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GERBES FOR THE CHOW

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Abstract

The definition of the coherence relations in non-abelian cohomology is a difficult problem studied by many authors. The purpose of this paper is to simplify the solution provided by the author which uses the notion of sequences of fibred categories and to apply the resulting theory to higher divisors and Chow theory.

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1 Introduction

Non-Abelian cohomology has been created by Grothendieck and his collaborators with the purpose of giving a geometric interpretation of characteristic classes. Let (C,J) be a Grothendieck site, and L a sheaf defined on (C,J), we know that $H^0(X,L)$ is the set of global sections of L, and $H^1(X,L)$ is the set of isomorphism classes of torsors bounded by L. In [5], Giraud has defined the notion of gerbes bounded by a sheaf L, objects which are classified by $H^2(X,L)$. Many concrete problems have created the need to provide a geometric interpretation of higher cohomology classes. The main difficulties to solve this problem are related to the combinatoric which arise when one try to define coherence relations for n-gerbes, $n>2$. In [17] is developed the notion of sequences of gerbes which provides a partial answer by attaching to a sequence of fibred categories endowed with suitable properties a cohomology class, this construction can also be viewed as a geometric interpretation of the connecting morphism in cohomology. Remark that this approach gives a complete satisfaction in the geometric study of the Brauer group as shown in [14]. This theory has been successfully applied in many areas, like in symplectic geometry, where it has enabled to give new insights on quantization and symplectic fibrations. It has also been applied to the study of moduli spaces in differential geometry [19].

The purpose of this paper is to simplify the notion of sequences of fibred categories studied in [17] since the classifying classes here are easier to compute. The main tool used here is the topos of the site of sheaves $Sh(C,J)$ defined on the Grothendieck site (C,J). In [7], Grothendieck defines on $Sh(C,J)$ a Grothendieck topology, which can be used to define notions of varieties and algebraic spaces for any Grothendieck site (see also [16]). This topology allows us to define fibred categories on the basis (C,J) for which the objects of the fibres are varieties, thus are naturally endowed with a Grothendieck topology; we

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study such 2-sequences of fibred categories, and apply our results to define and study higher divisors in algebraic geometry.

Notations. In this paper all categories are stable by finite limits and colimits. Let C be a category. We denote by C/X, the comma category of morphisms of C whose target is X. Let $U_{i_1},...,U_{i_p}$ be objects of C, we denote by $U_{i_1...i_p}$, the fiber product of $U_{i_1},...,U_{i_p}$ over the terminal object of C. Let $p: F \rightarrow C$ be a fibred category (see definition 1, paragraph 2), and a morphism $h_{i_1...i_p}: e_{i_1...i_p} \rightarrow e'_{i_1...i_p}$ between objects of the fibre $F_{U_{i_1...i_p}}$, we denote by $h^{j_1...j_l}_{i_1...i_p}: e^{j_1...j_l}_{i_1...i_p} \rightarrow e'^{j_1...j_l}_{i_1...i_p}$ the restriction of $h_{i_1...i_p}$ between the respective restrictions $e^{j_1...j_l}_{i_1...i_p}$ and $e'^{j_1...j_l}_{i_1...i_p}$ of $e_{i_1...i_p}$ and $e'_{i_1...i_p}$ to $U_{i_1..i_pj_1..j_l}$.

2 Grothendieck topologies, varieties and geometric spaces

Definition 2.1. Let C be a category, a sieve S defined on C, is a subclass of the class $ob(C)$, of objects of C such that: if $X \in S$, and $Y \rightarrow X$ is a morphism of C, then $Y \in S$.

A Grothendieck topology J, defined on the category C, is a correspondence which assigns to every object X of C, a non-empty class of sieves $J(X)$ of C/X which satisfies the following properties:

- Let $S \in J(Y)$, and $f: X \rightarrow Y$ a morphism of C, the pullback $S^f = \{h: Z \rightarrow X, f \circ h \in S\}$ is an element of $J(X)$.

- A sieve S of C/Y is an element of $J(Y)$ if for every morphism $f: X \rightarrow Y$, $S^f \in J(X)$.

A category equipped with a Grothendieck topology is called a site or a Grothendieck site. We denote it by (C,J).

Definition 2.2. Let C be a category, a presheaf F on C is a functor $F: C^0 \rightarrow Set$, where C^0 is the opposite category of C, and Set the category of sets. We denote by $PreSh(C)$ the category of presheaves defined on the category C.

A presheaf on the Grothendieck site (C,J) is called a sheaf if and only if for every object X of C, and every element S of $J(X)$, $Lim_{Y \rightarrow X \in S} F(Y) = F(X)$.

Let C be a site, a trivial sheaf F defined on C, is a sheaf F such that there exists a set E such that for every object X of C, $F(X) = E$, and the restriction maps are the identity of E.

We denote by $Sh(C,J)$ the category of sheaves defined on the Grothendieck site (C,J). We have an full embedding $C \rightarrow PreSh(C)$ defined by the Yoneda embedding which associates to the object X of C, the presheaf h_X defined by $h_X(Z) = Hom_C(Z,X)$. We say that the topology is subcanonical if the presheaf h_X is a sheaf for every $X \in C$. In the sequel, we will consider only subcanonical topologies. If there is no confusion, we will often denote h_X by X.

Example 2.3. Let Op be the category whose objects are open subsets of $\mathbb{R}^n, n \in \mathbb{N}$, and whose morphisms are local homeomorphisms; for every object X of Op, an element of $J(X)$ is a family of local homeomorphisms $(h_i: U_i \rightarrow X)_{i\in I}$ such that $\bigcup_{i\in I} h(U_i) = X$. Every topological space T defines a sheaf h_T of Op by assigning to X the set of continuous maps: $h_T(X) = Hom(X,T)$.

Example 2.4. Let Aff be the category of affine schemes: it is the category opposite to the category of commutative rings with a unit. We endow Aff with the etale topology. For every object X of Aff, an element of $J(X)$ is a finite family of etale morphisms $(h_i : U_i \to X)_{i\in I}$ such that $\bigcup_{i\in I} h(U_i) = X$. Every scheme S defines a sheaf h_S of Aff by assigning to X the set of morphisms of schemes: $h_S(X) = Hom(X,S)$. See [8], VIII. proposition 5.1.

