give a sense of the power provided by the existence of adjoints. The first result extends theorems due to Weyl [WY], Horn [HO], Lalesco [LE] and Lidskii [LI]. (The methods of proof for Hilbert spaces carry over without difficulty.)

Theorem 6.4. *Let $\mathbf{A} \in \mathbb{K}(\mathcal{B})$, the set of compact operators on $\mathcal{B}$, and let $\{\lambda_n\}$ be the eigenvalues of $\mathbf{A}$ counted up to algebraic multiplicity. If Φ is a mapping on $[0,\infty]$ which is nonnegative and monotone increasing, then we have:*

1. *(Weyl)*
$$\sum\nolimits_{n=1}^{\mathbf{N}}\Phi\left(|\lambda_n(\mathbf{A})|\right) \leqslant \sum\nolimits_{n=1}^{\mathbf{N}}\Phi\left(\mu_n(\mathbf{A})\right)$$
and

2. *(Horn)*
$$\sum\nolimits_{n=1}^{\mathbf{N}}\Phi\left(|\lambda_n(\mathbf{A}_1\mathbf{A}_2)|\right) \leqslant \sum\nolimits_{n=1}^{\mathbf{N}}\Phi\left(\mu_n(\mathbf{A}_1)\mu_n(\mathbf{A}_2)\right).$$
In case $\mathbf{A} \in \mathbb{S}_1(\mathcal{B})$, we have:

3. *(Lalesco)*
$$\sum\nolimits_{n=1}^{\mathbf{N}}|\lambda_n(\mathbf{A})| \leqslant \sum\nolimits_{n=1}^{\mathbf{N}}\mu_n(\mathbf{A})$$
and

4. *(Lidskii)*
$$\sum\nolimits_{n=1}^{\mathbf{N}}\lambda_n(\mathbf{A}) = Tr(\mathbf{A}).$$

6.1 Discussion

In a Hilbert space $\mathcal{H}$, the Schatten classes $\mathbb{S}_p(\mathcal{H})$ are the only ideals in $\mathbb{K}(\mathcal{H})$, and $\mathbb{S}_1(\mathcal{H})$ is minimal. In a Banach space, this is far from true. A complete history of the subject can be found in the recent book by Pietsch [PI1] (see also Retherford [RE], for a nice review). We limit this discussion to a few major topics in the subject. First, Grothendieck [GR] defined an important class of nuclear operators as follows:

Definition 6.5. *If $A \in \mathbb{F}(\mathcal{B})$ (the operators of finite rank), define the ideal $\mathbf{N}_1(\mathcal{B})$ by:*

$$\mathbf{N}_1(\mathcal{B}) = \{A \in \mathbb{F}(\mathcal{B}) \mid \mathbf{N}_1(A) < \infty\},$$

where

$$\mathbf{N}_1(A) = \operatorname{glb}\left\{\sum\nolimits_{n=1}^m \|f_n\|\,\|\phi_n\| \;\middle|\, f_n \in \mathcal{B}',\ \phi_n \in \mathcal{B},\ A = \sum\nolimits_{n=1}^m \phi_n \langle\cdot\,, f_n\rangle\right\}$$

and the greatest lower bound is over all possible representations for A.

Grothendieck has shown that $\mathbf{N}_1(\mathcal{B})$ is the completion of the finite rank operators. $\mathbf{N}_1(\mathcal{B})$ is a Banach space with norm $\mathbf{N}_1(\cdot)$, and is a two-sided ideal in $\mathbb{K}(\mathcal{B})$. It is easy to show that:

Corollary 6.6. *$\mathbb{M}(\mathcal{B}), \mathbb{N}(\mathcal{B})$ and $\mathbf{N}_1(\mathcal{B})$ are two-sided *ideals.*

In order to compensate for the (apparent) lack of an adjoint for Banach spaces, Pietsch [PI2], [PI3] defined a number of classes of operator ideals for a given $\mathcal{B}$. Of particular importance for our discussion is the class $\mathbb{C}_p(\mathcal{B})$, defined by

$$\mathbb{C}_p(\mathcal{B}) = \left\{A \in \mathbb{K}(\mathcal{B}) \;\middle|\, \mathbb{C}_p(A) = \sum\nolimits_{i=1}^\infty [s_i(A)]^p < \infty\right\},$$

where the singular numbers $s_n(A)$ are defined by:

$$s_n(A) = \inf\{\|A-K\|_{\mathcal{B}} \mid \text{rank of } K \leqslant n\}.$$

Pietsch has shown that, $\mathbb{C}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$, while Johnson et al [JKMR] have shown that for each $A \in \mathbb{C}_1(\mathcal{B})$, $\sum_{n=1}^\infty |\lambda_n(A)| < \infty$. On the other hand, Grothendieck [GO] has provided an example of an operator A in $\mathbf{N}_1(L^\infty[0,1])$ with $\sum_{n=1}^\infty |\lambda_n(A)| = \infty$ (see Simon [SI], pg. 118). Thus, it follows that, in general, the containment is strict. It is known that, if $\mathbb{C}_1(\mathcal{B}) = \mathbf{N}_1(\mathcal{B})$, then $\mathcal{B}$ is isomorphic to a Hilbert space (see Johnson et al). It is clear from the above discussion, that:

