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Asymptotically Almost Automorphic Solutions to Nonautonomous Semilinear Evolution Equations

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Abstract

This paper is concerned with a class of nonautonomous semilinear evolution equation in a Banach space. We establish an existence theorem about asymptotically almost automorphic mild solution to the addressed evolution equation. An example is given to illustrate our abstract result.

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1 Introduction

The aim of this paper is to study the existence of asymptotically almost automorphic mild solution to the following nonautonomous semilinear evolution equations

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

in a Banach space X.

Recently, the existence of almost automorphic type solutions for evolution equations has attracted more and more attention. We refer the reader to [3, 5–13, 15–17] for some

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of recent developments on this topic. Especially, there is a large literature on the existence of pseudo almost automorphic solution to equation (1.1) (see, e.g., [5, 8, 12, 16, 17] and references therein). However, it seems that there are seldom results available in the literature about the existence of asymptotically almost automorphic solution to Equation (1.1). That is the main motivation of this paper.

Throughout the rest of this paper, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, by X a Banach space, and by Ω a subset of X. Next, let us recall some definitions of almost automorphic functions and asymptotically almost automorphic functions (for more details, see [7, 12, 14, 16, 17]).

Definition 1.1. A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R}, X)$ the set of all such functions.

It is well known that the range $\mathcal{R}_f = \{f(t) : t \in \mathbb{R}\}$ of an almost automorphic function f is relatively compact in X, thus bounded in norm (c.f. [14, Theorem 1.31]).

Definition 1.2. A continuous function $f : \mathbb{R}\times\Omega \to X$ is called almost automorphic in t uniformly for x in compact subsets of Ω if for every compact subset K of Ω and every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t,x) = \lim_{n\to\infty} f(t+s_n, x)$$

is well defined for each $t \in \mathbb{R}$, $x \in K$ and

$$\lim_{n\to\infty} g(t-s_n, x) = f(t,x)$$

for each $t \in \mathbb{R}$, $x \in K$. Denote by $AA(\mathbb{R}\times\Omega, X)$ the set of all such functions.

We denote by $C_0(\mathbb{R}, X)$ the space of all continuous functions $h : \mathbb{R} \to X$ such that $\lim_{t\to\infty} h(t) = 0$, and by $C_0(\mathbb{R}\times\Omega, X)$ the space of all continuous functions $h : \mathbb{R}\times\Omega \to X$ such that $\lim_{t\to\infty} h(t,x) = 0$ uniformly for x in any compact subset of Ω.

Definition 1.3. A continuous function $f : \mathbb{R} \to X$ is called asymptotically almost automorphic if it admits a decomposition

$$f(t) = g(t) + h(t), \quad t \in \mathbb{R},$$

where $g \in AA(\mathbb{R}, X)$ and $h \in C_0(\mathbb{R}, X)$. Denote by $AAA(\mathbb{R}, X)$ the set of all such functions.

Definition 1.4. A continuous function $f : \mathbb{R}\times\Omega \to X$ is called asymptotically almost automorphic in t uniformly for x in compact subsets of Ω if it admits a decomposition

$$f(t,x) = g(t,x)+h(t,x), \quad t \in \mathbb{R},\ x \in X,$$

where $g \in AA(\mathbb{R}\times\Omega, X)$ and $h \in C_0(\mathbb{R}\times\Omega, X)$. Denote by $AAA(\mathbb{R}\times\Omega, X)$ the set of all such functions.

Remark 1.5. It is worth to note that the notion of asymptotically almost automorphic functions is different from the notion of asymptotically almost automorphic functions in [14], where an asymptotically almost automorphic function is defined on $\mathbb{R}^+$.

Now, let us recall a concept, which is recently introduced in [17].

Definition 1.6. A continuous function $f(t,s) : \mathbb{R}\times\mathbb{R} \to X$ is called bi-almost automorphic if for every sequence of real numbers (s_m), we can extract a subsequence (s_n) such that

$$g(t,s) = \lim_{n\to\infty} f(t+s_n, s+s_n)$$

is well defined for each $t,s \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n, s-s_n) = f(t,s)$$

for each $t,s \in \mathbb{R}$. $bAA(\mathbb{R}\times\mathbb{R}, X)$ stands for the set of all such functions.

Example 1.7. ([17])

(i) If $f(t,s) = g(t-s)$ for some $g \in C(\mathbb{R}, X)$, then $f \in bAA(\mathbb{R}\times\mathbb{R}, X)$.

