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WELL-POSEDNESS RESULT FOR A NONLINEAR ELLIPTIC PROBLEM INVOLVING VARIABLE EXPONENT AND ROBIN TYPE BOUNDARY CONDITION

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Abstract

In this work we study the following nonlinear elliptic boundary value problem, $b(u) - div\, a(x,\nabla u) = f$ in Ω, $a(x,\nabla u).\eta = -|u|^{p(x)-2}u$ on $\partial\Omega$, where Ω is a smooth bounded open domain in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary $\partial\Omega$. We prove the existence and uniqueness of a weak solution for $f \in L^\infty(\Omega)$, the existence and uniqueness of an entropy solution for L^1-data f. The functional setting involves Lebesgue and Sobolev spaces with variable exponent

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1 Introduction

This paper is motived by phenomena which are described by Robin type boundary problem of the form

$$\begin{cases} b(u) - div\, a(x,\nabla u) = f \text{ in } \Omega \\ \\ a(x,\nabla u).\eta = -\,|u|^{p(x)-2}\, u \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded open domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial\Omega$ and η the outer unit normal vector on $\partial\Omega$. When $p(.) \equiv 2$, we obtain an homogeneous Robin condition. Therefore, (1.1) includes a Robin boundary problem.

The study of problems involving variable exponent has received considerable attention in recent years (cf. [4,5,7-17,19-27, 29-34]) due to the fact that they can model various phenomena which arise in the study of elastic mechanics (see [4]), electrorheological fluids (see [11,22,29,30]) or image restauration (see [9]).

When the boundary value condition is a Neumann or Robin boundary condition in the context of variable exponent, we must work in general with the space $W^{1,p(.)}(\Omega)$ instead of the common space $W_0^{1,p(.)}(\Omega)$. The main difficulty which appears in this case of existence and also uniqueness of solutions is that the famous Poincar inequality does not apply (see [8]). We must use the Poincar-Wirtinger inequality instead of the Poincar inequality but due to the average number, it is not easy to use the Poincar-Wirtinger inequality to obtain appropriate properties for problem involving more general operator and data considered in this paper. We use in this paper a Poincar-Sobolev type inequality to get the main apriori estimate for the proof of the existence and uniqueness of entropy solution (see the proof of proposition 4.7 below). Recently, Ouaro (see [25]) studied the following problem

$$\begin{cases} -\text{div}\; a(x,\nabla u) + |u|^{p(x)-2}\, u = f \text{ in } \Omega, \\ \\ a(x,\nabla u).\eta = \varphi \;\text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

under the following assumptions:

$$\begin{cases} p(.) : \Omega \to \mathbb{R} \text{ is a measurable function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{1.3}$$

where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$.

For the vector fields $a(.,.)$, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathodory and is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, i.e. $a(x,\xi) = \nabla_\xi A(x,\xi)$ such that:

- The following equality holds

$$A(x,0) = 0, \tag{1.4}$$

for almost every $x \in \Omega$.

- There exists a positive constant C_1 such that

$$|a(x,\xi)| \leq C_1 \left(j(x) + |\xi|^{p(x)-1} \right) \tag{1.5}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $1/p(x) + 1/p'(x) = 1$.

- There exists a positive constant C_2 such that for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$,

$$(a(x,\xi) - a(x,\eta)) \,.\, (\xi - \eta) > 0. \tag{1.6}$$

- The following inequalities hold

$$|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi) \tag{1.7}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

Under assumptions (1.3)-(1.7), Ouaro (see [25]) proved the existence and uniqueness of entropy solutions of problem (1.2) for L^1-data f and φ. Assumption on the function A and the use of the quantity $|u|^{p(x)-2}\, u$ allowed Ouaro, in particular, to exploit a minimization method for the proof of existence of a weak solution for (1.2) when the data f and φ are in $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$ respectively [25]. Note also that the uniqueness of weak and entropy solutions of (1.2) in [25] is due to the fact that $s \mapsto |s|^{p(x)-2}\, s$ is increasing.

In this paper, we improve the result in [25] by making less regularity on the data a and b. More precisely:

$$\begin{cases} p(.) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{1.8}$$

and

$$\begin{cases} b : \mathbb{R} \to \mathbb{R} \text{ is continuous, surjective, nondecreasing function} \\ \text{such that } b(0) = 0. \end{cases} \tag{1.9}$$

For the vector field $a(.,.)$, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory such that:

- there exists a positive constant C_2 with

$$|a(x,\xi)| \leq C_2 \left(j(x) + |\xi|^{p(x)-1} \right) \tag{1.10}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

- there exists a positive constant C_3 such that for every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, the following inequalities hold:

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0 \tag{1.11}$$

and

$$a(x,\xi).\xi \geq C_3 |\xi|^{p(x)} \tag{1.12}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

The remaining part of the paper is the following: in section 2, we introduce some notations/functional spaces. In section 3, we prove the existence and the uniqueness of weak solution of (1.1) when the data $f \in L^\infty(\Omega)$. Using the results of section 3, we study in section 4, the question of the existence and the uniqueness of entropy solution of (1.1) when the data $f \in L^1(\Omega)$.

2 Assumptions and preliminaries

As the exponent $p(.)$ appearing in (1.10) and (1.12) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} = \inf\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-} \right) |u|_{p(.)} |v|_{p'(.)}, \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Let

$$W^{1,p(.)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equiped with the following norm

$$||u||_{1,p(.)} = |u|_{p(.)} + |(|\nabla u|)|_{p(.)}.$$

The space $(W^{1,p(.)}(\Omega),||.||_{1,p(.)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result (see [16]):

Lemma 2.1 *If $u_n, u \in L^{p(.)}(\Omega)$ and $p_+ < +\infty$, then the following properties hold:*
(i) $|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p_-} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p_+}$;
(ii) $|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p_+} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p_-}$;
(iii) $|u|_{p(.)} < 1$ $(respectively = 1; > 1) \Leftrightarrow \rho_{p(.)}(u) < 1$ $(respectively = 1; > 1)$;
(iv) $|u_n|_{p(.)} \to 0$ $(respectively \ \to +\infty) \Leftrightarrow \rho_{p(.)}(u_n) \to 0$ $(respectively \ \to +\infty)$;
(v) $\rho_{p(.)}\left(u/|u|_{p(.)}\right) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the following notation:

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

We have the following lemma (see [32,34]):

Lemma 2.2 *If $u \in W^{1,p(.)}(\Omega)$ then the following properties hold*:
(i) $\|u\|_{1,p(.)} < 1(respectively = 1; > 1) \Leftrightarrow \rho_{1,p(.)}(u) < 1(respectively = 1; > 1)$;
(ii) $\|u\|_{1,p(.)} < 1 \Leftrightarrow \|u\|_{1,p(.)}^{p_+} \le \rho_{1,p(.)}(u) \le \|u\|_{1,p(.)}^{p_-}$;
(iii) $\|u\|_{1,p(.)} > 1 \Leftrightarrow \|u\|_{1,p(.)}^{p_-} \le \rho_{1,p(.)}(u) \le \|u\|_{1,p(.)}^{p_+}$.
(iv) $\|u_n\|_{1,p(.)} \to 0$ $(respectively \ \to +\infty) \Leftrightarrow \rho_{1,p(.)}(u_n) \to 0$ $(respectively \ \to +\infty)$;

Put

$$p^\partial(x) := (p(x))^\partial := \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}, \text{ if } p(x) < N \\ \infty, \text{ if } p(x) \ge N; \end{cases}$$

then we have the following embedding result:

Proposition 2.3 *Let $p \in C(\bar{\Omega})$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 \le q(x) < p^\partial(x),\ \forall\, x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.

Let us introduce the following notation: given two bounded measurable functions $p(.), q(.)$: $\Omega \to \mathbb{R}$, we write

$$q(.) \ll p(.) \ \text{ if } \ ess \inf_{x\in\Omega} (p(x) - q(x)) > 0.$$

Remark 2.4. Observe that we use the same notation f for f and its trace when convenient.

3 Existence and uniqueness of weak solution

In this part, we study the existence and the uniqueness of a weak solution of (1.1) when the data $f \in L^\infty(\Omega)$.

Definition 3.1 *A weak solution of* (1.1) *is a measurable function u such that*

$$u \in W^{1,p(.)}(\Omega),\ b(u) \in L^{\infty}(\Omega), |u|^{p(.)-2}u \in L^{\infty}(\partial\Omega)$$

and

$$\int_{\Omega} a(x,\nabla u).\nabla\varphi\,dx + \int_{\Omega} b(u)\varphi\,dx + \int_{\partial\Omega} |u|^{p(x)-2}u\varphi\,d\sigma = \int_{\Omega} f\varphi\,dx,\ \ \forall\,\varphi \in W^{1,p(.)}(\Omega), \quad (3.1)$$

where $d\sigma$ is the surface measure on $\partial\Omega$.

