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Abstract

In this paper, the Adomian decomposition method for solving nonlinear partial differential equations (NPDEs) is revisited. Then we show how this method can be extended and used to solve under-determined systems of NPDEs. The examples of Kompaneets, Novikov and Ginzburg-Landau equations are considered as illustration.

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1 Introduction

The description of an important number of phenomena and dynamic processes in physics, mechanics, chemistry, biology and in other life sciences requires the use of nonlinear partial differential equations. In most cases, finding analytic solutions to these equations remains a very difficult task. Commonly used analytic approaches linearize the equations by assuming that the nonlinearities are relatively insignificant. Such procedures change the actual problem to make it tractable by conventional methods. The solutions obtained in this way sometimes seriously and drastically influence the comprehension of described phenomena. In other side, the numerical methods are based on discretization techniques and require intensive computer time to solve the identified equations. Moreover, these methods only permit to obtain approximate solutions for some finite values of independent variables

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and sometimes overlook some important phenomena. All these aspects justify the need to search for alternative techniques to solve nonlinear differential equations, such as the homotopy perturbation method, the iteration variational method [1, 2, 3, 4] and the Adomian decomposition method [5, 6, 7].

The Adomian decomposition method consists of: (*i*) splitting the given equation into linear and nonlinear parts, (*ii*) identifying the initial and/or boundary conditions and the terms involving the independent variables alone as initial approximation, (*iii*) decomposing the unknown function into a series whose components are to be determined, (*iv*) decomposing the nonlinear function in terms of special polynomials called Adomian polynomials, and (*v*) finding the successive terms of the series solution by recurrent relation. It was shown in [8] that the Adomian decomposition method is a special case of the homotopy analysis method. In addition the criteria for the convergence of the homotopy analysis method was shown to be the same criteria for the convergence of the Adomian decomposition method and that one has a variety of choices for the linear operator and therefore a variety of choices for the initial estimation to start the Adomian decomposition iteration process.

It was formally shown by many researchers that if an exact solution exists for a given problem, then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. Cherruault examined the convergence of the Adomian method in [9]. In addition, Cherruault and Adomian presented a new proof of convergence of the same method in [10]. For more details about the proofs on the rapid convergence, see the above mentioned references and references therein. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. It was also shown by many authors that the series obtained by evaluating a few terms gives an approximation of high degree of accuracy if compared with other numerical techniques [11].

Although the Adomian decomposition method has been intensively used to solve nonlinear problems, to our best knowledge of the literature, it is not still tested to handle under-determined systems of nonlinear partial differential equations. The present work is an attempt to give a mathematical formulation to this purpose. After setting some useful notations in section 2, we describe in the third and fourth sections the general principle of the Adomian decomposition method for nonlinear differential equations and systems of nonlinear differential equations. Finally, we perform, in the last section, an extension of the Adomian method to under-determined systems of differential equations.

2 Principle of the Adomian method

The Adomian decomposition method allows the resolution of various functional equations of algebraic, integral, differential, integro-differential and partial differential types. It is adapted as well to linear as to nonlinear problems. Let E be a functional Banach space. Consider the problem of finding a function $u \in E$ which satisfies the functional equation (written into the canonical form [12])

$$u - Nu = f, \tag{2.1}$$

where $N : E \longrightarrow E$ is a nonlinear operator and $f \in E$ is a known function. The Adomian method consists of looking for a solution u as a series

$$u = \sum_{n=0}^{+\infty} u_n \tag{2.2}$$

and in also decomposing the nonlinear term $N u$ into a series

$$N u = \sum_{n=0}^{+\infty} A_n. \tag{2.3}$$

The A_n are called Adomian polynomials and are obtained by the following formula [13, 14, 15, 16]