(ii) Let $h(t,s) = \sin t \cos s,\ t,s \in \mathbb{R}$. Then $h \in bAA(\mathbb{R}\times\mathbb{R}, \mathbb{R})$.

We also need to recall some notations about evolution family and exponential dichotomy (cf., e.g., [2, 4]).

Definition 1.8. A set $\{U(t,s) : t \geq s,\ t,s \in \mathbb{R}\}$ of bounded linear operators on X is called an evolution family if

(a) $U(s,s) = I$, $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$ and $t,r,s \in \mathbb{R}$,

(b) $\{(\tau,\sigma) \in \mathbb{R}^2 : \tau \geq \sigma\} \ni (t,s) \longmapsto U(t,s)$ is strongly continuous.

Definition 1.9. An evolution family $U(t,s)$ is called hyperbolic (or has exponential dichotomy) if there are projections $P(t)$, $t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t, and constants M, $\omega > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$,

(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$),

(c) $\|U(t,s)P(s)\| \leq Me^{-\omega(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Me^{-\omega(t-s)}$ for all $t \geq s$.

Here and below $Q := I - P$.

Exponential dichotomy is a classical concept in the study of long-term behaviour of evolution equations, see e.g. [4]. If $U(t,s)$ is hyperbolic, then

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}, \end{cases}$$

is called Green's function corresponding to $U(t,s)$ and $P(\cdot)$, and

$$\|\Gamma(t,s)\| \leq \begin{cases} Me^{-\omega(t-s)}, & t \geq s,\ t,s \in \mathbb{R}, \\ Me^{-\omega(s-t)}, & t < s,\ t,s \in \mathbb{R}. \end{cases}$$

At last, let us recall some notions of Stepanov bounded functions (for more details, see [15]).

Definition 1.10. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by

$$f^b(t,s) := f(t+s).$$

Definition 1.11. Let $p \in [1,\infty)$. The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in X such that $f^b : \mathbb{R} \to L^p(0,1;X)$ is bounded. We denote

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \Big)^{1/p}.$$

2 Main result

In this paper, we assume that $\{A(t)\}_{t\in\mathbb{R}}$ satisfy the 'Acquistapace-Terreni' conditions introduced in [1], that is

(H1) There exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L_1, L_2 \geq 0$, and $\gamma_1, \gamma_2 \in (0,1]$ with $\gamma_1 + \gamma_2 > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{L_1}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L_2 |t-s|^{\gamma_1} |\lambda|^{-\gamma_2}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

Remark 2.1. If (H1) holds, then it follows from [2, Theorem 2.3] that there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\leq t<\infty}$ on X, which governs the linear version of (1.1).

We further suppose that

(H2) The evolution family $U(t,s)$ generated by $A(t)$ has an exponential dichotomy with constants M, $\omega > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function $\Gamma(t,s)$.

(H3) $\Gamma(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},X)$ uniformly for $x \in X$.

(H4) $f \in AAA(\mathbb{R}\times X,X)$ and

$$L := \sup_{t\in\mathbb{R},\ u\neq v} \frac{\|f(t,u)-f(t,v)\|}{\|u-v\|} < +\infty.$$

Before presenting our main results, let us establish some basic properties for asymptotically almost automorphic functions.

Lemma 2.2. *The following hold true:*

(a) Assume $f \in AAA(\mathbb{R},X)$ admits a decomposition $f = g+h$, where $g \in AA(\mathbb{R},X)$ and $h \in C_0(\mathbb{R},X)$. Then

$$\overline{\{g(t): t\in\mathbb{R}\}} \subset \overline{\{f(t): t\in\mathbb{R}\}}.$$

(b) $AAA(\mathbb{R},X)$ is a Banach space with the norm

$$\|f\| = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

(c) Let $f \in AAA(\mathbb{R},X)$. Then the range $\mathcal{R}_f = \{f(t): t\in\mathbb{R}\}$ is relatively compact in X.