Notice that the integrals in (3.1) are well defined since for the third integral in the left-hand side, we can use the fact that the trace of $\varphi \in W^{1,p(.)}(\Omega)$ on $\partial\Omega$ is well defined in $L^p(\partial\Omega)$, for $1 \leq p < +\infty$. The main result of this part is the following:

Theorem 3.2. *Assume that (1.8)-(1.12) hold and* $f \in L^{\infty}(\Omega)$. *Then there exists a unique weak solution of* (1.1).
Proof.
Part 1: Existence
For $k > 0$, we consider the following approximated problem:

$$\begin{cases} T_k(b(u_k)) - div\, a(x,\nabla u_k) = f \text{ in } \Omega \\ a(x,\nabla u_k).\eta = T_k(-|u_k|^{p(x)-2}u_k) \text{ on } \partial\Omega, \end{cases} \quad (3.2)$$

where for any $k > 0$, the truncation function T_k is defined by $T_k(s) := \max\{-k, \min\{k,s\}\}$. Note that as $T_k(b(u_k)) \in L^{\infty}(\Omega)$ and $T_k(|u_k|^{p(x)-2}u_k) \in L^{\infty}(\partial\Omega)$, thanks to [21, Theorem 3.1], there exists $u_k \in W^{1,p(.)}(\Omega)$ which is a weak solution of (3.2).
We recall that for any $\varepsilon > 0$,

$$H_{\varepsilon}(s) = \min\left\{\frac{s^+}{\varepsilon}, 1\right\},$$

$$sign_0^+(s) = \begin{cases} 1 \text{ if } s > 0 \\ 0 \text{ if } s \leq 0 \end{cases}$$

and if γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ, i.e.

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \emptyset, \\ +\infty \text{ if } [s,+\infty) \cap D(\gamma) = \emptyset, \\ -\infty \text{ if } (-\infty,s] \cap D(\gamma) = \emptyset. \end{cases}$$

We now show that $|b(u_k)| \leq \|f\|_{L^{\infty}(\Omega)}$ a.e. in Ω and $|u_k| \leq b_0^{-1}\left(\|f\|_{L^{\infty}(\Omega)}\right)$ a.e. in $\partial\Omega$ for all $k > 0$.
We take $\varphi = H_{\varepsilon}(u_k - M)$ as a test function in (3.1) for the weak solution u_k and $M > 0$ a

constant to be chosen later.
We have

$$\int_\Omega a(x,\nabla u_k).\nabla H_\varepsilon(u_k-M)\,dx+\int_\Omega T_k(b(u_k))H_\varepsilon(u_k-M)\,dx+ \int_{\partial\Omega} T_k(|u_k|^{p(x)-2}u_k)H_\varepsilon(u_k-M)\,d\sigma=\int_\Omega fH_\varepsilon(u_k-M)dx. \tag{3.3}$$

Let $J:=\int_\Omega a(x,\nabla u_k).\nabla H_\varepsilon(u_k-M)\,dx$.
We deduce that $J=\frac{1}{\varepsilon}\int_{\{0<u_k-M<\varepsilon\}} a(x,\nabla u_k).\nabla H_\varepsilon(u_k-M)\,dx\geq 0$ then, according to (3.3), we obtain:

$$\int_\Omega T_k(b(u_k))H_\varepsilon(u_k-M)\,dx \quad +\int_{\partial\Omega} T_k(|u_k|^{p(x)-2}u_k)H_\varepsilon(u_k-M)\,d\sigma \leq \int_\Omega fH_\varepsilon(u_k-M)\,dx, \tag{3.4}$$

which is equivalent to say

$$\int_\Omega (T_k(b(u_k))-T_k(b(M)))\,H_\varepsilon(u_k-M)\,dx \quad +\int_{\partial\Omega} T_k(|u_k|^{p(x)-2}u_k)H_\varepsilon(u_k-M)\,d\sigma \leq \int_\Omega (f-T_k(b(M)))\,H_\varepsilon(u_k-M)\,dx. \tag{3.5}$$

As the two terms in the left-hand side in (3.5) are nonnegative then we deduce that

$$\int_\Omega (T_k(b(u_k))-T_k(b(M)))H_\varepsilon(u_k-M)\,dx\leq \int_\Omega (f-T_k(b(M)))H_\varepsilon(u_k-M)\,dx \tag{3.6}$$

and

$$\int_{\partial\Omega} T_k(|u_k|^{p(x)-2}u_k)H_\varepsilon(u_k-M)\,d\sigma\leq \int_\Omega (f-T_k(b(M)))H_\varepsilon(u_k-M)\,dx. \tag{3.7}$$

We now let ε goes to 0 in (3.6) and (3.7) to get:

$$\int_\Omega (T_k(b(u_k))-T_k(b(M)))^+\,dx\leq \int_\Omega (f-T_k(b(M)))sign_0^+(u_k-M)\,dx \tag{3.8}$$

and

$$\int_{\partial\Omega} T_k(|u_k|^{p(x)-2}u_k)sign_0^+(u_k-M)\,d\sigma\leq \int_\Omega (f-T_k(b(M)))sign_0^+(u_k-M)\,dx. \tag{3.9}$$

Choosing now $M=b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right)$ in (3.8) and (3.9)(M is a constant since b is onto) to obtain:

$$\int_\Omega (T_k(b(u_k))-T_k(\|f\|_{L^\infty(\Omega)}))^+\,dx \leq \int_\Omega (f-T_k(\|f\|_{L^\infty(\Omega)}))sign_0^+(u_k-b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right))\,dx, \tag{3.10}$$

and

$$\int_{\partial\Omega} T_k(|u_k|^{p(x)-2}u_k)sign_0^+(u_k - b_0^{-1}(\|f\|_{L^\infty(\Omega)}))\,d\sigma$$
$$\leq \int_\Omega (f - T_k(\|f\|_{L^\infty(\Omega)}))sign_0^+(u_k - b_0^{-1}(\|f\|_{L^\infty(\Omega)}))\,dx. \tag{3.11}$$

Hence, for all $k > \|f\|_{L^\infty(\Omega)}$, it follows that

$$T_k(b(u_k)) \leq \|f\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega \tag{3.12}$$

and

$$u_k \leq b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right) \text{ a.e. in } \partial\Omega. \tag{3.13}$$

It remains to prove that $T_k(b(u_k)) \geq -\|f\|_{L^\infty(\Omega)}$ *a.e* in Ω and $u_k \geq -b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right)$ a.e. in $\partial\Omega$ for all $k > \|f\|_{L^\infty(\Omega)}$.

Let us remark that as u_k is a weak solution of (3.2), then $(-u_k)$ is a weak solution of the following problem

$$\begin{cases} T_k(\tilde{b}(u_k)) - div\,\tilde{a}(x,\nabla u_k) = \tilde{f} \text{ in } \Omega \\ \tilde{a}(x,\nabla u_k).\eta = T_k\left(-|u_k|^{p(x)-2}u_k\right) \text{ on } \partial\Omega, \end{cases} \tag{3.14}$$

where $\tilde{a}(x,\xi) = -a(x,-\xi)$, $\tilde{b}(s) = -b(-s)$, $\tilde{f} = -f$.
According to (3.12) and (3.13), we deduce that

$$T_k(-b(u_k)) \leq \|f\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega, \text{ for all } k > \|f\|_{L^\infty(\Omega)}$$

and

$$-u_k \leq b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right) \text{ a.e. in } \partial\Omega.$$

Therefore, we get

$$T_k(b(u_k)) \geq -(\|f\|_{L^\infty(\Omega)}) \;\; \forall\, k > \|f\|_{L^\infty(\Omega)} \tag{3.15}$$

and

$$u_k \geq -b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right) \text{ a.e. in } \partial\Omega \;\; \forall\, k > \|f\|_{L^\infty(\Omega)}. \tag{3.16}$$

It follows from (3.12), (3.13), (3.15) and (3.16) that for all $k > \|f\|_{L^\infty(\Omega)}$,

$$|b(u_k)| \leq \|f\|_{L^\infty(\Omega)} \quad \text{a.e. in } \Omega \tag{3.17}$$

and

$$|u_k| \leq b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right) \text{ a.e. in } \partial\Omega. \tag{3.18}$$

We now fix $k = \|f\|_{L^\infty(\Omega)} + \left(b_0^{-1}\left(\|f\|_{L^\infty(\Omega)}\right)\right)^{p_+-1} + 2$ in (3.2) to end the prove of the existence result.

Part 2: Uniqueness. Let u_1 and u_2 be two weak solutions of (1.1).
Let us take $\varphi = u_1 - u_2$ as test function in (3.1) for u_1 and also for u_2, to get

$$\int_\Omega a(x,\nabla u_1).\nabla(u_1-u_2)\,dx + \int_\Omega b(u_1)(u_1-u_2)dx \quad + \int_{\partial\Omega} |u_1|^{p(x)-2}u_1(u_1-u_2)\,d\sigma = \int_\Omega f(u_1-u_2)\,dx,$$

and

$$\int_\Omega a(x,\nabla u_2).\nabla(u_1-u_2)\,dx + \int_\Omega b(u_2)(u_1-u_2)dx \quad + \int_{\partial\Omega} |u_2|^{p(x)-2}u_2(u_1-u_2)\,d\sigma = \int_\Omega f(u_1-u_2)\,dx.$$

Substracting the two preceding relations, we obtain

$$\int_\Omega (a(x,\nabla u_1)-a(x,\nabla u_2)).\nabla(u_1-u_2)\,dx + \int_\Omega (b(u_1)-b(u_2))(u_1-u_2)\,dx + \int_{\partial\Omega}(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2)(u_1-u_2)\,d\sigma = 0. \tag{3.19}$$

From (3.19) we deduce that

$$\int_\Omega (a(x,\nabla u_1)-a(x,\nabla u_2)).\nabla(u_1-u_2)\,dx = 0, \tag{3.20}$$

$$\int_\Omega (b(u_1)-b(u_2))(u_1-u_2)\,dx = 0 \tag{3.21}$$

and

$$\int_{\partial\Omega}(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2)(u_1-u_2)\,d\sigma = 0. \tag{3.22}$$

Since $p_- > 1$, the following relation is true for any $\xi, \eta \in \mathbb{R}$, $\xi \neq \eta$ (cf. [15])

$$\left(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta\right)(\xi-\eta) > 0. \tag{3.23}$$

Thanks to (3.20), (3.22), (3.23) and assumption (1.11), we get that there exists a constant c such that

$$u_1 - u_2 = c \ \ a.e. \text{ in } \Omega \text{ and } u_1 - u_2 = 0 \text{ a.e. in } \partial\Omega. \tag{3.24}$$

From (3.24), it follows that

$$u_1 = u_2 \text{ a.e. in } \Omega. \ \square$$

4 Entropy solutions

In this section, we study the existence and uniqueness of entropy solution to problem (1.1) when the right-hand side $f \in L^1(\Omega)$. We first recall some notations.