Definition 2.5. Let (C,J) be a site, we say that the morphism $F \to G$ between elements of $Sh(C,J)$ is a covering morphism if and only if for every object X in C, and every morphism $X \to G$, the canonical projection $X \times_G F \to X$ is a covering sieve of X; the family of morphisms $(F_i \to G)_{i\in I}$ is a covering family of G if and only if the morphism $\bigcup_{i\in I} F_i \to G$ is a covering morphism, see [7], p 251-252. These covering families define on $Sh(C,J)$ a Grothendieck topology (See [7] proposition 5.4 p. 254).

Definition 2.6. Let C be a category, a monomorphism of C is a morphism $f : X \to Y$, such that for every object Z of C, The map $Hom(Z,X) \to Hom(Z,Y)$ which sends the element $h \in Hom(Z,X)$ to $f \circ h$ is injective.

Suppose that (C,J) is a site, denote by e the final object of $Sh(C,J)$. The object X of C is an open subset of e if and only if there exists a monomorphism $i : X \to e$ (see [1] p. 20 and [7] definition 8.3 p. 421). The morphism i is called an open immersion

The object U of C/X is an open subset of X, if and only if it is an open subset of the final object of C/X for the induced topology.

Definition 2.7. Let (C,J) be a site, we suppose that for every object X of C, every open subset $f : U \to X$, of C/X is contained in a sieve of X, a geometric space (See also [16]) is a sheaf F of (C,J) which satisfies such that there exists a family $(U_i)_{i\in I}$ of objects of C and a sieve $p : \bigcup_{i\in I} U_i \to F$ of F, for the Grothendieck topology on $Sh(C,J)$.

The family $(U_i)_{i\in I}$ is called an *atlas*.

Let $p_i : U_i \to F$ be the composition of the canonical embedding $U_i \to \bigcup_{i\in I} U_i$ and p. If for every i, the map p_i is an open immersion, then F is called a *variety*.

Example 2.8. A geometric space F in Op is defined by a sheaf F on Op, and a covering morphism $p : \bigcup_{i\in I} U_i \to F$. In particular a topological manifold is a geometric space, a more carefully investigation shows that it is a variety (see [16]).

Example 2.9. A geometric space F in Aff is defined by a sheaf F on Aff, and a covering morphism $p : \bigcup_{i\in I} U_i = Spec(A_i) \to F$. In particular a scheme is a geometric space, a more careful investigation shows that it is a variety.

Let F, be a geometric space, suppose that the covering $p : \bigcup_{i\in I} U_i \to F$ is 1-connected; (this is equivalent to saying that for every $i \in I$, every sheaf on U_i is trivial). The pullback F_i of F by p_i is trivial. Let F_i^j be the pullback of F_i on $U_i \times_F U_j$ by the projection $U_i \times_F U_j \to U_i$. There exists an isomorphism $g_{ij} : F_j^i \to F_i^j$. The morphism $c_{ijk} = g_{ki}^j g_{ij}^k g_{jk}^i$ is an automorphism of F_k^{ij}, the restriction of F_k to $U_i \times_F U_j \times_F U_k$ which can be identified with an automorphism of $F(U_i \times_F U_j \times_F U_k)$. We call a 2-cocycle a family c_{ijk} which verifies the relation: $c_{ikl}^j g_{lk}^{ij} c_{ijk}^l g_{kl}^{ij} = c_{ijl}^k c_{jkl}^i$.

Suppose that F is a variety; recall that i.e that $U_i \to F$ is an open immersion for every i. Thus F_i is the restriction of F to U_i. Let $h_i : F_i \to E_i$, the trivialization of the restriction of F on U_i, on $U_i \times_F U_j$, we can define the map $h_i^{j^{-1}} \circ h_j^i$ which is an automorphism of E_{ij}, the fibre of the restriction of F to $U_i \times_F U_j$. We have the relation: $c_{ik}^j = c_{ij}^k c_{jk}^i$

Let C be a category, X,P objects of C, a *P-point* of X (see [1] p. 17, or [7] p. 385) is a morphism $x : P \to X$.

In a category stable by finite limits, and colimits, a group object (See [1] p.35) is defined by:

- An object G endowed with a morphism $p : G \times G \to G$ called the product, which is associative,

- The neutral element, which is a global point, that is a morphism $e : 1 \to G$, (where 1 is the final object).

- The inverse is a morphism $i : G \to G$.

This data must satisfy the following conditions:

Let $x,y : P \to G$ be two P-points of G, by the universal property of the product, x and y define a morphism $(x,y) : P \to G \times G$, we write $p \circ (x,y) = xy$, we must have $x(yz) = (xy)z$.

Let $i(x)$ be the point $i \circ x$, the fact that i is the inverse map is equivalent to saying that $p(x,i(x))$ is the composition of the unique map $p_P : P \to 1$ with e, we must also have $p(x, e \circ p_P) = p(e \circ p_P, x) = x$.

An action of the group object G on X is defined by a morphism $A : G \times X \to X$. The universal property of the product and the action induces a morphism: $h_{T,X} : Hom(T,G) \times Hom(T,X) \to Hom(T,X)$; for every points $g,g' : T \to G$, and $x : T \to X$, $(gg')x = g(g'x)$. The action is free if and only if for every T-point, g of G, $h_{T,X}(g,.)$ is injective. Remark that by using the Yoneda embedding, if h_G is a group object of $Sh(C,J)$, then G is also a group object of C.