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{2.4}$$

The substitution of (2.2)-(2.3) into (2.1) gives

$$\sum_{n=0}^{+\infty} u_n = f + \sum_{n=0}^{+\infty} A_n. \tag{2.5}$$

This yields by identification

$$u_0 = f, \quad u_1 = A_0, \quad u_{n+1} = A_n,\ n \geq 1. \tag{2.6}$$

In practice, all terms of the series (2.2) cannot be determined, and the solution is approximated by the truncated series

$$\varphi_n = \sum_{i=0}^{n-1} u_i. \tag{2.7}$$

However, one can ask the following question: Which conditions does the method converge to? One deduces from the relation (2.5) that

$$\sum_{n=0}^{+\infty} A_n < +\infty \quad \Longleftrightarrow \quad \sum_{n=0}^{+\infty} u_n < +\infty. \tag{2.8}$$

Therefore, the convergence conditions are in connection with the nonlinear operator N from which the Adomian polynomials A_n are defined. In addition, an analysis of several results concerning the convergence of the Adomian method [9, 10, 17, 18, 19, 20, 21] allows us to conclude that the properties satisfied by the operator N so that there exists a solution to the problem (2.1) also ensure the convergence of the series (2.3) and hence the convergence of the series (2.2). In other words, the Adomian decomposition method converges whenever a solution to the problem exists. Before describing the use of this method for the solution investigation of partial differential equations, we define some important notations used in the sequel.

3 Basic notations

Consider X, an n-dimensional independent variable space, and U, an m-dimensional dependent variable space. Let $x = \left(x^1, \cdots, x^n\right) \in X$ and $u = \left(u^1, \cdots, u^m\right) \in U$. We define the space $U^{(s)}$, $s \in \mathbb{N}$ as:

$$U^{(s)} := \left\{ u^{(s)} \, : \, u^{(s)} = \bigotimes_{j=1}^{m} \left(\bigotimes_{k=0}^{s} u^j_{(k)} \right) \right\}, \tag{3.1}$$

where $u^j_{(k)}$ is the

$$p_k = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \text{-tuple} \tag{3.2}$$

of all distinct k-order partial derivatives of u^j. The $u^j_{(k)}$ vector components are recursively obtained as follows:

i) $u^j_{(0)} = u^j$ and $u^j_{(1)} = \left(u^j_{x^1}, u^j_{x^2}, \cdots, u^j_{x^n}\right)$.

ii) Assume that $u^j_{(k)}$ is known.

- Form the tuples $\widehat{u}^j_{(k+1)}(l)$ as follows

$$\widehat{u}^j_{(k+1)}(l) = \left(\frac{\partial}{\partial x^1} u^j_{(k)}[l], \frac{\partial}{\partial x^2} u^j_{(k)}[l], \cdots, \frac{\partial}{\partial x^n} u^j_{(k)}[l] \right), \quad l = 1, 2, \cdots, p_k;$$

  where $u^j_{(k)}[l]$ is the l-th component of the vector $u^j_{(k)}$.
- Form, by iteration, the tuples $\widetilde{u}^j_{(k+1)}(l)$ following the scheme $\widetilde{u}^j_{(k+1)}(1) = \widehat{u}^j_{(k+1)}(1)$ and for $l = 2, 3, \cdots, p_k$, the vector $\widetilde{u}^j_{(k+1)}(l)$ is nothing but the tuple $\widehat{u}^j_{(k+1)}(l)$ in which all components already present in $\widetilde{u}^j_{(k+1)}(i)$, $i = 1, 2, \cdots, l-1$, are excluded.
- Finally, form the vector

$$u^j_{(k+1)} = \left(\widetilde{u}^j_{(k+1)}(1), \widetilde{u}^j_{(k+1)}(2), \cdots, \widetilde{u}^j_{(k+1)}(p_k) \right).$$

As a matter of clarity, let us immediately illustrate this construction by the following