For any $u \in W^{1,p(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.
Set

$$\mathcal{T}^{1,p(.)}(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R}, \text{ measurable such that } T_k(u) \in W^{1,p(.)}(\Omega), \text{ for any } k > 0 \Big\}.$$

As $W^{1,p(.)}(\Omega) \subset W^{1,p_-}(\Omega)$ and since Ω is bounded, then by [6, Lemma 2.1] (see also [1]), we have the following result:

Proposition 4.1. *Let $u \in \mathcal{T}^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(.)}(\Omega)$, then $v \in \left(L^{p(.)}(\Omega)\right)^N$ and $v = \nabla u$ in the usual sense.*

We define $\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:
(C_1) $u_n \to u$ a.e. in Ω.
(C_2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$.
(C_3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

The function v is the trace of u in the generalized sense. In the sequel the trace of $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ then $(u-\varphi) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and $tr(u-\varphi) = tr(u) - tr(\varphi)$ (see [2,3]).

We can now introduce the notion of entropy solution of (1.1).

Definition 4.2. *A measurable function u is an entropy solution to problem (1.1) if $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $b(u) \in L^1(\Omega)$, $|u|^{p(x)-2}\, u \in L^1(\partial\Omega)$ and for every $k > 0$,*

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx + \int_{\partial\Omega} |u|^{p(x)-2}\, uT_k(u-\varphi)d\sigma \le \int_\Omega fT_k(u-\varphi)dx \tag{4.1}$$

for all $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Notice that the integrals in (4.1) are well defined. Indeed, since $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$, then $(u-\varphi) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, hence $T_k(u-\varphi) \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and consequently the first, the second and the fourth integral in (4.1) are well defined. Moreover, in the third integral, we can use the fact that the trace of $g \in W^{1,p}(\Omega)$ on $\partial\Omega$ is well defined in $L^p(\partial\Omega)$.

Our main result in this section is the following:

Theorem 4.3. *Assume (1.8)-(1.12) and $f \in L^1(\Omega)$, then there exists a unique entropy solution u to problem* (1.1).

In order to prove Theorem 4.3, we need the following propositions among which, some can be proved following [7,26,27] with necessary changes in detail. But those which are new will be proved.

Proposition 4.4. *Assume (1.8)-(1.12) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1.1). If there exists a positive constant M such that*

$$\int_{\{|u|>k\}} k^{q(x)} dx \leq M \tag{4.2}$$

then

$$\int_{\{|\nabla u|^{\alpha(.)}>k\}} k^{q(x)} dx \leq \|f\|_{L^1(\Omega)} + M, \text{ for all } k > 0,$$

where $\alpha(.) = p(.)/(q(.)+1)$ and $q(.) : \overline{\Omega} \to (0,+\infty)$ is mesurable and such that $q_- > 0$.

Proposition 4.5. *Assume (1.8)-(1.12) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1.1), then*

$$\int_{\Omega} |\nabla T_k(u)|^{p(x)} dx \leq k\|f\|_{L^1(\Omega)} \text{ for all } k > 0, \tag{4.3}$$

$$\|b(u)\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)} \tag{4.4}$$

and

$$\left\| |u|^{p(x)-2}\, u \right\|_{L^1(\partial\Omega)} = \left\| |u|^{p(x)-1} \right\|_{L^1(\partial\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{4.5}$$

Proof. We will only prove relation (4.5) since the proof of relations (4.3) and (4.4) can be found in [7,26,27]. For this, we take $\varphi = 0$ in relation (4.1) to get for all $k > 0$

$$\int_{\partial\Omega} |u|^{p(x)-2}\, u T_k(u) d\sigma \leq k\|f\|_{L^1(\Omega)}. \tag{4.6}$$

We deduce from (4.6) that

$$\int_{\partial\Omega\cap\{|u|\geq k\}} |u|^{p(x)-2}\, u T_k(u) d\sigma \leq k\|f\|_{L^1(\Omega)}$$

which is equivalent to

$$\int_{\partial\Omega\cap\{u\geq k\}} |u|^{p(x)-2}\, u d\sigma - \int_{\partial\Omega\cap\{u\leq -k\}} |u|^{p(x)-2}\, u d\sigma \leq \|f\|_{L^1(\Omega)}. \tag{4.7}$$

It follows from (4.7) that

$$\int_{\partial\Omega\cap\{|u|\geq k\}} |u|^{p(x)-1} d\sigma \leq \|f\|_{L^1(\Omega)}. \tag{4.8}$$

Finally, we let $k \to 0$ in (4.8) by using Fatou's lemma to obtain relation (4.5). □

Proposition 4.6. *Assume (1.8)-(1.12) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1.1), then*

$$\int_{\Omega} |\nabla T_k(u)|^{p_-} dx \leq const(\|f\|_1, \Omega)(k+1) \text{ for all } k > 0 \tag{4.9}$$

and

$$\int_{\partial\Omega} |T_k(u)|^{p_-} d\sigma \leq const(\|f\|_1, \Omega)(k+1) \text{ for all } k > 0. \tag{4.10}$$

Proof. We easily deduce (4.9) from (4.3). Now, let us prove (4.10). We take $\varphi = 0$ in relation (4.1) to get

$$\int_{\partial\Omega} |u|^{p(x)-2}\, uT_k(u) d\sigma \leq k\|f\|_1. \tag{4.11}$$

The inequality (4.11) is equivalent to

$$\int_{\partial\Omega\cap\{|u|\leq k\}} |T_k(u)|^{p(x)} d\sigma + \int_{\partial\Omega\cap\{|u|>k\}} |u|^{p(x)-2}\, uT_k(u) d\sigma \leq k\|f\|_1.$$

Therefore,

$$\int_{\partial\Omega\cap\{|u|\leq k\}} |T_k(u)|^{p(x)} d\sigma \leq k\|f\|_1. \tag{4.12}$$

Furthermore, for all $k > 0$ we use (4.12) to obtain

$$\begin{aligned}
\int_{\partial\Omega\cap\{|u|\leq k\}} |T_k(u)|^{p_-} d\sigma &= \int_{\partial\Omega\cap\{|u|\leq k\}} |u|^{p_-} d\sigma \\
&= \int_{\partial\Omega\cap\{|u|\leq k, |u|>1\}} |u|^{p_-} d\sigma + \int_{\partial\Omega\cap\{|u|\leq k, |u|\leq 1\}} |u|^{p_-} d\sigma \\
&\leq \int_{\partial\Omega\cap\{|u|\leq k, |u|>1\}} |u|^{p(x)} d\sigma + \text{meas}_{N-1}(\partial\Omega) \\
&\leq k\|f\|_1 + \text{meas}_{N-1}(\partial\Omega) \\
&\leq \text{const}(\|f\|_1, \Omega)(k+1).
\end{aligned} \tag{4.13}$$

Similarly, it follows that for all $k > 0$,

$$\begin{aligned}
\int_{\partial\Omega\cap\{|u|>k\}} |T_k(u)|^{p_-} d\sigma &= k\int_{\partial\Omega\cap\{|u|>k\}} |T_k(u)|^{p_--1} d\sigma \\
&\leq k\int_{\partial\Omega} |u|^{p_--1} d\sigma \\
&\leq k\int_{\partial\Omega\cap\{|u|>1\}} |u|^{p(x)-1} d\sigma + k\int_{\partial\Omega\cap\{|u|\leq 1\}} |u|^{p_--1} d\sigma \\
&\leq k\int_{\partial\Omega} |u|^{p(x)-1} d\sigma + k\,\text{meas}_{N-1}(\partial\Omega).
\end{aligned} \tag{4.14}$$

Adding relations (4.13) and (4.14) and using (4.5), we get (4.10). □

Proposition 4.7. *Assume (1.8)-(1.12) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1.1). Then*

$$\text{meas}\{|u| > k\} \leq \frac{const\left(\|f\|_{L^1(\Omega)}, p_-, (p_-)^*, \Omega\right)}{k^{\alpha}} \text{ for all } k \geq 1, \tag{4.15}$$

and

$$\text{meas}\{|\nabla u| > k\} \leq \frac{const\left(\|f\|_{L^1(\Omega)}, p_-\right)}{k^{p_--1}} \text{ for all } k \geq 1, \tag{4.16}$$

where $(p_-)^* = \frac{1}{p_-} - \frac{1}{N}$ and $\alpha = (p_-)^*\left(1 - \frac{1}{p_-}\right)$

Proof. We only prove relation (4.15). The proof of (4.16) can be found in [7]. Using Proposition 4.6 (relation (4.9)), we obtain for all $k \geq 1$ that

$$\int_\Omega |\nabla T_k(u)|^{p_-} dx \leq K_1 k, \tag{4.17}$$

where K_1 is a positive real constant depending on $\|f\|_1$ and meas(Ω).