Corollary 6.7. *$\mathbb{C}_p(\mathcal{B})$ is a two-sided *ideal in $\mathbb{K}(\mathcal{B})$, and $\mathbb{S}_1(\mathcal{B}) \subset \mathbf{N}_1(\mathcal{B})$.*

For a given separable Banach space, it is not clear how the spaces $\mathbb{C}_p(\mathcal{B})$ of Pietsch relate to our Schatten Classes $\mathbb{S}_p(\mathcal{B})$ (clearly $\mathbb{S}_p(\mathcal{B}) \subseteq \mathbb{C}_p(\mathcal{B})$). Thus, the interesting question is that of the equality of $\mathbb{S}_p(\mathcal{B})$ and $\mathbb{C}_p(\mathcal{B})$.

7 Conclusion

In this paper, we have refined and extended the work in [GBZS] and [GZ3] to develop a reasonably complete view of the theory of adjoints for bounded linear operators on separable Banach spaces. We have further identified the obstacles to a similar program for closed densely defined linear operators. A major result in this case is that all operators of Baire class one have an adjoint. (This corrects an error in [GZ3].) As applications, we first used the polar decomposition property to extend the Poincaré inequality. Then, the polar decomposition property, along with a few results for vector measures and vector-valued functions allowed us to extend the spectral theorem to all operators of Baire class one on a Banach space and closed densely defined linear operator on a separable Hilbert space. We were also able to extend the spectral theorem to all linear operators of Baire class one on separable Banach spaces. Finally, we showed how the polar decomposition property allowed us to provide a natural Banach space version of the Schatten class of compact operators.

acknowledgements

We would like to thank an anonymous referee for providing us with Example 4.3, which showed that our original version of Theorem 4.2 was incorrect and for making a number of important observations and corrections that have improve our presentation.

We thank Professor Anatolij Pliczko for pointing out an error in our proof in [GZ3], that all closed densely defined linear operators on $\mathcal{B}$ have an adjoint.

During the course of the development of this work, we have benefited from important critical remarks from Professors Jerome Goldstein and Ioan I. Vrabie. They also identified a few errors in an earlier draft, which have led to an improvement.

References

[AL] A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*, Colloq. Math. **1** (1948), 289-293.

[B] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monographs in Mathematics, New York, (2010).

[CF] I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon Breach, (1968).

[DA] E. B. Davies, *The Functional Calculus*, J. London Mat. Soc. Vol. **52** (1995) 166-176.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators Part II: Spectral Theory*, Wiley Classics edition, Wiley Interscience (1988).

[DU] J. Diestel and J. J. Uhl, Jr, *Vector Measures* , Math. Surveys 15, Amer. Math. Soc. Providence, RI, (1977).

[EN] K. J. Engel, R. Nagel et al, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math., **194** Springer New York, (2000).

[EV] L. C. Evans, *Partial Differential Equations,* AMS Graduate Studies in Math. **18**, Providence, R.I, 1998.

[FH] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, (1965).

[FR] K. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren I; II,* Math. Ann. **109** (1934), 465-487, 685-713.

[GBZS] T. Gill, S. Basu, W. W. Zachary and V. Steadman, *Adjoint for operators in Banach spaces*, Proceedings of the American Mathematical Society, **132** (2004), 1429-1434.

[G] L. Gross, *Abstract Wiener spaces,* Proc. Fifth Berkeley Symposium on Mathematics Statistics and Probability, (1965), 31-42.

[GMZ] T. L. Gill, F. Mensah, and W. W. Zachary, The Square-Root Equation, Afr. Diaspora J. Math. 8 (2009), No. 2, pp. 55-62.

[GRA] L. Grafakos, *Classical and Modern Fourier Analysis,* Person Prentice-Hall, New Jersey, 2004.

[GR] A. Grothendieck, *Products tensoriels topologiques et espaces nucleaires*, Memoirs of the American Mathematical Society, **16** (1955).

[GZ] T. L. Gill and W. W. Zachary, *Foundations for relativistic quantum theory I: Feynman's operator calculus and the Dyson conjectures*, Journal of Mathematical Physics **43** (2002), 69-93.

[GZ1] T. L. Gill and W. W. Zachary, *Banach Spaces for the Feynman integral*, Real Analysis Exchange **34**(2) (2008)/(2009), 267-310.

[GZ2] T. L. Gill and W. W. Zachary, *A New Class of Banach Spaces*, Journal of Physics A: Math. and Gen. **41** (2008), 495206.

[GZ3] T. L. Gill and W. W. Zachary, *The Adjoint Problem on Banach Spaces*, Communications in Mathematical Analysis **10** (2009), 1-11.

[GZ4] T. L. Gill and W. W. Zachary, *Feynman operator calculus: The constructive theory*, Expositiones Mathematicae **29** (2011), 165-203.