Proof. The proof is similar to that of [7, Lemmas 1.7-1.9]. So we omit the details. □

Theorem 2.3. *Let (H1)-(H4) hold. Then, Equation* (1.1) *admits a unique asymptotically almost automorphic mild solution provided that one of the following two conditions holds:*

(H5) $L < \frac{\omega}{2M}$;

(H6) there exists a constant $p > 1$ such that

$$\|L(\cdot)\|_{S^p} < \frac{1-e^{-\omega}}{2M}\cdot\left(\frac{\omega q}{1-e^{-\omega q}}\right)^{\frac{1}{q}},$$

where

$$L(t) := \sup_{u\neq v} \frac{\|f(t,u)-f(t,v)\|}{\|u-v\|}, \quad \frac{1}{p}+\frac{1}{q} = 1.$$

Proof. Define a nonlinear operator $\mathcal{F}$ on $AAA(\mathbb{R},X)$ by

$$\mathcal{F}(u)(t) = \int_{\mathbb{R}} \Gamma(t,s)f(s,u(s))ds, \quad t\in\mathbb{R}.$$

Now, we first show that

$$\mathcal{F}(AAA(\mathbb{R},X)) \subset AAA(\mathbb{R},X).$$

Let $\phi \in AAA(\mathbb{R},X)$. Suppose

$$f = g+h, \quad \phi = \alpha+\beta,$$

where

$$g \in AA(\mathbb{R}\times X,X),\ \ h \in C_0(\mathbb{R}\times X,X),\ \ \alpha \in AA(\mathbb{R},X),\ \ \beta \in C_0(\mathbb{R},X).$$

Then

$$f(t,\phi(t)) \quad = \quad f(t,\phi(t)) - f(t,\alpha(t)) + g(t,\alpha(t)) + h(t,\alpha(t)).$$

Noting that $\|f(t,\phi(t)) - f(t,\alpha(t))\| \le L\|\beta(t)\|$, we know that

$$f(\cdot,\phi(\cdot)) - f(\cdot,\alpha(\cdot)) \in C_0(\mathbb{R},X).$$

By (a) of Lemma 2.2, we know that

$$\|g(t,u) - g(t,v)\| \le L\|u-v\|,$$

which yields that $g(\cdot,\alpha(\cdot)) \in AA(\mathbb{R},X)$. Since $\overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, $h(\cdot,\alpha(\cdot)) \in C_0(\mathbb{R},X)$. Now, we conclude that $f(\cdot,\phi(\cdot)) \in AAA(\mathbb{R},X)$. Let

$$f(t,\phi(t)) = F(t) + G(t),$$

where $F \in AA(\mathbb{R},X)$ and $G \in C_0(\mathbb{R},X)$. Then

$$\mathcal{F}(\phi)(t) = \int_{\mathbb{R}} \Gamma(t,s)F(s)ds + \int_{\mathbb{R}} \Gamma(t,s)G(s)ds := I(t) + J(t),$$

where

$$I(t) = \int_{\mathbb{R}} \Gamma(t,s)F(s)ds, \quad J(t) = \int_{\mathbb{R}} \Gamma(t,s)G(s)ds.$$

Since (H2) and (H3) hold, by a similar proof to that of [17, Theorem 3.1], one can get $I \in AA(\mathbb{R},X)$. For J, since $G \in C_0(\mathbb{R},X)$, $\forall \varepsilon > 0$, there exists a constant $K > 0$ such that

$$\|G(t)\| \le \varepsilon, \quad |t| > K,$$

which yields that

$$\begin{aligned}
\|J(t)\| &\le \left\| \int_{|s|>K} \Gamma(t,s)G(s)ds \right\| + \left\| \int_{|s|\le K} \Gamma(t,s)G(s)ds \right\| \\
&\le \varepsilon \int_{\mathbb{R}} \|\Gamma(t,s)\|ds + \int_{|s|\le K} \|\Gamma(t,s)\|ds \cdot \|G\|_0 \\
&\le \frac{2M}{\omega}\varepsilon + \int_{|s|\le K} \|\Gamma(t,s)\|ds \cdot \|G\|_0,
\end{aligned}$$

where $\|G\|_0$ is the supremum norm of G, and

$$\int_{|s|\le K} \|\Gamma(t,s)\|ds \le \begin{cases} \int_{|s|\le K} Me^{-\omega(t-s)}ds = Me^{-\omega t}\int_{|s|\le K} e^{\omega s}ds \to 0, & t \to +\infty, \\ \int_{|s|\le K} Me^{-\omega(s-t)}ds = Me^{\omega t}\int_{|s|\le K} e^{-\omega s}ds \to 0, & t \to -\infty. \end{cases}$$

Therefore, $J \in C_0(\mathbb{R},X)$ and $\mathcal{F}(\phi) \in AAA(\mathbb{R},X)$.