We now use a Poincar-Sobolev type inequality (see [28, Lemma in p. 308]) to get (since $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$) that there exists a positive real constant K_2 depending on Ω such that

$$\left(\int_\Omega |T_k(u)|^{(p_-)^*} dx\right)^{\frac{p_-}{(p_-)^*}} \leq K_2 \left(\left(\int_{\partial\Omega} |T_k(u)| d\sigma\right)^{p_-} + \int_\Omega |\nabla T_k(u)|^{p_-} dx\right), \tag{4.18}$$

where $(p_-)^*$ is the Sobolev exponent with respect to p_-. By Hlder inequality, we have the following

$$\left(\int_{\partial\Omega} |T_k(u)| d\sigma\right)^{p_-} \leq \left(\|T_k(u)\|_{L^{p_-}(\partial\Omega)} \times (meas_{N-1}(\partial\Omega))^{\frac{1}{(p_-)'}}\right)^{p_-}. \tag{4.19}$$

We deduce from (4.19) by using Proposition 4.6 (relation (4.10)) that for all $k \geq 1$

$$\left(\int_{\partial\Omega} |T_k(u)| d\sigma\right)^{p_-} \leq K_3 k \tag{4.20}$$

where K_3 is a positive real constant which depends on $\|f\|_1$, p_-, meas(Ω) and meas($\partial\Omega$).
By (4.17), (4.18) and (4.20), we deduce that for all $k \geq 1$,

$$\left(\int_\Omega |T_k(u)|^{(p_-)^*} dx\right)^{\frac{p_-}{(p_-)^*}} \leq K_4 k, \tag{4.21}$$

where K_4 is a positive real constant depending only on $\|f\|_1$, p_-, $(p_-)^*$, meas(Ω) and meas($\partial\Omega$).
It follows from (4.21) that

$$\int_\Omega |T_k(u)|^{(p_-)^*} dx \leq K_5 k^{\frac{(p_-)^*}{p_-}}, \tag{4.22}$$

where K_5 is a positive real constant depending only on $\|f\|_1$, p_-, $(p_-)^*$, meas(Ω) and meas($\partial\Omega$).
Note that (4.22) implies that

$$\int_{\{|u|>k\}} |T_k(u)|^{(p_-)^*} dx \leq K_5 k^{\frac{(p_-)^*}{p_-}}. \tag{4.23}$$

The inequality (4.23) is equivalent to the following

$$\int_{\{|u|>k\}} k^{(p_-)^*} dx \leq K_5 k^{\frac{(p_-)^*}{p_-}},$$

which in turn is also equivalent to

$$k^{(p_-)^*}\operatorname{meas}(\{|u|>k\}) \leq K_5 k^{\frac{(p_-)^*}{p_-}}. \tag{4.24}$$

We deduce from (4.24), the following relation

$$\operatorname{meas}(\{|u|>k\}) \leq K_5 k^{(p_-)^*\left(\frac{1}{p_-}-1\right)}. \tag{4.25}$$

From (4.25), we deduce (4.15). □

We are now ready to give the proof of Theorem 4.3.

Proof of Theorem 4.3.

$*$ **Uniqueness of entropy solution.** Let $h>0$ and u_1,u_2 be two entropy solutions of (1.1). We write the entropy inequality (4.1) corresponding to the solution u_1 with $T_h(u_2)$ as a test function and to the solution u_2 with $T_h(u_1)$ as a test function. Upon addition, we get

$$\begin{cases} \displaystyle\int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx + \int_{\{|u_2-T_h(u_1)|\leq k\}} a(x,\nabla u_2).\nabla(u_2-T_h(u_1))dx \\ \displaystyle + \int_{\partial\Omega} |u_1|^{p(x)-2}\, u_1 T_k(u_1-T_h(u_2))d\sigma + \int_{\partial\Omega} |u_2|^{p(x)-2}\, u_2 T_k(u_2-T_h(u_1))d\sigma \\ \displaystyle + \int_\Omega b(u_1)T_k(u_1-T_h(u_2))dx + \int_\Omega b(u_2)T_k(u_2-T_h(u_1))dx \\ \displaystyle \leq \int_\Omega f(x)\Big(T_k(u_1-T_h(u_2)) + T_k(u_2-T_h(u_1))\Big)dx. \end{cases} \tag{4.26}$$

Now, define

$$E_1 := \{|u_1-u_2|\leq k, |u_2|\leq h\}, \quad E_2 := E_1\cap\{|u_1|\leq h\}, \text{ and } E_3 := E_1\cap\{|u_1|>h\}.$$

We start with the first integral in (4.26). By (1.12), we have

$$\begin{cases} \displaystyle\int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \\ \displaystyle = \int_{\{|u_1-T_h(u_2)|\leq k\}\cap\{|u_2|\leq h\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \\ \displaystyle + \int_{\{|u_1-T_h(u_2)|\leq k\}\cap\{|u_2|>h\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \\ \displaystyle = \int_{\{|u_1-T_h(u_2)|\leq k\}\cap\{|u_2|\leq h\}} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \\ \displaystyle \int_{\{|u_1-h\times \mathrm{sign}(u_2)|\leq k\}\cap\{|u_2|>h\}} a(x,\nabla u_1).\nabla u_1 dx \\ \displaystyle \geq \int_{\{|u_1-T_h(u_2)|\leq k\}\cap\{|u_2|\leq h\}} a(x,\nabla u_1).\nabla(u_1-u_2)dx = \int_{E_1} a(x,\nabla u_1).\nabla(u_1-u_2)dx \\ \displaystyle = \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_{E_3} a(x,\nabla u_1).\nabla(u_1-u_2)dx \\ \displaystyle = \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_{E_3} a(x,\nabla u_1).\nabla u_1 dx - \int_{E_3} a(x,\nabla u_1).\nabla u_2 dx \\ \displaystyle \geq \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx - \int_{E_3} a(x,\nabla u_1).\nabla u_2 dx. \end{cases} \tag{4.27}$$

Using (1.10) and (2.1), we estimate the last integral in (4.27) as follows:

$$\begin{cases} \left|\int_{E_3} a(x,\nabla u_1).\nabla u_2 dx\right| \le C_1 \int_{E_3} \left(j(x)+|\nabla u_1|^{p(x)-1}\right)|\nabla u_2|dx \\ \le C_1 \left(|j|_{p'(.)}+\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}}\right)|\nabla u_2|_{p(.),\{h-k<|u_2|\le h\}}, \end{cases} \tag{4.28}$$

where $\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}} = \left\||\nabla u_1|^{p(x)-1}\right\|_{L^{p'(\cdot)}(\{h<|u_1|\le h+k\})}$.

Now, since u_1 is an entropy solution to problem (1.1), by taking $\varphi = T_h(u_1)$ in the entropy inequality (4.1) we get (using (1.12)) that

$$\int_{\{h<|u_1|\le h+k\}} |\nabla u_1|^{p(x)} dx \le k\|f\|_1.$$

So, by Lemma 2.1, $\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}} \le C < +\infty$, where C is a constant which does not depend on h.

Therefore,

$$C_1 \left(|j|_{p'(.)}+\left||\nabla u_1|^{p(x)-1}\right|_{p'(.),\{h<|u_1|\le h+k\}}\right) \le C_1\left(|j|_{p'(.)}+C\right) < +\infty.$$

Since u_2 is an entropy solution to problem (1.1), by taking $\varphi = T_h(u_2)$ in the entropy inequality (4.1) we get (using (1.12)) that

$$\int_{\{h<|u_2|\le h+k\}} |\nabla u_2|^{p(x)} dx \le k\int_{\{|u_2|>h\}} |f|dx.$$

Using inequality (4.15) of Proposition 4.7, we have $\text{meas}\{|u_2|>h\} \longrightarrow 0$ as $h \to +\infty$. As $f \in L^1(\Omega)$ we get

$$k\int_{\{|u_2|>h\}} |f|dx \longrightarrow 0 \text{ as } h \to +\infty \text{ for any fixed number } k>0.$$

From the above convergence we deduce that

$$\lim_{h\to+\infty} \int_{\{h<|u_2|\le h+k\}} |\nabla u_2|^{p(x)} dx = 0, \text{ for any fixed number } k>0.$$

Hence,

$$\lim_{h\to+\infty} \int_{\{h-k<|u_2|\le h\}} |\nabla u_2|^{p(x)} dx = \lim_{l\to+\infty} \int_{\{l<|u_2|\le l+k\}} |\nabla u_2|^{p(x)} dx = 0,$$

for any fixed number $k>0$ with $l = h-k$.