Proposition 2.10. *Let (C,J) be a site and G a group object of $Sh(C,J)$ which acts freely on the geometric space X, and such that for every object $Y \in C$, the projection $Y \times G \to Y \in J(Y)$ then, the sheaf h_X/G is a geometric space.*

Proof. Let $(U_i)_{i\in I}$ be an atlas of X; Denote by p'_i the composition of p_i with the map $X \to X/G$ (see definition 5 for p_i). Then $(U_i, p'_i)_{i\in I}$ defines an atlas of X/G. To show this, firstly we consider an object Z of C, and a morphism $h : h_Z \to X/G$. The pullback of $X \to X/G$ by h is $h_Z \times h_G$; to show this, consider an element of $Hom(T,Z) \times_{Hom(T,X)/G} Hom(T,X)$ which is defined by an element u of $Hom(T,Z)$, and an element v of $Hom(T,X)$ which have the same image in $Hom(T,X)/G$. The elements of $Hom(T,Z) \times_{Hom(T,X)/G} Hom(T,X)$ whose image by the first projection is u are of the form $(u, gv), g \in Hom(T,G)$. Since the action of G is free, we deduce that $h_Z \times_{X/G} h_X = h_Z \times h_G$. Since $(U_i)_{i\in I}$ is an atlas of X, the pullback of $h_{Z\times G} \to X$ by $\bigcup_{i\in I} h_{U_i} \to X$ is in $J(Z \times G)$, since the map $Z \times G \to Z \in J(Z)$. We deduce that $(U_i, p'_i)_{i\in I}$ is an atlas of X/G. ◻

3 Sheaves of categories, $(2,2)$-gerbes, $(2,1)$-gerbes, $(1,2)$-gerbes

Let (C,J) be a category equipped with a Grothendieck topology, and $p : F \to C$ a functor. For every object X of C, we denote F_X, the sub-category of F obtained by restricting to objects $x \in F$ such that $p(x) = X$, and to arrows mapped by p to the identity arrow of X. A morphism $h : x \to x'$ between objects of F_X is an element of $h \in Hom_F(x,x')$, such that $p(h) = Id_X$. The category F_X is called the fibre of X. Let $f : X \to Y$ be a morphism of C, and $x \in F_X, y \in F_Y$. We denote by $Hom_f(x,y)$ the subset of the set of $Hom_F(x,y)$ such that for every element $h \in Hom_f(x,y)$, $p(h) = f$.

Definition 3.1. The morphism $h \in Hom_f(x,y)$ is Cartesian if and only if for every element $z \in F_X$, the canonical map $Hom_{Id_X}(z,x) \to Hom_f(z,y)$ which sends $l \to h \circ l$ is bijective. We say that the category F is a fibred category over C, if and only if for every morphism $f : X \to Y$ in C, and every element $y \in F_Y$, there exists a Cartesian morphism $c_f : x \to y$ such that $p(c_f) = f$.

Example 3.2. The forgetful functor $C/X \to C$, which sends $Y \to X$ to Y, is Cartesian (i.e takes Cartesian morphisms to Cartesian morphisms), as well as its restriction to any sieve of X.

Let $p : F \to C$ and $p' : F' \to C$ be Cartesian functors, we denote by $Cart(F,F')$ the class of morphisms between F and F' such that for every element $h \in Cart(F,F')$, we have $p' \circ h = p$, and h sends Cartesian morphisms to Cartesian morphisms.

Definition 3.3. Let $p : F \to C$ be a Cartesian functor. We say that the pair (F,p) is a sheaf of categories, if and only if:

- Let X be an object of C, for every sieve $R \in J(X)$, the forgetful functor $Cart(E/X,F) \to Cart(R,F)$ is an equivalence of categories.

We say that the sheaf of categories is connected if for every object X of C, there exists a sieve $R \in J(X)$, such that for every morphism $Y \to X \in R$, F_Y is not empty, and the objects of F_Y are isomorphic to each others. We are going to study only connected sheaves of categories here.

Let $f : X \to Y$ be a morphism of C, and y an object of F_Y, a restriction map of f is a Cartesian map $c_f : x \to y$, we say often that x is a restriction of y.

Suppose that the topology of C is generated by the family $(U_i)_{i\in I}$ (see [7] p. 221), we can assume that for every $i \in I$, the object of the fibre of U_i are isomorphic to each other. Choose an object $x_i \in F_{U_i}$, on U_{ij}, there exists a morphism $g_{ij} : x^i_j \to x^j_i$; the morphism $c_{ijk} = g^j_{ki} g^k_{ij} g^i_{jk}$ is an automorphism of x^{ij}_k. Which satisfies the relation:

$$c^j_{ikl} g^{ij}_{lk} c^l_{ijk} g^{ij}_{kl} = c^k_{ijl} c^i_{jkl}$$

We have seen that geometric spaces satisfy the condition above, in fact, there are examples of sheaves of categories.

Definition 3.4. A sheaf of categories on the Grothendieck site (C,J) is a gerbe, if and only if there exists a sheaf L on (C,J) such that for every object $x \in F_X$, $Aut_{Id_X}(x) \simeq L(U)$, and this identification commutes with morphisms between objects and with restrictions. We say that the sheaf of categories $p : F \to C$ is bounded by L, and L is its band.

Definition 3.5. *Suppose that the site (C,J) has a final object e; a gerbe is trivial if and only if it has a global section. This is equivalent to saying that the fibre F_e is not empty.*

A global section is called a torsor; equivalently a torsor is a gerbe $p : F \to C$ such that for every object X of C, the fibre F_X contains a unique object.

Definition 3.6. Let (C,J) be a Grothendieck site. Consider an object of $Sh(C,J)$ which is a variety X defined on (C,J). An n-sequence of fibred categories over X, is a sequence of functors, $p_n : F_n \to F_{n-1} ... p_1 : F_1 \to C/X = F_0$ which satisfies the following conditions:

- The functors $p_l, l = 1, ..n$ are fibred categories.