Example 3.1. • For $n = 2$, $x = \left(x^1, x^2\right)$ and we have:

$$u^j_{(1)} = \left(u^j_{x^1}, u^j_{x^2}\right),$$

$$\widehat{u}^j_{(2)}(1) = \left(\frac{\partial}{\partial x^1} u^j_{(1)}[1], \frac{\partial}{\partial x^2} u^j_{(1)}[1] \right) = \left(u^j_{2x^1}, u^j_{x^1 x^2}\right),$$

$$\widehat{u}^j_{(2)}(2) = \left(\frac{\partial}{\partial x^1} u^j_{(1)}[2], \frac{\partial}{\partial x^2} u^j_{(1)}[2] \right) = \left(u^j_{x^2 x^1}, u^j_{2x^2}\right),$$

$$\widetilde{u}^j_{(2)}(1) = \widehat{u}^j_{(2)}(1) = \left(u^j_{2x^1}, u^j_{x^1 x^2}\right), \quad \widetilde{u}^j_{(2)}(2) = \left(\check{u}^j_{x^2 x^1}, u^j_{2x^2}\right) = \left(u^j_{2x^2}\right),$$

$$u^j_{(2)} = \left(\widetilde{u}^j_{(2)}(1), \widetilde{u}^j_{(2)}(2)\right) = \left(u^j_{2x^1}, u^j_{x^1 x^2}, u^j_{2x^2}\right).$$

- For $n = 3$, $x = \left(x^1, x^2, x^3\right)$ and the same scheme leads to

$$u^j_{(2)} = \left(u^j_{2x^1}, u^j_{x^1x^2}, u^j_{x^1x^3}, u^j_{2x^2}, u^j_{x^2x^3}, u^j_{2x^3}\right),$$

$$u^j_{(3)} = \left(u^j_{3x^1}, u^j_{2x^1x^2}, u^j_{2x^1x^3}, u^j_{x^12x^2}, u^j_{x^1x^2x^3}, u^j_{x^12x^3}, u^j_{3x^2}, u^j_{2x^2x^3}, u^j_{x^22x^3}, u^j_{3x^3}\right),$$

  for $k = 2$ and $k = 3$, respectively.

An element $u^{(s)}$, in the space $U^{(s)}$, is the

$$q_s = m(1 + p_1 + p_2 + \cdots + p_s) = m\begin{pmatrix} n+s \\ s \end{pmatrix}\text{-tuple} \tag{3.3}$$

defined by

$$u^{(s)} = \left(u^1_{(0)}, u^1_{(1)}, \cdots, u^1_{(s)}, u^2_{(0)}, u^2_{(1)}, \cdots, u^2_{(s)}, \cdots, u^m_{(0)}, u^m_{(1)}, \cdots, u^m_{(s)}\right). \tag{3.4}$$

We denote by $X \times U^{(s)}$, the total space whose coordinates are denoted by $(x, u^{(s)})$, encompassing the independent variables x and the dependent variables with their derivatives up to order s, globally denoted by $u^{(s)}$.

In the sequel, a q_s-uple $u^{(s)}$ is referred to (3.4), whereas the integers p_k and q_s are defined by (3.2) and (3.3), respectively.

4 Adomian method for differential equations

In this section, we briefly review the Adomian method and apply it to solve the Kompaneets and Novikov equations.

4.1 General scheme

Consider the partial differential equation

$$G\left(x, u^{(s)}(x)\right) = f(x) \tag{4.1}$$

where the nonzero positive integer s is the order of equation, $x = \left(x^1, \cdots, x^n\right)$ are independent variables, $u = u(x)$ is the dependent variable, f is a continuous function and G is a differentiable function with respect to its arguments. Suppose that (4.1) can be written into the form

$$u_{(k_0)}[h_0](x) + F\left(x, u_{(k_1)}[h_1](x), u_{(k_2)}[h_2](x), \cdots, u_{(k_r)}[h_r](x)\right) = f(x), \tag{4.2}$$

where $r, k_i, h_i \in \mathbb{N}$ with $\max\{k_i, i = 0, \cdots, r\} = s$, $h_i \in \{1, 2, \cdots, p_{k_i}\}$ and F is a differential function with respect to its r arguments. Let L be the linear differential operator such that $Lu = u_{(k_0)}[h_0]$ and