So by Lemma 2.1,

$$|\nabla u_2|_{p(.),\{h-k<|u_2|\le h\}} \longrightarrow 0 \text{ as } h \to +\infty, \text{ for any fixed number } k>0.$$

Therefore, from (4.27) and (4.28), we obtain that

$$\int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1).\nabla(u_1-T_h(u_2))dx \geq I_h+\int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx, \quad (4.29)$$

where I_h converges to zero as $h\to+\infty$.
We may adopt the same procedure to treat the second term in (4.26) to obtain

$$\int_{\{|u_2-T_h(u_1)|\leq k\}} a(x,\nabla u_2).\nabla(u_2-T_h(u_1))dx \geq J_h-\int_{E_2} a(x,\nabla u_2).\nabla(u_1-u_2)dx, \quad (4.30)$$

where J_h converges to zero as $h\to+\infty$.

Now, set for all $h,k>0$,

$$K_h=\int_\Omega b(u_1)T_k(u_1-T_h(u_2))dx+\int_\Omega b(u_2)T_k(u_2-T_h(u_1))dx$$

and

$$P_h=\int_{\partial\Omega}|u_1|^{p(x)-2}\,u_1T_k(u_1-T_h(u_2))d\sigma+\int_{\partial\Omega}|u_2|^{p(x)-2}\,u_2T_k(u_2-T_h(u_1))d\sigma.$$

We have

$$b(u_1)T_k(u_1-T_h(u_2))\longrightarrow b(u_1)T_k(u_1-u_2) \text{ a.e. in } \Omega \text{ as } h\to+\infty$$

and

$$|b(u_1)T_k(u_1-T_h(u_2))|\leq k|b(u_1)|\in L^1(\Omega).$$

Then by Lebesgue Theorem, we deduce that

$$\lim_{h\to+\infty}\int_\Omega b(u_1)T_k(u_1-T_h(u_2))dx=\int_\Omega b(u_1)T_k(u_1-u_2)dx. \quad (4.31)$$

Similarly, we have

$$\lim_{h\to+\infty}\int_\Omega b(u_2)T_k(u_2-T_h(u_1))dx=\int_\Omega b(u_2)T_k(u_2-u_1)dx. \quad (4.32)$$

Using (4.31) and (4.32), we get

$$\lim_{h\to+\infty}K_h=\int_\Omega (b(u_1)-b(u_2))\,T_k(u_1-u_2)dx. \quad (4.33)$$

By the same procedure as above, we use the Lebesgue theorem to obtain

$$\lim_{h\to+\infty}P_h=\int_{\partial\Omega}\Big(|u_1|^{p(x)-2}\,u_1-|u_2|^{p(x)-2}\,u_2\Big)\,T_k(u_1-u_2)d\sigma. \quad (4.34)$$

We next examine the right-hand side of (4.26).
For all $k>0$,

$$f(x)\Big(T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))\Big)\longrightarrow f(x)\Big(T_k(u_1-u_2)+T_k(u_2-u_1)\Big)=0$$

a.e. in Ω as $h \to +\infty$ and

$$\left| f(x)\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)\right| \leq 2k|f(x)| \in L^1(\Omega).$$

Lebesgue Theorem allows us to write

$$\lim_{h\to+\infty} \int_\Omega f(x)\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)dx = 0. \tag{4.35}$$

Using (4.29), (4.30), (4.33), (4.34) and (4.35), we get from (4.26) the following inequality:

$$\begin{cases} \displaystyle\int_{\{|u_1-u_2|\leq k\}} \Big(a(x,\nabla u_1) - a(x,\nabla u_2)\Big).\Big(\nabla u_1 - \nabla u_2\Big)dx + \\ \displaystyle\int_\Omega (b(u_1) - b(u_2))\, T_k(u_1-u_2)dx + \int_{\partial\Omega} \Big(|u_1|^{p(x)-2}\, u_1 - |u_2|^{p(x)-2}\, u_2\Big)\, T_k(u_1-u_2)d\sigma \leq 0. \end{cases} \tag{4.36}$$

It follows also from (4.36) that

$$\int_{\{|u_1-u_2|\leq k\}} \Big(a(x,\nabla u_1) - a(x,\nabla u_2)\Big).\Big(\nabla u_1 - \nabla u_2\Big)dx = 0, \tag{4.37}$$

$$\int_\Omega (b(u_1) - b(u_2))\, T_k(u_1-u_2)dx = 0 \tag{4.38}$$

and

$$\int_{\partial\Omega} \Big(|u_1|^{p(x)-2}\, u_1 - |u_2|^{p(x)-2}\, u_2\Big)\, T_k(u_1-u_2)d\sigma = 0, \tag{4.39}$$

for all $k > 0$.
From (4.37) and (1.11), it follows that

$$u_1 - u_2 = c \ a.e. \text{ in } \Omega, \text{ where } c \text{ is a real constant.} \tag{4.40}$$

By (4.39), we deduce that for all $k \in \mathbb{N}^*$ there exists $C_k \subset \partial\Omega$, $\text{meas}(C_k) = 0$ such that for all $x \in \partial\Omega \backslash C_k$,

$$\Big(|u_1(x)|^{p(x)-2}\, u_1(x) - |u_2(x)|^{p(x)-2}\, u_2(x)\Big)\, T_k(u_1(x) - u_2(x)) = 0.$$

Therefore,

$$\Big(|u_1(x)|^{p(x)-2}\, u_1(x) - |u_2(x)|^{p(x)-2}\, u_2(x)\Big)\,(u_1(x) - u_2(x)) = 0, \text{ for all } x \in \partial\Omega\backslash \bigcup_{k\in\mathbb{N}^*} C_k. \tag{4.41}$$

Now, we use (3.23) and (4.41) to get

$$u_1 - u_2 = 0 \ a.e. \text{ on } \partial\Omega. \tag{4.42}$$

Finally, (4.40) and (4.42) give

$$u_1 = u_2 \ a.e. \text{ in } \Omega.$$

$*$ **Existence of entropy solution.** Let $f_n = T_n(f)$; then $(f_n)_{n\in\mathbb{N}}$ is a sequence of bounded functions which strongly converges to f in $L^1(\Omega)$ and such that

$$\|f_n\|_1 \leq \|f\|_1, \text{ for all } n \in \mathbb{N}. \tag{4.43}$$

We consider the problem

$$\begin{cases} b(u_n) - \text{div } a(x, \nabla u_n) = f_n \text{ in } \Omega, \\ \\ a(x, \nabla u_n).\eta = -|u_n|^{p(x)-2} \; u_n \;\; \text{on } \partial\Omega. \end{cases} \tag{4.44}$$

It follows from Theorem 3.2 that there exists a unique $u_n \in W^{1,p(.)}(\Omega)$ with $b(u_n) \in L^\infty(\Omega)$ and $|u_n|^{p(x)-2} \; u_n \in L^\infty(\partial\Omega)$ so that

$$\int_\Omega a(x, \nabla u_n).\nabla\varphi dx + \int_\Omega b(u_n)\varphi dx + \int_{\partial\Omega} |u_n|^{p(x)-2} \; u_n \varphi d\sigma = \int_\Omega f_n \varphi dx, \tag{4.45}$$

for all $\varphi \in W^{1,p(.)}(\Omega)$.

Our aim is to prove that these approximated solutions u_n tend to a measurable function u (as n goes to $+\infty$) which is an entropy solution to the limit problem (1.1). To start with, we first prove the following lemma:

Lemma 4.8. *For any* $k > 0$, $\|T_k(u_n)\|_{1,p(.)} \leq 1 + C$ *where* $C = const(k, f, p_-, p_+, \text{meas}(\Omega))$ *is a positive constant.*

Proof. By taking $\varphi = T_k(u_n)$ in (4.45), we get

$$\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n) + \int_\Omega b(u_n) T_k(u_n) dx + \int_{\partial\Omega} |u_n|^{p(x)-2} \; u_n T_k(u_n) d\sigma = \int_\Omega f_n T_k(u_n) dx.$$

Since all the terms in the left-hand side of the equality above are nonnegative and

$$\int_\Omega f_n T_k(u_n) dx \leq k \|f_n\|_1 \leq k \|f\|_1,$$

by using (1.12) we obtain

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \leq Ck \|f\|_1. \tag{4.46}$$

We also have that

$$\int_\Omega |T_k(u_n)|^{p(x)} dx = \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p(x)} dx + \int_{\{|u_n| > k\}} |T_k(u_n)|^{p(x)} dx.$$

Furthermore,

$$\begin{aligned} \int_{\{|u_n| > k\}} |T_k(u_n)|^{p(x)} dx &= \int_{\{|u_n| > k\}} k^{p(x)} dx \\ &\leq \begin{cases} k^{p_+} \text{meas}(\Omega) \text{ if } k \geq 1, \\ \text{meas}(\Omega) \text{ if } k < 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{p(x)} dx &\leq \int_{\{|u_n| \leq k\}} k^{p(x)} dx \\ &\leq \begin{cases} k^{p_+} \text{meas}(\Omega) \text{ if } k \geq 1, \\ \text{meas}(\Omega) \text{ if } k < 1. \end{cases} \end{aligned}$$

This allows us to write

$$\int_\Omega |T_k(u_n)|^{p(x)}dx \leq 2(1+k^{p_+})\text{meas}(\Omega). \tag{4.47}$$

Hence, adding (4.46) and (4.47) yields

$$\rho_{1,p(.)}(T_k(u_n)) \leq Ck\|f\|_1 + (1+k^{p_+})\text{meas}(\Omega) = const(k,f,p_+,\text{meas}(\Omega)). \tag{4.48}$$

For $\|T_k(u_n)\|_{1,p(.)} \geq 1$, we have according to Lemma 2.2 that

$$\|T_k(u_n)\|_{1,p(.)}^{p_-} \leq \rho_{1,p(.)}(T_k(u_n)) \leq const(k,f,p_+,\text{meas}(\Omega)),$$

which is equivalent to

$$\|T_k(u_n)\|_{1,p(.)} \leq \Big(const(k,f,p_+,\text{meas}(\Omega)))\Big)^{\frac{1}{p_-}} = const(k,f,p_+,p_-,\text{meas}(\Omega)).$$

The above inequality gives

$$\|T_k(u_n)\|_{1,p(.)} \leq 1 + const(k,f,p_+,p_-,\text{meas}(\Omega)).$$

Then, the proof of Lemma 4.8 is complete.