- For every object U of F_l, the fibre F_{l+1U} is a category whose objects are varieties of C, and its morphisms are morphisms of varieties over U (i.e on the site obtained by restricting (C,J) to C/U)

To define the notion of n-sequence of gerbes, we are going to associate firstly, to an automorphism above the identity of a gerbe bounded by a commutative sheaf, a 1-cocycle. Let h be an automorphism above the identity of the gerbe $p : F \to C/X$. Let $(U_i)_{i\in I}$ be 1-connected cover of X recall that this is equivalent to saying that the restriction of every sheaf to U_i is trivial. Let x_i be an object of F_{U_i}, there exists an arrow $l_i : x_i \to h(x_i)$. Let $u_{ij} : x^i_j \to x^j_i$ be a connecting morphism, on U_{ij} we have the morphism $h_{ji} = {l^i_j}^{-1} \circ h(u_{ij})^{-1} \circ l^j_i \circ u_{ij}$ of x^i_j. The following computation show that $h^i_{kj}u_{kj}h^k_{ji}u_{jk} = h^j_{ki}$:

$h^i_{kj}u_{kj}h^k_{ji}u_{jk} = {l^{ij}_k}^{-1}h(u^i_{kj})^{-1}l^{ik}_j u^i_{jk}u^i_{kj}{l^{ik}_j}^{-1}h(u^k_{ij})^{-1}l^{jk}_i u^k_{ij}u^i_{jk} = {l^{ij}_k}^{-1}h(u^i_{ij}u^k_{jk})^{-1}l^{jk}_i u^k_{ij}u^i_{jk}$.
By writing that $c_{ijk} = u^j_{ki}u^k_{ij}u^i_{jk}$, we obtain:

$h^i_{kj}u_{kj}h^k_{ji}u_{jk} = {l^{ij}_k}^{-1}h^{-1}(u^j_{ik}c_{ijk})l^{jk}_i u_{ik}c_{ijk}$. Since the group L is commutative and h commutes with morphisms between objects, $h^{-1}(u^j_{ik}c_{ijk})l^{jk}_i u_{ik}c_{ijk} = c^{-1}_{ijk}h^{-1}(u^j_{ik})l^{jk}_i c_{ijk}u_{ik} = h^{-1}(u^j_{ik})l^{jk}_i u_{ik}$.
This implies that $h^i_{kj}u_{kj}h^k_{ji}u_{jk} = h^j_{ki}$.

The cohomology (see S.G.A 4-2 for the definition of $\hat{C}ech$ cohomology on sites) class of the cocycle that we have just defined does not depend of the choices made. Suppose that we fix the x_i, but replace l_i by l'_i, then there exists $u_i \in L(U_i)$ such that $l'_i = u_i l_i$, and h_{ij} is replaced by ${u^i_j}^{-1}h_{ij}u^j_i$.

Suppose that we replace x_i by x'_i, let $v_i : x'_i \to x_i$ be a connecting morphism, $h(v_i)^{-1}l_i v_i$ is a connecting morphism l'_i between x'_i and $h(x'_i)$, $u'_{ij} = {v^j_i}^{-1}u_{ij}v^i_j$ is a connecting morphism between ${x'}^i_j$ and ${x'}^j_i$. We can write ${l'}^{i\,-1}_j h^{-1}(u'_{ij}){l'}^j_i u'_{ij} =$

$(h(v^i_j)^{-1}l^i_j v^i_j)^{-1}h({v^j_i}^{-1}u_{ij}v^i_j)^{-1}h(v^j_i)^{-1}l^j_i v^j_i {v^j_i}^{-1}u_{ij}v^i_j = {v^i_j}^{-1}h_{ij}v^i_j = h_{ij}$ since the elements of the band commute with morphisms between objects.

Definition 3.7. An n-sequence of fibred categories, $p_n : F_n \to F_{n-1}...p_1 : F_1 \to C/X = F_0$ is a n-sequence of gerbes, if and only if:

- For every object U of F_{n-2}, and e_U of F_{n-1U}, the fiber F_{ne_U} is a gerbe bounded by a sheaf L_{e_U} defined on C/e_U.

- Let U be an object of F_l. There exists a cover $(U_i)_{i\in I}$ of U, such that for every object e_i, e'_i of F_{l+1U_i}, and there exists an isomorphism between F_{l+2e_i} and $F_{l+2e'_i}$.

- There exists a commutative sheaf L on C called the band such that the trivial automorphisms (those corresponding to trivial torsors) of F_{ne_U} are the sections of L.

3.1 The classifying 4-cocycle

In the sequel, we will consider only 2-sequences of fibred categories that we call also $(2,2)$-gerbes, the general situation will be studied in a forthcoming paper.

We are going to associate a 4-cocycle to a 2-sequence $p_2 : F_2 \to p_1 : F_1 \to X$ bounded by a sheaf of commutative groups L. Let $(U_i)_{i\in I}$ be a cover of X, and x_i an object of F_{1U_i}, we denote by $g_{ij} : x^i_j \to x^j_i$ a connecting morphism. The morphism $c_{ijk} = g^j_{ki}g^k_{ij}g^i_{jk}$ is an automorphism of x_k^{ij}. Let U be an object of C/x_k^{ij}, for every object $U' \in F_{2U}$ we can lift the pullback of c_{ijk} by $U \to x_{ij}^k$, to a Cartesian morphism $U" \to U'$. If $h' : U' \to V'$ is a morphism above the morphism $h : U \to V \to x_k^{ij}$ between objects of C/x_k^{ij}, we can lift the pullback of h by c_{ijk} to a morphism $h" : U" \to V"$ in such a way that $h" : U" \to V" \to V'$ coincide with $U" \to h' : U' \to V'$. This shows that the correspondence which associates $U"$ to U' defines an automorphism c'_{ijk}, of the gerbe $F_{2x_k^{ij}}$. (See also Giraud [6] Scholie 1.6 p.3)

On U_{ijkl}, we have the morphisms $c^i_{jkl}, c^j_{ikl}, c^k_{ijl}, u^{ij}_{lk} c^l_{ijk} u^{ij}_{kl} = c'{}_{ijk}$ of x_l^{ijk}. The automorphism ${c'}^{l}_{ijk}{}^{-1} {c^j_{ikl}}^{-1} c^k_{ijl} c^i_{jkl}$ is an automorphism above the identity of x_l^{ijk}. We identify it with an element $c_{ijkl} \in C^1(x_l^{ijk}, L)$ up to a coboundary. The cohomology class of the $\hat{C}ech$ boundary c_{ijklm} of c_{ijkl} is trivial. Thus we can identify c_{ijklm} with an element of $L(U_{ijklm})$ The family c_{ijklm} is the classifying 4-cocycle of the $(2,2)$-gerbe.