$$L = \frac{\partial^{k_0}}{\left(\partial x^{i_1}\right)^{l_1}\left(\partial x^{i_2}\right)^{l_2}\cdots\left(\partial x^{i_{s_0}}\right)^{l_{s_0}}}$$

with $1 \leq i_1 < i_2 < \cdots < i_{s_0} \leq n$, $l_1 + l_2 + \cdots + l_{s_0} = k_0$ and $l_i \neq 0$. In order to minimize the number of operations to perform in practice, it is suitable that L be the smallest order linear differential operator in G. Set $I = \{i_1, i_2, \cdots, i_{s_0}\}$. Define L^{-1}, the inverse of the operator L by

$$\begin{aligned} L^{-1}v(x) &= \left(\int_{x_1^{i_1}}^{x^{i_1}} \int_{x_2^{i_1}}^{z_{l_1-1}^{i_1}} \cdots \int_{x_{l_1}^{i_1}}^{z_1^{i_1}}\right) \cdot \left(\int_{x_1^{i_2}}^{x^{i_2}} \int_{x_2^{i_2}}^{z_{l_2-1}^{i_2}} \cdots \int_{x_{l_2}^{i_2}}^{z_1^{i_2}}\right) \\ &\cdots \left(\int_{x_1^{i_{s_0}}}^{x^{i_{s_0}}} \int_{x_2^{i_{s_0}}}^{z_{l_{s_0}-1}^{i_{s_0}}} \cdots \int_{x_{l_{s_0}}^{i_{s_0}}}^{z_1^{i_{s_0}}}\right) v\left(y^1, y^2, \cdots, y^n\right) \\ &\left(\mathrm{d}z_0^{i_1}\,\mathrm{d}z_1^{i_1} \cdots \mathrm{d}z_{l_1-1}^{i_1}\right) \cdot \left(\mathrm{d}z_0^{i_2}\,\mathrm{d}z_1^{i_2} \cdots \mathrm{d}z_{l_2-1}^{i_2}\right) \cdots \left(\mathrm{d}z_0^{i_{s_0}}\,\mathrm{d}z_1^{i_{s_0}} \cdots \mathrm{d}z_{l_{s_0}-1}^{i_{s_0}}\right), \end{aligned}$$

where $y^j = z_0^j$ if $j \in I$ and $y^j = x^j$ otherwise. The constants x_k^i, $k = 1, 2, \cdots, l_i$ as well as the order of integration are closely related to the choice of boundary conditions in such a way that $L^{-1}(Lu) = u(x) + g(x)$. Here, g is a function completely determined by the boundary conditions. Thus, applying L^{-1} to both sides of (4.2) yields

$$u(x) = -g(x) + L^{-1} f(x) - L^{-1} F. \tag{4.3}$$

The Adomian decomposition method consists in searching for the function u as an infinite series

$$u(x) = \sum_{\tau=0}^{+\infty} u_\tau(x), \tag{4.4}$$

where the u_τ are functions to determine. Express the differential function F as a series

$$F = \sum_{\tau=0}^{+\infty} A_\tau, \tag{4.5}$$

where the A_τ are multivariate polynomials also called Adomian polynomials

$$A_\tau = \frac{1}{\tau!}\left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau} F\left(x, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_1)}[h_1], \sum_{i=0}^{\tau} \lambda^i u_{i,(k_2)}[h_2], \cdots, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_r)}[h_r]\right)\right]_{\lambda=0}. \tag{4.6}$$

Notice that for a linear term, i.e. $F = u_{(k)}[h]$, its Adomian polynomials are $A_\tau = u_{\tau,(k)}[h]$. The substitution in (4.3) of the expressions (4.4) and (4.5) along with (4.6) yields

$$\sum_{\tau=0}^{+\infty} u_\tau = -g(x) + L^{-1} f(x) - L^{-1}\left(\sum_{\tau=0}^{+\infty} A_\tau\right). \tag{4.7}$$