From Lemma 4.8, we deduce that for any $k > 0$, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p(.)}(\Omega)$ and so in $W^{1,p_-}(\Omega)$. Then, up to a subsequence we can assume that for any $k>0$, $T_k(u_n)$ converges weakly to σ_k in $W^{1,p_-}(\Omega)$, and so $T_k(u_n)$ strongly converges to σ_k in $L^{p_-}(\Omega)$.

We next prove the following proposition:

Proposition 4.9. *Assume that (1.8)-(1.12) hold and $u_n \in W^{1,p(\cdot)}(\Omega)$ is the weak solution of problem (4.44), then the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy in measure. In particular, there exists a measurable function u and a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ such that $u_n \longrightarrow u$ in measure.*

Proof. Let $s>0$ and define

$$E_n := \{|u_n| > k\}, \quad E_m := \{|u_m| > k\} \quad \text{and} \quad E_{n,m} := \{|T_k(u_n) - T_k(u_m)| > s\}$$

where $k>0$ is to be fixed. We note that

$$\{|u_n - u_m| > s\} \subset E_n \cup E_m \cup E_{n,m}$$

and hence

$$\text{meas}\{|u_n - u_m| > s\} \leq \text{meas}(E_n) + \text{meas}(E_m) + \text{meas}(E_{n,m}). \tag{4.49}$$

Let $\varepsilon > 0$. Using Proposition 4.7 (relation (4.15)), we choose $k = k(\varepsilon)$ such that

$$\text{meas}(E_n) \leq \varepsilon/3 \quad \text{and } \text{meas}(E_m) \leq \varepsilon/3. \tag{4.50}$$

Since $T_k(u_n)$ strongly converges in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$.
Thus,

$$\text{meas}(E_{n,m}) \leq \frac{1}{s^{p_-}} \int_\Omega |T_k(u_n) - T_k(u_m)|^{p_-} dx \leq \frac{\varepsilon}{3}, \tag{4.51}$$

for all $n, m \geq n_0(s, \varepsilon)$.
Finally, from (4.49), (4.50) and (4.51), we obtain

$$\text{meas}\{|u_n - u_m| > s\} \leq \varepsilon \text{ for all } n, m \geq n_0(s, \varepsilon). \tag{4.52}$$

Relations (4.52) mean that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and the proof of Proposition 4.9 is complete.

Note that as $u_n \longrightarrow u$ in measure, up to a subsequence, we can assume that $u_n \longrightarrow u$ a.e. in Ω.
In the sequel, we need the following two technical lemmas (see [18,31]).

Lemma 4.10. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 \ll p(.) \in L^\infty(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

The second technical lemma is a well known result in measure theory (see [18]):

Lemma 4.11. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \longrightarrow [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) < \varepsilon \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma d\mu < \delta.$$

We now set to prove that the function u in the Proposition 4.9 is an entropy solution of (1.1).

Let $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. For any $k > 0$, choose $T_k(u_n - \varphi)$ as a test function in (4.45).
We get

$$\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - \varphi)dx + \int_\Omega b(u_n)T_k(u_n - \varphi)dx$$
$$+ \int_{\partial\Omega} |u_n|^{p(x)-2} u_n T_k(u_n - \varphi)d\sigma = \int_\Omega f_n(x)T_k(u_n - \varphi)dx. \tag{4.53}$$

The following proposition is useful to pass to the limit in the first term of (4.53).

Proposition 4.12. *Assume that $(1.8) - (1.12)$ hold and $u_n \in W^{1,p(.)}(\Omega)$ be the weak solution of the problem (4.44), then*
(i) ∇u_n converges in measure to the weak gradient of u;
(ii) for all $k > 0$, $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in $(L^1(\Omega))^N$;

(iii) for all $t > 0$, $a(x, \nabla T_t(u_n))$ strongly converges to $a(x, \nabla T_t(u))$ in $\left(L^1(\Omega)\right)^N$ and weakly in $\left(L^{p'(\cdot)}(\Omega)\right)^N$;
(iv) u_n converges to some function v a.e. on $\partial\Omega$.

Proof.
(i) We claim that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is Cauchy in measure.
Let $s > 0$ and consider

$$A_{n,m} := \{|\nabla u_n| > h\} \cup \{|\nabla u_m| > h\}, \;\; B_{n,m} := \{|u_n - u_m| > k\}$$

and

$$C_{n,m} := \{|\nabla u_n| \leq h, |\nabla u_m| \leq h, \; |u_n - u_m| \leq k, \; |\nabla u_n - \nabla u_m| > s\},$$

where h and k will be chosen later.

Note that

$$\{|\nabla u_n - \nabla u_m| > s\} \subset A_{n,m} \cup B_{n,m} \cup C_{n,m}. \tag{4.54}$$

Let $\varepsilon > 0$. By Proposition 4.7 (relation (4.16)), we may choose $h = h(\varepsilon)$ large enough such that

$$\text{meas}(A_{n,m}) \leq \varepsilon/3, \tag{4.55}$$

for all $n, m \geq 0$.
On the other hand, by Proposition 4.9

$$\text{meas}(B_{n,m}) \leq \varepsilon/3, \tag{4.56}$$

for all $n, m \geq n_0(k, \varepsilon)$.

Moreover, since $a(x, \xi)$ is continuous with respect to ξ for a.e. $x \in \Omega$, by assumption (1.11) there exists a real valued function $\gamma : \Omega \longrightarrow [0, +\infty]$ such that $\text{meas}(\{x \in \Omega : \gamma(x) = 0\}) = 0$, and

$$(a(x, \xi) - a(x, \xi')).(\xi - \xi') \geq \gamma(x), \tag{4.57}$$

for all $\xi, \xi' \in \mathbb{R}^N$ such that $|\xi| \leq h, \; |\xi'| \leq h, \; |\xi - \xi'| \geq s$, for a.e $x \in \Omega$.

Let $\delta = \delta(\varepsilon)$ be given by Lemma 4.11, replacing ε and A by $\varepsilon/3$ and $C_{n,m}$ respectively.
As u_n is a weak solution of (4.44), using $T_k(u_n - u_m)$ as a test function in (4.45), we get

$$\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - u_m)dx + \int_\Omega b(u_n)T_k(u_n - u_m)dx$$
$$+ \int_{\partial\Omega} |u_n|^{p(x)-2} \, u_n T_k(u_n - u_m)d\sigma = \int_\Omega f_n T_k(u_n - u_m)dx \leq k\|f\|_1.$$

Similarly, we have for u_m that

$$\int_\Omega a(x, \nabla u_m).\nabla T_k(u_m - u_n)dx + \int_\Omega b(u_m)T_k(u_m - u_n)dx$$
$$+ \int_{\partial\Omega} |u_m|^{p(x)-2} \, u_m T_k(u_m - u_n)d\sigma = \int_\Omega f_m T_k(u_m - u_n)dx \leq k\|f\|_1.$$

Adding the last two inequalities yields

$$\int_{\{|u_n-u_m|\leq k\}} (a(x,\nabla u_n)-a(x,\nabla u_m)).(\nabla u_n-\nabla u_m)dx+\int_{\Omega}\Big(b(u_n)-b(u_m)\Big)T_k(u_n-u_m)dx$$
$$+\int_{\partial\Omega}\Big(|u_n|^{p(x)-2}\,u_n-|u_m|^{p(x)-2}\,u_m\Big)\,T_k(u_n-u_m)d\sigma\leq 2k\|f\|_1.$$

Since the second and the third term of the above inequality are nonnegative, we obtain by using (4.57) that

$$\int_{C_{n,m}}\gamma(x)dx\leq\int_{C_{n,m}}(a(x,\nabla u_n)-a(x,\nabla u_m)).(\nabla u_n-\nabla u_m)dx\leq 2k\|f\|_1<\delta,$$

where $k=\delta/4\|f\|_1$.

From Lemma 4.11, it follows that

$$\text{meas}(C_{n,m})\leq\varepsilon/3. \tag{4.58}$$

Thus, using (4.54), (4.55), (4.56) and (4.58), we get

$$\text{meas}(\{|\nabla u_n-\nabla u_m|>s\})\leq\varepsilon,\ \text{for all } n,m\geq n_0(s,\varepsilon) \tag{4.59}$$

and then the claim is proved.