3.2 $(2,1)$-gerbe, and $(1,2)$-gerbe

We will often need a particular 2-sequence of gerbes:

Definition 3.8. A gerbe-torsor or a $(2,1)$-gerbe is a $(2,2)$-gerbe $p_2 : F_2 \to p_1 : F_1 \to X$, which satisfies the following properties:

- For every object U of C/X, there exists a covering $(U_i)_{i\in I}$ of X such that for every object e_{U_i} of F_{1U_i}, the category $F_{2e_{U_i}}$ is a trivial gerbe over e_{U_i}.

- There exists a sheaf L such for every global section V of $F_{2e_{U_i}}$, there exists an isomorphism between $Aut_{e_{U_i}}(V)$, the group of automorphisms of V over the identity of e_{U_i} with $L(U_i)$, and this isomorphism commutes with morphisms between objects and with restrictions.

The classifying cocycle of a $(2,1)$-gerbe.

We are going to associate to a gerbe-torsor, a 3-cocycle defined as follows:

Let x_i be an object of F_{1U_i}, and $u_{ij} : x^i_j \to x^j_i$ a morphism, we can define the cocyccle $c_{ijk} = u^j_{ki} u^k_{ij} u^i_{jk}$ of x_k^{ij}. Since the gerbe $F_{2x_k^{ij}}$ is trivial, we can pick V, a global section over x_k^{ij}, in this situation, let c'_{ijk} be a morphism of V above c_{ijk}. The $\hat{C}$ech coboundary of c'_{ijk} is an automorphism above the identity of x_l^{ijk} that we identify with an element of $L(U_{ijkl})$. The family of morphisms c_{ijkl} defines a 3-cocycle which is L-valued.

Remark 3.9. Suppose that the cohomology class of c_{ijkl} is zero, we can assume that $c_{ijkl} = 0$. This is equivalent to saying that c'_{ijk} is the classifying cocycle of a gerbe $p' : F' \to C/X$, such that for every object U of C/X, the objects of F'_U are torsors over e_U, where e_U is an object of F_{1U}. The classifying cocycle of the $(2,1)$-gerbe can be viewed as an obstruction to obtain such a gerbe, that is, to reduce the trivial gerbe F_{2e_U} to a torsor.

There is a natural manner to associate to a $(2,2)$-gerbe $p_2 : F_2 \to p_1 : F_1 \to C/X$ a $(2,1)$-gerbe $p'_2 : F'_2 \to p_1 : F_1 \to C/X$ defined as follows: Let U, be an object of C/X, and e_U an object of F_{1U}, the trivial gerbe F'_{2e_U} is the gerbe whose objects are the automorphisms of F_{2e_U} above morphisms of e_U. Remark that the classifying cocycle of this $(2,1)$-gerbe is the class c_{ijkl} that we have used to define the classifying cocycle of (p_2, p_1). Thus (p_2, p_1) is trivial if and only if (p'_2, p_1) is trivial. This allows to interpret a $(2,2)$-gerbe as a geometric obstruction.

Definition 3.10. A $(1,2)$-gerbe is a $(2,2)$-gerbe $p_2 : F_2 \to p_1 : F_1 \to C$, such that the gerbe $p_1 : F_1 \to C$ is trivial.

The classifying 3-cocycle of a $(1,2)$-gerbe.

Suppose that the covering $(U_i)_{i\in I}$ is a good covering, and $x_i = F_{1U_i}$ is isomorphic to the trivial $L(U_i)$-torsor t_i on U_i. Let $h_i : x_i \to t_i$ be an isomorphism, consider the automorphism $c_{ij} = h_i^{j^{-1}} h_j^i$ of x_{ij}. Let U be an object of C/x_{ij}. We can lift c^i_{jk} to an automorphism c'^i_{jk} of $F_{2x_{ijk}}$. The $\hat{C}ech$ boundary of c'^i_{jk} is an automorphism of the gerbe $F_{2x_{ijk}}$ above the identity that we identify with an element of $c_{ijk} \in C^1(U_{ijk}, L)$ up to a coboundary. The cohomology class of the boundary c_{ijkl} of c_{ijk} is trivial. We can thus identify c_{ijkl} to an element of $L(U_{ijkl})$. The family c_{ijkl} is the classifying cocycle of the $(1,2)$-gerbe.

3.3 Examples: The lifting obstruction

Let (C,J) be a site, and $0 \to L \to M \to N \to 0$ an exact sequence of commutative sheaves defined on C. It defines the following exact sequence in cohomology:

$$H^n(X,L) \to H^n(X,M) \to H^n(X,N) \to H^{n+1}(X,L)$$

Let $[c^n]$ be an element of $H^n(X,N)$, represented by the n-cocycle c^n of the sheaf N. A natural problem is to find obstructions to lift the n-cocycle c^n to a class in $H^n(X,M)$.

Recall the construction of the coboundary operator $H^n(X,N) \to H^{n+1}(X,L)$. Let $(U_i)_{i\in I}$ be a good cover of (C,J), the restriction of M and N on U_i are trivial. This implies the existence of a global section $b^n_i \in M(U_{i_1..i_{n+1}})$ over c^n_i, the restriction of c_n to $U_{i_1..i_{n+1}}$. We can write the coboundary c^{n+1} of the chain b^n_i it is an $(n+1)$-L-cocycle whose cohomology class is the image of $[c^n]$ by the boundary operator. The cohomology classes of the $(n+1)$-L-cocycles which are in the image of the connecting morphism $H^n(X,N) \to H^{n+1}(X,L)$ are in bijection with the quotient of $H^n(X,N)$ by the image of the morphism $H^n(X,M) \to H^n(X,N)$.