The functions u_τ can be obtained using the following recurrence relation

$$u_0 = -g(x) + L^{-1} f(x), \quad u_{\tau+1} = -L^{-1} A_\tau, \quad \tau \geq 0 \tag{4.8}$$

since the A_τ depend only on $u_0, u_1, \cdots, u_\tau$. If the functions u_τ are determined for all $\tau \in \mathbb{N}$, then a solution u of equation (4.2) is immediately formed using the series (4.4). Otherwise, if only a finite number, says ν, of the functions u_τ is found, one can use the partial sum

$$\phi_\nu(x) = \sum_{\tau=0}^{\nu-1} u_\tau(x) \tag{4.9}$$

as an analytical expression of an approximate solution to equation (4.2).

4.2 Application to the Kompaneets equation

The Kompaneets equation [22], also known as the photon diffusion equation, is written as

$$\frac{\partial u}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial u}{\partial x} + u + u^2\right)\right], \tag{4.10}$$

where $u = u(t,x)$. The expanded form of equation (4.10) is

$$u_t = x^2 u_{xx} + \left(4x + x^2\right)u_x + 4xu + 2x^2 uu_x + 4xu^2. \tag{4.11}$$

Using Adomian decomposition method, we solve this Kompaneets equation subject to the initial condition

$$u(0,x) = \frac{1}{x}. \tag{4.12}$$

In an operator form, equation (4.11) becomes

$$Lu = F(x,u,u_x,u_{xx}) = x^2 u_{xx} + \left(4x + x^2\right)u_x + 4xu + 2x^2 uu_x + 4xu^2, \tag{4.13}$$

where L is defined by

$$L = \frac{\partial}{\partial t}. \tag{4.14}$$

The inverse operator L^{-1} is identified by

$$L^{-1}\, v(t,x) = \int_0^t v(s,x)\,\mathrm{d}s. \tag{4.15}$$

Applying L^{-1} to both sides of (4.13) and using the initial condition (4.12), we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{x} + L^{-1}F(x,u,u_x,u_{xx}) \\ &= \frac{1}{x} + L^{-1}\left[x^2 u_{xx} + \left(4x + x^2\right)u_x + 4xu + 2x^2 uu_x + 4xu^2\right]. \end{aligned} \tag{4.16}$$

The Adomian decomposition method suggests that $u(t,x)$ can be defined by

$$u(t,x) = \sum_{\tau=0}^{+\infty} u_\tau(t,x) \tag{4.17}$$

and the nonlinear term $F(x,u,u_x,u_{xx})$ by

$$F = \sum_{\tau=0}^{+\infty} A_\tau, \tag{4.18}$$

where the A_τ are Adomian polynomials obtained by the formula

$$A_\tau = \frac{1}{\tau!}\left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau}F\left(x, \sum_{i=0}^{\tau}\lambda^i u_{i,(0)}[1], \sum_{i=0}^{\tau}\lambda^i u_{i,(1)}[2], \sum_{i=0}^{\tau}\lambda^i u_{i,(2)}[3]\right)\right]_{\lambda=0} \tag{4.19}$$

explicitly giving:

$$
\begin{aligned}
A_0 &= 4xu_{0,x}+4xu_0+4xu_0^2+x^2u_{0,2x}+x^2u_{0,x}+2x^2u_0u_{0,x},\\
A_1 &= 4xu_{1,x}+4xu_1+8xu_0u_1+x^2u_{1,2x}+x^2u_{1,x}+2x^2u_1u_{0,x}+2x^2u_0u_{1,x},\\
A_2 &= 4xu_{2,x}+4xu_2+4xu_1^2+2x^2u_0u_{2,x}+x^2u_{2,2x}+8xu_0u_2\\
&+ 2x^2u_2u_{0,x}+x^2u_{2,x}+2x^2u_1u_{1,x},\\
A_3 &= 4xu_{3,x}+4xu_3+8xu_1u_2+2x^2u_2u_{1,x}+2x^2u_3u_{0,x}+8xu_0u_3\\
&+ 2x^2u_1u_{2,x}+2x^2u_0u_{3,x}+x^2u_{3,2x}+x^2u_{3,x},\\
A_4 &= 4xu_{4,x}+4xu_4+4xu_2^2+2x^2u_2u_{2,x}+x^2u_{4,2x}+x^2u_{4,x}+2x^2u_4u_{0,x}\\
&+ 8xu_1u_3+2x^2u_1u_{3,x}+2x^2u_0u_{4,x}+2x^2u_3u_{1,x}+8xu_0u_4,\\
A_5 &= 4xu_{5,x}+4xu_5+8xu_2u_3+x^2u_{5,2x}+8xu_1u_4+2x^2u_5u_{0,x}+2x^2u_1u_{4,x}\\
&+ 2x^2u_4u_{1,x}+2x^2u_0u_{5,x}+x^2u_{5,x}+8xu_0u_5+2x^2u_2u_{3,x}+2x^2u_3u_{2,x};
\end{aligned}
$$

and so on. Using the above assumptions gives

$$\sum_{\tau=0}^{+\infty} u_\tau = \frac{1}{x}+L^{-1}\left(\sum_{\tau=0}^{+\infty} A_\tau\right) \tag{4.20}$$

from which one derives the recursive relation

$$u_0(t,x)=\frac{1}{x}, \quad u_{\tau+1}(t,x)=L^{-1}A_\tau, \quad \tau\geq 0. \tag{4.21}$$

The first components are determined by:

$$
\begin{aligned}
u_1(t,x) &= 3t,\\
u_2(t,x) &= (9+6x)t^2,\\
u_3(t,x) &= \left(18+48x+10x^2\right)t^3,\\
u_4(t,x) &= \left(27+216x+155x^2+15x^3\right)t^4,\\
u_5(t,x) &= \left(\frac{162}{5}+1232x^2+\frac{3456}{5}x+\frac{1956}{5}x^3+21x^4\right)t^5,\\
u_6(t,x) &= \left(\frac{162}{5}+1728x+\frac{19988}{3}x^2+4874x^3+\frac{4242}{5}x^4+28x^5\right)t^6
\end{aligned}
$$

permitting to write an approximate solution to Kompaneets equation (4.11) under (4.12) as

$$
\begin{aligned}
\phi(t,x) &= u_0(t,x)+u_1(t,x)+u_2(t,x)+u_3(t,x)+u_4(t,x)+u_5(t,x)+u_6(t,x)\\
&= \frac{1}{x}+3t+(9+6x)t^2+\left(18+48x+10x^2\right)t^3+\left(27+216x+155x^2+15x^3\right)t^4\\
&+ \left(\frac{162}{5}+1232x^2+\frac{3456}{5}x+\frac{1956}{5}x^3+21x^4\right)t^5\\
&+ \left(\frac{162}{5}+1728x+\frac{19988}{3}x^2+4874x^3+\frac{4242}{5}x^4+28x^5\right)t^6
\end{aligned}
$$

whose behavior is shown in the Figure 1.

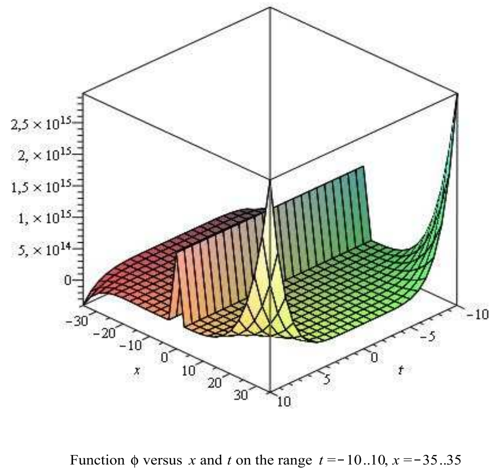


Figure 1. Function ϕ, approximate solution of the Kompaneets equation, versus x and t.