Consequently, $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function v.
In order to end the proof of (i), we need the following lemma:

Lemma 4.13
(a) For a.e. $t\in\mathbb{R}$, $\nabla T_t(u_n)$ converges in measure to $v\chi_{\{|u|<t\}}$;
(b) for a.e. $t\in\mathbb{R}$, $\nabla T_t(u)=v\chi_{\{|u|<t\}}$;
(c) $\nabla T_t(u)=v\chi_{\{|u|<t\}}$ holds for all $t\in\mathbb{R}$.

Proof.
• Proof of (a).
We know that $\nabla u_n\to v$ in measure. Thus, $\chi_{\{|u|<t\}}\nabla u_n\to\chi_{\{|u|<t\}}v$ in measure.
Now, let us show that $\left(\chi_{\{|u_n|<t\}}-\chi_{\{|u|<t\}}\right)\nabla u_n\to 0$ in measure. For that, it is sufficient to show that $\left(\chi_{\{|u_n|<t\}}-\chi_{\{|u|<t\}}\right)\to 0$ in measure. Now, for all $\delta>0$,

$$\left\{\left|\chi_{\{|u_n|<t\}}-\chi_{\{|u|<t\}}\right||\nabla u_n|>\delta\right\}\subset\left\{\left|\chi_{\{|u_n|<t\}}-\chi_{\{|u|<t\}}\right|\neq 0\right\}$$
$$\subset\{|u|=t\}\cup\{u_n<t<u\}\cup\{u<t<u_n\}\cup\{u_n<-t<u\}\cup\{u<-t<u_n\}.$$

Thus,

$$\left\{\begin{array}{l}\text{meas}\left\{\left|\chi_{\{|u_n|<t\}}-\chi_{\{|u|<t\}}\right||\nabla u_n|>\delta\right\}\leq\text{meas}\{|u|=t\}+\text{meas}\{u_n<t<u\}+\\ \text{meas}\{u<t<u_n\}+\text{meas}\{u_n<-t<u\}+\text{meas}\{u<-t<u_n\}.\end{array}\right. \tag{4.60}$$

Note that
meas $\{|u|=t\} \leq$ meas $\{t-h<u<t+h\}+$ meas $\{-t-h<u<-t+h\} \to 0$ as $h \to 0$ for a.e. t, since u is a fixed function. Next,

$$\text{meas}\{u_n<t<u\} \leq \text{meas}\{t<u<t+h\}+\text{meas}\{|u-u_n|>h\}, \text{ for all } h>0.$$

Due to Proposition 4.9, we have for all fixed $h>0$, meas $\{|u-u_n|>h\} \to 0$ as $n \to +\infty$. Since meas $\{t<u<t+h\} \to 0$ as $h \to 0$, for all $\varepsilon>0$, one can find N such that for all $n>N$, meas $\{u_n<t<u\}<\varepsilon/2+\varepsilon/2=\varepsilon$ by choosing h and then N. Each of the other terms in the right-hand side of (4.60) can be treated in the same way as for meas $\{u_n<t<u\}$. Thus, meas $\left\{\left|\chi_{\{|u_n|<t\}}-\chi_{\{|u|<t\}}\right| |\nabla u_n|>\delta\right\} \to 0$ as $n \to +\infty$. Since $\nabla T_t(u_n)=\nabla u_n \chi_{\{|u_n|<t\}}$, the claim (a) follows.

• Proof of (b).
Let ψ_t be the weak $W^{1,p(.)}$-limit of $T_t(u_n)$, then it is also the strong L^1-limit of $T_t(u_n)$. But, as T_t is a Lipschitz function, the convergence in measure of u_n to u implies the convergence in measure of $T_t(u_n)$ to $T_t(u)$. Thus, by the uniqueness of the limit in measure, ψ_t is identified with $T_t(u)$, we conclude that $\nabla T_t(u_n) \to \nabla T_t(u)$ weakly in $L^{p(.)}(\Omega)$.

The previous convergence also ensures that $\nabla T_t(u_n)$ converges to $\nabla T_t(u)$ weakly in $L^1(\Omega)$. On the other hand, by (a), $\nabla T_t(u_n)$ converges to $v\chi_{\{|u|<t\}}$ in measure. By Lemma 4.10, since $\nabla T_t(u_n)$ is uniformly bounded in $L^{p_-}(\Omega)$, the convergence is actually strong in $L^1(\Omega)$; thus it is also weak in $L^1(\Omega)$. By the uniqueness of a weak L^1-limit, $v\chi_{\{|u|<t\}}$ coincides with $\nabla T_t(u)$.

• Proof of (c)
Let $0<t<s$, and s be such that $v\chi_{\{|u|<s\}}$ coincides with $\nabla T_s(u)$. Then

$$\nabla T_t(u)=\nabla T_t(T_s(u))=\nabla T_s(u)\chi_{\{|T_s(u)|<t\}}=v\chi_{\{|u|<s\}}\chi_{\{|u|<t\}}=v\chi_{\{|u|<t\}}.$$

Now, we can end the proof of (i). Indeed, combining Lemma 4.13-(c) and Proposition 4.1, (i) follows.

(ii) Let $s>0$, $k>0$ and consider

$$F_{n,m}=\{|\nabla u_n-\nabla u_m|>s,|u_n|\leq k,|u_m|\leq k\},\ G_{n,m}=\{|\nabla u_m|>s,|u_n|>k,|u_m|\leq k\},$$

$$H_{n,m}=\{|\nabla u_n|>s,|u_m|>k,|u_n|\leq k\} \text{ and } I_{n,m}=\{0>s,|u_m|>k,|u_n|>k\}.$$

Note that

$$\{|\nabla T_k(u_n)-\nabla T_k(u_m)|>s\} \subset F_{n,m}\cup G_{n,m}\cup H_{n,m}\cup I_{n,m}. \tag{4.61}$$

Let $\varepsilon>0$. By Proposition 4.7, we may choose $k(\varepsilon)$ such that

$$\text{meas}(G_{n,m}) \leq \frac{\varepsilon}{4},\ \text{meas}(H_{n,m}) \leq \frac{\varepsilon}{4} \text{ and meas}(I_{n,m}) \leq \frac{\varepsilon}{4}. \tag{4.62}$$

Therefore, using (4.59), (4.61) and (4.62) we get

$$\text{meas}(\{|\nabla T_k(u_n)-\nabla T_k(u_m)|>s\}) \leq \varepsilon, \text{ for all } n,m \geq n_1(s,\varepsilon). \tag{4.63}$$

Consequently, $\nabla T_k(u_n)$ converges in measure to $\nabla T_k(u)$.
Then, using lemmas 4.8 and 4.10, (ii) follows.

(iii) By lemmas 4.10 and 4.13, we have that for all $t>0$, $a(x,\nabla T_t(u_n))$ strongly converges to $a(x,\nabla T_t(u))$ in $\left(L^1(\Omega)\right)^N$ (as n goes to $+\infty$) and $a(x,\nabla T_t(u_n))$ weakly converges to $\chi_t \in (L^{p'(.)}(\Omega))^N$ (as n goes to $+\infty$) in $(L^{p'(.)}(\Omega))^N$. Since each of the convergences implies the weak L^1-convergence, χ_t can be identified with $a(x,\nabla T_t(u))$; thus, $a(x,\nabla T_t(u)) \in (L^{p'(.)}(\Omega))^N$. The proof of (iii) is then complete.

(iv) As u_n is a weak solution of (4.44), using $T_k(u_n)$ as a test function in (4.45), we get

$$\int_{\partial\Omega} |T_k(u_n)|^{p(x)}\,dx \le \int_{\partial\Omega} |u_n|^{p(x)-2}\,u_n T_k(u_n)dx \le k\,\|f\|_1 .$$

and

$$\int_{\Omega} |\nabla T_k(u_n)|^{p(x)}\,dx \le Ck\,\|f\|_1 .$$

We deduce from the inequalities above that

$$\int_{\partial\Omega} |T_k(u_n)|^{p_-}\,dx \le C(f,\Omega)k. \tag{4.64}$$

and

$$\int_{\Omega} |\nabla T_k(u_n)|^{p_-}\,dx \le C(C_3,f,\Omega)k, \tag{4.65}$$

for $k\ge 1$.
Note also that

$$\int_{\Omega} |T_k(u_n)|^{p_-}dx \le 2(1+k^{p_+})\text{meas}(\Omega)+\text{meas}(\Omega),$$

for $k\ge 1$.
Furthermore, $T_k(u_n)$ converges weakly to $T_k(u)$ in $W^{1,p_-}(\Omega)$ and since for every $1\le p\le +\infty$,

$$\tau : W^{1,p}(\Omega)\to L^p(\partial\Omega), u\mapsto \tau(u)=u|_{\partial\Omega}$$

is compact, we deduce that $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^{p_-}(\partial\Omega)$ and so, up to a subsequence, we can assume that $T_k(u_n)$ converges to $T_k(u)$ a.e. on $\partial\Omega$. In other words, there exists $A\subset\partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega\backslash A$ with $\mu(A)=0$, where μ is the area measure on $\partial\Omega$.
Now, we use Hlder Inequality, (4.64) and (4.65) and the Poincar-Sobolev type inequality as in (4.18) to get

$$\int_{\Omega} |T_k(u_n)|\,dx \le (\text{meas}(\Omega))^{\frac{1}{((p_-)^*)'}}\,(Ck)^{\frac{1}{p_-}} \tag{4.66}$$

and

$$\int_{\Omega} |\nabla T_k(u_n)|\,dx \le (\text{meas}(\Omega))^{\frac{1}{(p_-)'}}\,(Ck)^{\frac{1}{p_-}}\,, \tag{4.67}$$

for $k\ge 1$.
By using Fatou's Lemma in (4.66) and (4.67) we get as n goes to $+\infty$ that

$$\int_{\Omega} |T_k(u)|\,dx \le (\text{meas}(\Omega))^{\frac{1}{((p_-)^*)'}}\,(Ck)^{\frac{1}{p_-}} \tag{4.68}$$

and

$$\int_{\Omega} |\nabla T_k(u)|\, dx \leq (\text{meas}(\Omega))^{\frac{1}{(p_-)'}} (Ck)^{\frac{1}{p_-}}, \tag{4.69}$$

for $k \geq 1$.