The space $H^0(X,N)$ classifies the global sections of the sheaf N, and $H^1(X,L)$ the L-torsors, we obtain that isomorphism classes L-torsors whose classifying cocycles are in the image of the connecting morphism $H^0(X,N) \to H^1(X,L)$ are in bijection with the quotient of $H^0(X,N)$ by the image of the morphism $H^0(X,M) \to H^0(X,N)$.

If $n = 1$, $H^1(X,N)$ classifies the torsors of the sheaf N, and $H^2(X,L)$ the L-gerbes, we obtain that isomorphism classes of L-gerbes whose classifying cocycles are in the image of the connecting morphism $H^1(X,N) \to H^2(X,L)$ are in bijection with the quotient of $H^1(X,N)$ by the image of the morphism $H^1(X,M) \to H^1(X,N)$.

Let $p^i_2 : F^i_2 \to p^i_1 : F^i_1 \to C/X, i = 1,2$ be two $(2,1)$-gerbes such that p^i_1 is a gerbe bounded by N, and (p^i_1, p^i_2) by L. We say that they are isomorphic if and only if their respective 3-cohomology classes associated to these gerbes are equal.

Proposition 3.11. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of sheaves defined on the site (C,J). Suppose that the morphism of sheaves $M \to N$ has local sections. Then the isomorphism classes of $(2,1)$-gerbes $p_2 : F_2 \to p_1 : F_1 \to C/X$ such that p_1 is a gerbe bounded by L, and (p_1, p_2) by N, whose classifying cocycles are in the image of the connecting morphism $l_2 : H^2(X,N) \to H^3(X,L)$ are in bijection with the quotient of $H^2(X,N)$ by the image of the morphism $H^2(X,M) \to H^2(X,N)$.*

Proof. We need only to construct for every cohomology class $[c_3] \in H^3(X,L)$ in the image of the connecting morphism $l_2 : H^2(X,N) \to H^3(X,L)$, a $(2,1)$-gerbe classified by $[c_3]$. Set $[c_3] = l_2([c_2])$. Let $p : F \to C$ be an N-gerbe bounded by $[c_2]$. By assumption, for every object U of C, the objects of the fiber F_U are N-torsors. Let e_U be an object of F_U, we define F_{2e_U} to be the category whose objects are M-torsors $p_U : V_U \to e_U$ whose quotient by L is e_U. A morphism between two objects of F_{2e_U} is a morphism of M-bundles which projects to the identity of e_U. If $(U_i)_{i\in I}$ is a good cover of C, and e_i an object of F_{U_i}. The objects of F_{2e_i} are isomorphic; they are trivial bundles since the map $M \to N$ has local sections. The projection $F_2 \to F_1$ is the projection which sends the M-bundle $V_U \to e_U$ to e_U, this projection is Cartesian. Let $p : V'_{U'} \to e_{U'}$ be an element of $F_{2e_{U'}}$, an $f : e_U \to e_{U'}$ a morphism, the pullback of p by f is a Cartesian morphism above f. □

Remark 3.12. Let $p : F \to C/X$ and $p' : F' \to C/X$ two gerbes bounded by N, we can define the summand $F + F'$ of F and F': The objects of $(F + F')_U$ are sum of N-bundles e_U and e'_U, where U is an object of C/X, and e_U (resp. e'_U) an object an object of F_U (resp. F'_U).

Consider the $(2,1)$-gerbes $p_2 : F_2 \to p_1 : F_1 \to C/X$ and $p'_2 : F'_2 \to p'_1 : F'_1 \to C/X$ whose classifying cocycles are image of l_2. Two such cocycles are isomorphic if and only if there exists a N-gerbe F_1" whose classifying cocycle is in the image of $H^2(X,M) \to H^2(X,N)$ and such that $F_1 = F'_1 + F"_1$.

4 Applications to algebraic geometry: Chow groups and higher divisors

In the sequel, X will be a quasi-projective variety of dimension n defined on the field k, L_X the sheaf of non zero rational functions defined on X. We endow X with the Zariski topology. Let U be an open subset of X, and $f \in L_X(U)$, we denote by (f) the principal divisor associated to f. The multiplicative group $L_X(U)$ is a $\mathbb{Z}$-group, for the action defined by $(a, f) \to f^a, a \in \mathbb{Z}, f \in L_X(U)$. Let h be an element of $L^l_X(U)$, $h = (h_1,...,h_l)$, where $h_i = \frac{a_i}{b_i}, i = 1,...,l$ and a_i, b_i are regular functions. Denote by $CH^l_X(U)$ the linear subspace generated by the set of irreducible closed subvarieties of U of codimension l which are local complete intersections; we define $ch_l(U) : L^l_X(U) \to CH^l_X(U)$ which sends h to the intersection product $(a_1 - b_1)...(a_l - b_l) \in CH^l_X(U)$. Remark that the theorem 1 V.21 of Serre [15] describes the elements of the image of $ch_l(U)$ as complete intersections codimension l subvarieties, since it implies that if a component of $(a_i - b_i)$ and a component of $(a_j - b_j)$ do not intersect properly, their coefficient in $(a_i - b_i).(a_j - b_j)$ is zero.

The map $ch_l(U)$ is l-multilinear for the multiplicative structure, it thus factors by a linear map $ch'_l(U) : L_X(U)^{\otimes l} \to CH^l_X(U)$ which factors by the quotient map $L^{\otimes l}_X(U) \to M^l_X(U)$, where $M^l_X(U)$ is the symmetric functions in l-variables on $L_X(U)$ for the multiplicative structure, that is is the quotient of $L^{l\otimes}_X(U)$ by its subset generated by elements $(x_1 \otimes .. \otimes x_l) - \sigma(x_1 \otimes .. \otimes x_l), \sigma \in S_l$. Since the element of $CH_X(U)$ are local complete intersections, for each integer l, we have an exact sequence of sheaves:

$$1 \to Z_X(l) \to M^l_X \to CH^l_X \to 1. \tag{1}$$

Where $Z_X(l)$ is the kernel of the morphism $M^l_X \to CH^l_X$; we deduce the existence of the following exact sequence in cohomology:

$$(2) \qquad H^p(X,Z_X(l)) \to H^p(X,M^l_X) \to H^p(X,CH^l_X) \to H^{p+1}(X,Z^X_X(l)).$$

Let $p = 0,1,2,3$, we define a p-gerbe bounded by the sheaf L, to be a global section of L if $p = 0$, a L-bundle if $p = 1$, a L-gerbe if $p = 2$, and a $(2,1)$-L-gerbe if $p = 3$. In the sequel p is an integer equal to $0,1$ or 2. This restriction is due to the fact that for $n > 3$, we cannot provide at this time a geometric interpretation of this notion.