4.3 Application to the Novikov equation

The Novikov equation is expressed as follows [23]:

$$u_t - u_{xxt} + 4u^2 u_x = 3u u_x u_{xx} + u^2 u_{xxx}, \tag{4.22}$$

where $u = u(t,x)$. Using Adomian decomposition method, we solve this equation subject to the initial condition

$$u(0,x) = x. \tag{4.23}$$

In an operator form, equation (4.22) becomes

$$Lu = F(x,u,u_x,u_{xx},u_{xxt},u_{xxx}) = u_{xxt} - 4u^2 u_x + 3u u_x u_{xx} + u^2 u_{xxx}, \tag{4.24}$$

where L is defined by

$$L = \frac{\partial}{\partial t}, \tag{4.25}$$

with its inverse L^{-1} given by

$$L^{-1} v(t,x) = \int_0^t v(s,x)\,\mathrm{d}s. \tag{4.26}$$

Applying L^{-1} to both sides of (4.24) and using the initial condition (4.23), we find

$$\begin{aligned} u(t,x) &= x + L^{-1} F(x,u,u_x,u_{xx},u_{xxt},u_{xxx}) \\ &= x + L^{-1}\left(u_{xxt} - 4u^2 u_x + 3u u_x u_{xx} + u^2 u_{xxx}\right). \end{aligned} \tag{4.27}$$

The Adomian polynomials A_τ are given by the formula

$$\begin{aligned} A_\tau &= \frac{1}{\tau!}\Bigg[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau}F\Bigg(x, \sum_{i=0}^{\tau}\lambda^i u_{i,(0)}[1], \sum_{i=0}^{\tau}\lambda^i u_{i,(1)}[2], \sum_{i=0}^{\tau}\lambda^i u_{i,(2)}[3], \\ &\quad \sum_{i=0}^{\tau}\lambda^i u_{i,(3)}[3], \sum_{i=0}^{\tau}\lambda^i u_{i,(3)}[4]\Bigg)\Bigg]_{\lambda=0} \end{aligned} \tag{4.28}$$

from which are deduced the explicit expressions:

$$\begin{aligned} A_0 &= u_{0,t2x} - 4u_0^2 u_{0,x} + 3u_0u_{0,x}u_{0,2x} + u_0^2 u_{0,3x}, \\ A_1 &= u_{1,t2x} - 8u_0u_{0,x}u_1 + u_0^2u_{1,3x} + 2u_0u_{0,3x}u_1 - 4u_0^2u_{1,x} + 3u_0u_{0,x}u_{1,2x} \\ &+ 3u_0u_{1,x}u_{0,2x} + 3u_1u_{0,x}u_{0,2x}, \\ A_2 &= u_{2,t2x} - 4u_1^2u_{0,x} + 3u_0u_{1,x}u_{1,2x} - 4u_0^2u_{2,x} + 2u_0u_{1,3x}u_1 + 3u_1u_{0,x}u_{1,2x} \\ &- 8u_0u_{1,x}u_1 + 3u_0u_{0,x}u_{2,2x} + 3u_2u_{0,x}u_{0,2x} + 2u_0u_{0,3x}u_2 + 3u_0u_{2,x}u_{0,2x} \\ &- 8u_0u_{0,x}u_2 + u_1^2u_{0,3x} + 3u_1u_{1,x}u_{0,2x} + u_0^2u_{2,3x}, \\ A_3 &= u_{3,t2x} + 3u_0u_{0,x}u_{3,2x} + 2u_0u_{2,3x}u_1 + 3u_1u_{2,x}u_{0,2x} + 3u_1u_{0,x}u_{2,2x} - 8u_0u_{2,x}u_1 \\ &+ u_0^2u_{3,3x} - 4u_0^2u_{3,x} + 2u_1u_{0,3x}u_2 + 2u_0u_{1,3x}u_2 + 3u_1u_{1,x}u_{1,2x} + 3u_0u_{2,x}u_{1,2x} \\ &- 8u_0u_{1,x}u_2 - 4u_1^2u_{1,x} + u_1^2u_{1,3x} - 8u_0u_{0,x}u_3 + 2u_0u_{0,3x}u_3 + 3u_0u_{1,x}u_{2,2x} \\ &+ 3u_0u_{3,x}u_{0,2x} + 3u_3u_{0,x}u_{0,2x} + 3u_2u_{0,x}u_{1,2x} + 3u_2u_{1,x}u_{0,2x} - 8u_1u_{0,x}u_2; \end{aligned}$$