For every $k \geq 1$, let $A_k := \{x \in \partial\Omega : |T_k(u(x))| < k\}$ and $B = \partial\Omega \setminus \bigcup_{k\geq 1} A_k$.

We have that

$$\begin{aligned} \mu(B) = \frac{1}{k}\int_B |T_k(u)|\, dx \quad & \leq \frac{1}{k}\int_{\partial\Omega} |T_k(u)|\, dx \\ & \leq \frac{C_1}{k} \|T_k(u)\|_{W^{1,1}(\Omega)} \\ & \leq \frac{C_1}{k} \|T_k(u)\|_{L^1(\Omega)} + \frac{C_1}{k} \|\nabla T_k(u)\|_{L^1(\Omega)}. \end{aligned}$$

According to (4.68) and (4.69), we deduce by letting $k \to +\infty$ that $\mu(B) = 0$.

Let us define in $\partial\Omega$ the function v by

$$v(x) := T_k(u(x)) \text{ if } x \in A_k.$$

We take $x \in \partial\Omega \setminus (A \cup B)$; then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x))).$$

Since $x \in A_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_n(x))| < k$, from which we deduce that $|u_n(x)| < k$.

Therefore,

$$u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0, \text{ as } n \to +\infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$. The proof of the proposition 4.12 is then complete.

To complete the proof of existence of entropy solution it remains to show that

$$|u_n|^{p(x)-2} u_n \to |u|^{p(x)-2} u \text{ in } L^1(\partial\Omega). \tag{4.70}$$

For this, let us see that $\left(|u_n|^{p(x)-2} u_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\partial\Omega)$. Indeed: As u_n is a weak solution of (4.44), using $\frac{1}{k}T_k(u_n - u_m)$ as a test function in (4.45), we get

$$\begin{aligned} & \int_\Omega \frac{1}{k} a(x, \nabla u_n).\nabla T_k(u_n - u_m)dx + \int_\Omega b(u_n)\frac{1}{k}T_k(u_n - u_m)dx \\ & + \int_{\partial\Omega} |u_n|^{p(x)-2}\, u_n \frac{1}{k} T_k(u_n - u_m)d\sigma = \int_\Omega f_n \frac{1}{k} T_k(u_n - u_m)dx. \end{aligned}$$

Similarly for u_m, with $\frac{1}{k}T_k(u_m - u_n)$ as test function, we have

$$\begin{aligned} & \int_\Omega \frac{1}{k} a(x, \nabla u_m).\nabla T_k(u_m - u_n)dx + \int_\Omega b(u_m)\frac{1}{k}T_k(u_m - u_n)dx \\ & + \int_{\partial\Omega} |u_m|^{p(x)-2}\, u_m \frac{1}{k} T_k(u_m - u_n)d\sigma = \int_\Omega f_m \frac{1}{k} T_k(u_m - u_n)dx. \end{aligned}$$

Adding the last two identities yields

$$\int_{\{|u_n-u_m|\leq k\}} \frac{1}{k}(a(x,\nabla u_n)-a(x,\nabla u_m)).(\nabla u_n-\nabla u_m)dx+$$
$$\int_{\Omega}\Big(b(u_n)-b(u_m)\Big)\frac{1}{k}T_k(u_n-u_m)dx$$
$$+\int_{\partial\Omega}\Big(|u_n|^{p(x)-2}\,u_n-|u_m|^{p(x)-2}\,u_m\Big)\frac{1}{k}T_k(u_n-u_m)d\sigma\leq\int_{\Omega}|f_n-f_m|\,dx. \quad (4.71)$$

Letting $k\to 0$ and as the first and the second term in the left-hand side of inequalitiy (4.71) are nonnegative, we get

$$\int_{\partial\Omega}\Big||u_n|^{p(x)-2}\,u_n-|u_m|^{p(x)-2}\,u_m\Big|\,d\sigma\leq\int_{\Omega}|f_n-f_m|\,dx. \quad (4.72)$$

Now, since $(f_n)_{n\in\mathbb{N}}$ is convergent in $L^1(\Omega)$, by (4.72) $\big(|u_n|^{p(x)-2}\,u_n\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\partial\Omega)$. As $L^1(\partial\Omega)$ is a Banach space and $s\longmapsto|s|^{p(x)-2}\,s$ is continuous and is a maximal monotone graph in $\mathbb{R}$, then (see [3])

$$|u_n|^{p(x)-2}u_n\to|u|^{p(x)-2}u \text{ in } L^1(\partial\Omega). \quad (4.73)$$

We are now able to pass to the limit in the identity (4.53).

For the right-hand side and the third term in the left-hand side of (4.53), the convergence is obvious since f_n strongly converges to f in $L^1(\Omega)$, $|u_n|^{p(x)-2}u_n$ strongly converges to $|u|^{p(x)-2}u$ in $L^1(\partial\Omega)$, $T_k(u_n-\varphi)$ converges weakly-$*$ to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and a.e in Ω, and $T_k(u_n-\varphi)$ converges weakly-$*$ to $T_k(u-\varphi)$ in $L^\infty(\partial\Omega)$ and a.e in $\partial\Omega$.
For the second term of (4.53), we have

$$\int_{\Omega}b(u_n)T_k(u_n-\varphi)dx=\int_{\Omega}(b(u_n)-b(\varphi))\,T_k(u_n-\varphi)dx$$
$$+\int_{\Omega}b(\varphi)T_k(u_n-\varphi)dx.$$

The quantity $(b(u_n)-b(\varphi))\,T_k(u_n-\varphi)$ is nonnegative and since for all $s\in\mathbb{R}$, $\;s\longmapsto b(s)$ is continuous, we get

$$(b(u_n)-b(\varphi))\,T_k(u_n-\varphi)\longrightarrow(b(u)-b(\varphi))\,T_k(u-\varphi) \text{ a.e. in } \Omega.$$

Then, it follows by Fatou's Lemma that

$$\liminf_{n\to+\infty}\int_{\Omega}(b(u_n)-b(\varphi))\,T_k(u_n-\varphi)dx\geq\int_{\Omega}(b(u)-b(\varphi))\,T_k(u-\varphi)dx. \quad (4.74)$$

We have $b(\varphi)\in L^1(\Omega)$.
Since $T_k(u_n-\varphi)$ converges weakly-$*$ to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and $b(\varphi)\in L^1(\Omega)$, it follows that

$$\lim_{n\to+\infty}\int_{\Omega}b(\varphi)T_k(u_n-\varphi)dx=\int_{\Omega}b(\varphi)T_k(u-\varphi)dx. \quad (4.75)$$

Next, we write the first term in (4.53) in the following form

$$\int_{\{|u_n-\varphi|\leq k\}} a(x,\nabla u_n).\nabla u_n dx - \int_{\{|u_n-\varphi|\leq k\}} a(x,\nabla u_n).\nabla \varphi dx. \tag{4.76}$$

Set $l = k + \|\varphi\|_\infty$. The second integral in (4.76) is equal to

$$\int_{\{|u_n-\varphi|\leq k\}} a(x,\nabla T_l(u_n)).\nabla \varphi dx.$$

Since $a(x,\nabla T_l(u_n))$ is uniformly bounded in $\left(L^{p'(\cdot)}(\Omega)\right)^N$ (by (1.10) and (4.46)), by Proposition 4.12$-(iii)$, it converges weakly to $a(x,\nabla T_l(u))$ in $\left(L^{p'(\cdot)}(\Omega)\right)^N$.
Therefore,

$$\lim_{n\to+\infty}\int_{\{|u_n-\varphi|\leq k\}} a(x,\nabla T_l(u_n)).\nabla \varphi dx = \int_{\{|u-\varphi|\leq k\}} a(x,\nabla T_l(u)).\nabla \varphi dx. \tag{4.77}$$

Moreover, $a(x,\nabla u_n).\nabla u_n$ is nonnegative and converges a.e. in Ω to $a(x,\nabla u).\nabla u$.
Thanks to Fatou's Lemma, we obtain

$$\liminf_{n\to+\infty}\int_{\{|u_n-\varphi|\leq k\}} a(x,\nabla u_n).\nabla u_n dx \geq \int_{\{|u-\varphi|\leq k\}} a(x,\nabla u).\nabla u dx. \tag{4.78}$$

By (4.74), (4.75), (4.77) and (4.78), we get

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx + \int_{\partial\Omega}|u|^{p(x)-2}uT_k(u-\varphi)d\sigma \leq \int_\Omega fT_k(u-\varphi)dx.$$

We conclude that u is an entropy solution of (1.1). □

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