[HO] A. Horn, *On the singular values of a product of completely continuous operators*, Proc. Nat. Acad. Sci. **36** (1950), 374–375.

[HP] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Pub. 31, Amer. Math. Soc. Providence, RI, (1957).

[HS] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, (1991).

[HSJ] B. Helffer, and J. Sjöstrand, *Équation de Schrödinger avec champ magnetique et équation de Harper, Schrödinger Operators*, (Snderborg, 2988) eds. H. Holden and A Jensen, Lecture Notes in Phys., vol. 345, Springer-Verlag, Berlin, (1989), 118-197.

[JKMR] W. B. Johnson, H. Konig, B. Maurey and J. R. Retherford, *Eigenvalues of p-summing and l_p type operators in Banach space*, J. Funct. Anal. **32** (1978), 353–380.

[K] T. Kato, *Perturbation Theory for Linear Operators,* second ed. Springer-Verlag, New York, (1976).

[KB] J. Kuelbs, *Gaussian measures on a Banach space,* Journal of Functional Analysis **5** (1970), 354–367.

[KW] J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner-Texte zür Mathematik, Band **26**, Teubner Verlagsgesellschaft, Leipzig, (1980).

[L] P. D. Lax, Symmetrizable linear tranformations. *Comm. Pure Appl. Math.* **7** (1954), 633–647.

[LE] T. Lalesco, *Une theoreme sur les noyaux composes*, Bull. Acad. Sci. **3** (1914/15), 271–272.

[LI] V. B. Lidskii, *Non-self adjoint operators with a trace*, Dokl. Akad. Nauk. SSSR **125** (1959), 485-487.

[LP] G. Lumer and R. S. Phillips, Dissipative operators in a Banach space, *Pacific J. Math.* **11** (1961), 679-698.

[LU] G. Lumer, Spectral operators, Hermitian operators and bounded groups, *Acta. Sci. Math.* (Szeged) **25** (1964), 75-85.

[PI1] A. Pietsch, *History of Banach Spaces and Operator Theory*, Birkhäuser, Boston, (2007).

[PI2] A. Pietsch, *Einige neue Klassen von kompacter linear Abbildungen*, Revue der Math. Pures et Appl. (Bucharest), **8** (1963), 423–447.

[PI3] A. Pietsch, *Eigenvalues and s-Numbers* , Cambridge University Press, (1987).

[PL] T. W. Palmer, Unbounded normal operators on Banach spaces, *Trans. Amer. Math. Sci.* **133** (1968), 385-414.

[PO] H. Poincaré, *Sur les équations aux dériées partielles de la physique mathématique,* Amer. J. Math. **12** (1890) 211-294.

[PZ] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* Applied Mathematical Sciences, **44**, Springer New York, (1983).

[RE] J. R. Retherford, *Applications of Banach ideals of operators*, Bull. Amer. Math. Soc. **81** (1975), 978-1012.

[RL] F. Rellich, *Störungsterie der Spektralzerlegung V.*, Math. Ann. **118** (1940), 462-484.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, (1972).

[RU] W. Rudin, *Functional Analysis*, McGraw-Hill Press, New York, (1973).

[SC] L. Schiff, *Quantum Mechanics,* International Series in Pure and Applied Math., McGraw-Hill, New York, (1955).

[SCH] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Harper & Row, New York, (1961).

[SI] B. Simon, *Trace Ideals and their Applications*, London Mathematical Society Lecture Notes Series **35**, Cambridge University Press, New York, (1979).

[SO] M. H. Stone, *Linear Transformations in Hilbert Space* , Math. Surveys 15, Amer. Math. Soc. Colloq. Publ. 15, Providence, RI, (1932).

[ST] V. Steadman, *Theory of operators on Banach spaces*, Ph.D thesis, Howard University, 1988.

[STE] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, NJ, (1970).

[STE2] E. M. Stein, *The characterization of functions arising as potentials*, Bull. Amer. Math. Soc., **67** (1961), 102-104.

[STT] V. A. Strauss and C. Trunk, *Spectralizable Operators*, Integr. Equ. Oper. Theory **61** (2008), 413-422.

[TI] E. C. Titchmarsh, *Some theorems on perturbation theory V.*, J. Analyse. Math. Soc. **4** (1954/56), 187-208.

[VN] J. von Neumann, *Über adjungierte Funktionaloperatoren,* Annals of Mathematics **33** (1932), 294-310.

[VPP] V. A. Vinokurov, Yu. Petunin and A. N. Pliczko, Measurability and Regularizability mappings inverse to continuous linear operators (in Russian), *Mat. Zametki.* **26** (1979), no. 4, 583-591. English translation: *Math. Notes* **26** (1980), 781-785.

[VR] I. Vrabie, C_0*-Semigroups and Applications*, North-Holland Math. Studies **191** Elsevier, New York, (2002).

[WY] H. Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. **35**, (1949), 408-11.

[YS] K. Yosida, *Functional Analysis,* second ed. Springer-Verlag, New York, (1968).