Definition 4.1. (p,l)-Cartier divisor, is defined by a p-chain $(U_{i_1..i_{p+1}}, f_{i_1..i_{p+1}})$ of sections of M^l_X such that the image of the $\hat{C}ech$ boundary $d(f_{i_1..i_{p+1}}) \in Z_X(l)(U_{i_1..i_{p+2}})$. This boundary is thus the classifying cocycle of a $p+1$-$Z_X(l)$-gerbe $A(p,l)$. To this gerbe, is associated the p-gerbe $B(p,l)$ bounded by $M^l_X/Z_X(l)$ whose classifying cocycle is defined by the classes of $f_{i_1..i_{p+1}}$ in $M^l_X(U_{i_1..i_{p+1}})/Z_X(l)(U_{i_1..i_{p+1}})$.

Proposition 4.2. *Let X be a quasi-projective variety of dimension n defined on the field k, the set isomorphism classes of (p,l)-Cartier divisors is the quotient of $H^p(X,CH^l_X)$ by the image of the morphism $h_{p,l} : H^p(X,M^l_X) \to H^p(X,CH^l_X)$ deduced from the exact sequence (2).*

Proof. The classifying cocycle of the $(p+1)$-gerbe $A(p,l)$, is the image of the classifying cocycle of $B(p,l)$ by the connecting morphism $H^p(X,M^l_X/Z_X(l)) \to H^{p+1}(X,Z_X(l))$. By comparing (2) with the exact sequence $H^p(X,Z_X(l)) \to H^p(X,M^l_X) \to H^p(X,M^l_X/Z_X(l)) \to H^{p+1}(X,Z_X(l))$ deduced from the exact sequence $1 \to Z_X(l) \to M^l_X \to M^l_X/Z^l_X$. We deduce that $H^p(X,M^l_X/Z_X(l))$ is isomorphic to $H^p(X,CH^l_X)$. The isomorphism classes of the (p,l)-Cartier divisors is the quotient $H^p(X,CH^l_X)$ by the image of the morphism $h_{p,l}$: $H^p(X,M^l_X) \to H^p(X,CH^l_X)$. □

Remark 4.3. The elements of $H^p(X,CH^l_X)$ are called (p,l)-Weil divisors. Two (p,l)-Weil divisors D_W and D'_W are equivalent if and only if $D_W - D'_W$ is an element of the image of $h_{p,l}$.

The $(p+1)$-chain $d(f_{i_1..i_{p+1}})$ is a boundary of elements of M^l_X. Thus correspond to a trivial M^l_X bundle if $p = 0$, a trivial M^l_X-gerbe if $p = 1$, and a trivial M^l_X-2-gerbe if $p = 2$, (See Brylinski-McLaughin [3] for the definition of 2-gerbe).

If $p = 0$, and $l = 1$ a $(0,1)$-Cartier (resp. a $(0,1)$-Weil divisor) divisor is nothing but a Cartier divisor (resp. a Weil divisor) in the classical sense. Two $(0,1)$-Weil divisors are equivalent if and only if they are equivalent in the classical sense.

More generally two $(0,l)$-divisors which are equivalent are rationally equivalent: this follows from the following argument: let D_W and D'_W be two Weil divisors, suppose that: $D'_W = D_W + (a_1)...(a_l)$, where $a_i,...,a_l$ are regular functions. Then $D'_W - D_W$ is a principal divisor of $(a_1)...(a_l)$, it follows from Hartshorne [9] p. 426, that D_W and D'_W are rationally equivalent.

Suppose that X is an affine variety X; since the sheaf of rational functions L_X is constant, we deduce that $H^p(X,M^l_X) = 0, p > 0$, and $H^0(X,M_X) = K(X)$ the field of rational functions of X. This implies that $H^p(X,CH^l_X) = H^{p+1}(X,Z_X(l))$.

Suppose that $l = 1$, then $Z_X(1) = O_X^*$ the sheaf of invertible regular functions, we have: $H^0(X, O_X^*) = O_X^*(X)$, $H^1(X, O_X^*) = Pic(X)$ the Picard group of X, and $H^p(X, O_X^*) = 0$ if $p > 1$.

4.1 The Cartier divisor associated to a local complete intersection subvariety

Let Y be a closed subvariety of the quasi-projective variety X of codimension l which is a local complete intersection, consider an open cover $(U_i)_{i\in I}$ by affine subsets, such that $U_i \cap Y$ is the locus l functions $(f_i^1, ..., f_i^l)$ which defines the element $F_i \in M_X^l(U)$ obtained by projecting the image of $(f_i^1, ..., f_i^l)$. The element $h_{ij} = F_j - F_i$ is in $Z_X^l(U_i \times_X U_j)$.

Proposition 4.4. *The element h_{ij} is a 1-$\hat{C}ech$ Z_X^l cocycle. If Y is irreducible, then its cohomological class vanishes if and only if Y is a global intersection.*

Proof. The $\hat{C}ech$ cocycle h_{ij} is the boundary of the M_X^l 0-cocycle $F_j - F_j$, this implies that $(h_{ij})_{i,j\in I}$ is a 1-$\hat{C}ech$ Z_X^l-cocycle. Suppose that the class $[h_{ij}]$ of (h_{ij}) vanishes, this implies the existence of a 0-chain f_i of $Z_X(l)$, such that $h_{ij} = f_j^i - f_i^j$. The boundary of $(F_i - f_i)$ is zero, this implies that $F_i - f_i$ is the restriction of a global section F of M_X^l; we can suppose that F is the class of $(h_1, ...h_c)$ since Y is irreducible. This implies that Y is the locus of $h_1, ..., h_c$. Conversely, suppose that Y is the complete intersection of $(f_1, ..., f_l)$. Then we can take F_i to be the restriction to U_i of the projection of $(f_1, ..., f_l)$ to M_X. This implies the result. □

Example 4.5. Suppose that $X = Spec(k)$, $L_X = k^*$ the set of non zero elements of k, for every element $a \in k^*$, $(a) = 0$. This implies that $Z_X(l) = k^{*\otimes l}$.