For $\phi,\psi \in AAA(\mathbb{R},X)$, we have

$$\|\mathcal{F}(\phi)(t) - \mathcal{F}(\psi)(t)\| \le \int_{\mathbb{R}} \|\Gamma(t,s)\|ds \cdot L\|\phi - \psi\|,$$

i.e.,

$$\|\mathcal{F}(\phi) - \mathcal{F}(\psi)\| \leq \frac{2ML}{\omega}.$$

Then, (H5) yields that $\mathcal{F}$ is a contraction. On the other hand, if (H6) holds, then

$$\begin{aligned}
& \|\mathcal{F}(\phi)(t) - \mathcal{F}(\psi)(t)\| \\
\leq \ & \int_{\mathbb{R}} \|\Gamma(t,s)\| L(s) ds \cdot \|\phi - \psi\| \\
\leq \ & \left[\int_{-\infty}^{t} M e^{-\omega(t-s)} L(s) ds + \int_{t}^{+\infty} M e^{-\omega(s-t)} L(s) ds \right] \cdot \|\phi - \psi\| \\
= \ & \sum_{k=1}^{\infty} \left[\int_{t-k}^{t-k+1} M e^{-\omega(t-s)} L(s) ds + \int_{t+k-1}^{t+k} M e^{-\omega(s-t)} L(s) ds \right] \cdot \|\phi - \psi\| \\
\leq \ & \sum_{k=1}^{\infty} \left[\left(\int_{t-k}^{t-k+1} M^q e^{-\omega q(t-s)} ds \right)^{1/q} + \left(\int_{t+k-1}^{t+k} M^q e^{-\omega q(s-t)} ds \right)^{1/q} \right] \cdot \|L(\cdot)\|_{S^p} \|\phi - \psi\| \\
\leq \ & \frac{2M}{1 - e^{-\omega}} \cdot \left(\frac{1 - e^{-\omega q}}{\omega q} \right)^{\frac{1}{q}} \cdot \|L(\cdot)\|_{S^p} \|\phi - \psi\|,
\end{aligned}$$

which means that $\mathcal{F}$ is also a contraction.

By (b) of Lemma 2.2, $AAA(\mathbb{R}, X)$ is a Banach space with the supremum norm. So $\mathcal{F}$ has a unique fixed point u in $AAA(\mathbb{R}, X)$. Then

$$u(t) = \int_{\mathbb{R}} \Gamma(t,s) f(s, u(s)) ds, \quad t \in \mathbb{R}.$$

Next, similar to the corresponding proof of [9, Theorem 3.5], one can show that $u(t)$ is just the unique asymptotically almost automorphic mild solution to (1.1). □

Next, by using the idea in [17], we give an example to illustrate our abstract results.

Example 2.4. Consider the following heat equation with Dirichlet boundary conditions:

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + u(t,x) \sin \dfrac{1}{2 + \cos t + \cos \pi t} + f(t, u(t,x)), \quad t \in \mathbb{R},\ x \in (0,1), \\ u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{2.1}$$

where

$$f(t,u) = u \sin \frac{1}{2 + \cos t + \cos \sqrt{2} t} + \frac{1}{1 + t^2} \cos u.$$

Let $X = L^2(0,1)$ and $Ax(\xi) = x''(\xi)$ for $\xi \in (0,1)$ and $x \in D(A)$, where

$$D(A) = \{x \in C^1[0,1] : x' \text{ is absolutely continuous on } [0,1], x'' \in X, x(0) = x(1) = 0\}.$$

It is well-known that A generates a C_0 semigroup $T(t)$ on X satisfying

$$\|T(t)\| \leq e^{-\pi^2 t}, \quad t \geq 0.$$

Now, define a family of linear operator $A(t)$ by

$$A(t)x = Ax + x\sin\frac{1}{2+\cos t+\cos\pi t}$$

with $D(A(t)) \equiv D(A)$. Then $A(t)$ generates an evolution family $U(t,s)$ satisfying

$$U(t,s) = \exp\left(\int_s^t \frac{1}{2+\cos\xi+\cos\pi\xi}d\xi\right)T(t-s)$$

and

$$\|U(t,s)\| \le e^{-(\pi^2-1)(t-s)}.$$

Now, we conclude that (H1)-(H3) hold with $M = 1$ and $\omega = \pi^2 - 1$. In addition, it is easy to see that (H4) holds with $L \le 2$, and thus (H5) holds. Thus, Theorem 2.3 yields that Equation (2.1) has a unique asymptotically almost automorphic mild solution.

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