Suppose that X is a curve; if $l > 1$, $CH_X^l = 0$.

Proposition 4.6. *Let $X = P^2k$ there exists a non trivial $Z_X(2)$-gerbe defined on X.*

Proof. First we construct a non trivial element of $H^1(X, CH_X^2)$. We can cover X with the three open subsets $U_i = \{[X_1, X_2, X_3], X_i \neq 0\}, i = 1, 2, 3$. On $U_i \cap U_j$, c_{ij} is the homogeneous element whose i and j coordinates are 1, and the other is 0; it is the intersection of the lines defined by $X_i - X_j$ and $X_k, k \neq i, j$. Since $U_{ijk} = \{[X_1, X_2, X_3] : X_i \neq 0, i = 1, 2, 3\}$, it implies that $c_{ij}^k = 0$. Thus the family (c_{ij}) defines a cocycle. This cocycle is not a boundary: Suppose that there exists a chain $(c_i)_{i\in I}$ such that $c_{ij} = c_j^i - c_i^j$. Write $c_i = l_i^1 h_i^1 + .. + l_i^{j_1} h_i^{j_i}, i = 1, 2, 3, l_i^1, .., l_i^{j_i} \in \mathbb{Z}, h_i^{j_n} \in U_i$. Suppose that the second homogeneous coordinate of a component $h_1^{j_l}$ is not zero, then its third homogeneous coordinate is not zero since $c_{13} = c_3^1 - c_1^3$, if its third homogeneous coordinate is not zero, then its second homogeneous coordinate is not zero also. Since its first coordinate is not zero, we deduce that the coordinates of $h_1^{j_l}$ are not zero, this argument implies that $h_i^{j_l} \in U_{ijk}, i = 1, 2, 3$. This is in contradiction with the fact that $c_{ij} = c_j^i - c_i^j$. Thus the class $(c_{ij})_{i,j\in I}$ defines a non trivial $Z_{P^2k}(2)$-gerbe on P^2k.

We have to show that the class c defined by $(c_{ij})_{i,j\in I}$ is not in the image of $H^1(X, M_X^2) \to H^1(X, CH_X^2)$. Suppose it is in that image, and let h be an element in its preimage. We denote by h_{ij}, the value of h on U_{ij}. Suppose $i = 1, j = 2$ we can represent it by a couple $(f_{12}, g_{12}), \in L_X^2$ such that the intersection of the divisors of f_{12} and g_{12} is $[1, 1, 0]$. Since on U_{123}, we can write h^3_{12} as a combination of h^2_{13} and h^1_{23} by applying the cocycle condition, this combination can be written in U_{12}, but this is impossible, since the locus of the components of h_{13} and h_{23} does not contain in c_{12} □

4.2 The Étale topology

Suppose that X is an integral quasi-compact scheme equipped with the étale topology, we denote respectively by U_X and Div_X the sheaves of non zero rational functions and the quotient of U_X by O^*_X, the sheaf of non zero regular functions defined on the étale topology. Let $Z_X(1)$ be the kernel of the map $U_X \to U_X/O^*_X$, we have an exact sequence: $H^n_{et}(X,Z_X(1)) \to H^n_{et}(X,U_X) \to H^n_{et}(X,Div_X) \to H^{n+1}_{et}(X,Z_X(1))$.

We can define the notion of p-Cartier étale gerbe to be the quotient of $H^p(X,Div_X)$ by the image of the map $H^p(X,U_X) \to H^p(X,Div_X)$. It is shown that if $p=1$, $H^1_{et}(X,Z_X(1)) = H_{Zar}(X,Z_X(1)) = Pic(X)$. If X is smooth, then the sheaf of étale Cartier divisors can be identified with the sheaf of Weil étale divisors whose sections are summands of irreducible codimension 1 varieties. This implies that $H^i(X,Div_X) = \sum_{x\in X^1} H^i(k(x),\mathbb{Z}), i=1,2$, where X^1 is a closed point of codimension 1, and $k(x)$ its residue field. The Hilbert 90 theorem implies that $H^1(k(x),\mathbb{Z}) = 0$. This implies that the 1-étale Cartier gerbes are trivial if X is smooth.

4.3 The Brauer group

Let k be a field, the Brauer group of k is $H^2_{et}(Spec(k),\bar{k}^*)$, where $\bar{k}$ is the algebraic closure of k. Let $Gl(n,\bar{k})$ be the linear group of invertible n-matrices, and $PGl(n,\bar{k})$ the corresponding projective group. The exact sequence $1 \to \bar{k}^* \to Gl(n,\bar{k}) \to PGl(n,\bar{k}) \to 1$, induces an exact sequence $H^n_{et}(Spec(k),\bar{k}^*) \to H^p_{et}(Spec(k),Gl(n,\bar{k})) \to H^p_{et}(Spec(k),PGl(n,\bar{k})) \to H^{p+1}_{et}(Spec(k),\bar{k}^*)$. If $p=1$, it is shown in Serre [14] (proposition 9. p. 166) that every class in $H^2_{et}(Spec(k),\bar{k}^*)$ is in the image of a morphism $H^1_{et}(Spec(k),PGl(n,\bar{k})) \to H^2_{et}(Spec(k),\bar{k}^*)$, for a given n, thus every element of the Brauer group is the classifying cocycle of a gerbe, which is the geometric obstruction to lift a torsor on the étale topos $Spec(k)$.

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