and so on. Use the relation

$$\sum_{\tau=0}^{+\infty} u_\tau = x + L^{-1}\left(\sum_{\tau=0}^{+\infty} A_\tau\right) \tag{4.29}$$

from which results the following recursive relation

$$u_0(t,x) = x, \quad u_{\tau+1}(t,x) = L^{-1}A_\tau, \quad \tau \geq 0 \tag{4.30}$$

generating the functions:

$$\begin{aligned} u_1(t,x) &= -4x^2t, \\ u_2(t,x) &= \left(-12x + 32x^3\right)t^2 - 8t, \\ u_3(t,x) &= \left(400x^2 - 320x^4\right)t^3 + 224xt^2, \\ u_4(t,x) &= \left(744x - 9856x^3 + 3584x^5\right)t^4 + \left(864 - 4992x^2\right)t^3, \\ u_5(t,x) &= \left(-\frac{326688}{5}x^2 + 214272x^4 - 43008x^6\right)t^5 + \left(-73728x + 101376x^3\right)t^4 - \frac{30208}{3}t^3, \\ u_6(t,x) &= \left(-\frac{407328}{5}x + \frac{16407808}{5}x^3 - 4347904x^5 + 540672x^7\right)t^6 \\ &+ \left(-\frac{695616}{5} + 3563520x^2 - 1955840x^4\right)t^5 + \frac{1908224}{3}xt^4. \end{aligned}$$

which serve to express an approximate analytical solution to the Novikov equation (4.22) with the initial condition (4.23) as follows:

$$\begin{aligned}
\phi(t,x) &= u_0(t,x)+u_1(t,x)+u_2(t,x)+u_3(t,x)+u_4(t,x)+u_5(t,x)+u_6(t,x)\\
&= x-\left(4x^2+8\right)t+\left(32x^3+212x\right)t^2-\left(320x^4+4592x^2+\frac{27616}{3}\right)t^3\\
&+ \left(\frac{1689272}{3}x+91520x^3+3584x^5\right)t^4\\
&+ \left(-43008x^6-1741568x^4+\frac{17490912}{5}x^2-\frac{695616}{5}\right)t^5\\
&+ \left(-\frac{407328}{5}x+\frac{16407808}{5}x^3-4347904x^5+540672x^7\right)t^6.
\end{aligned}$$

See Figure 2 for its graphic representation.

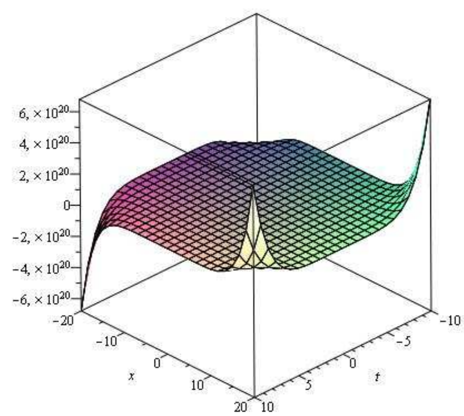


Figure 2. Function ϕ, approximate solution of the Novikov equation, versus x and t.

5 Adomian method for systems of differential equations

We consider here the cases of both determined and under-determined systems of NPDEs.

5.1 Case of determined systems

Consider the system of partial differential equations

$$G_j\left(x,u^{(s)}(x)\right)=f_j(x), \quad j=1,2,\cdots,m \tag{5.1}$$

where the nonzero positive integer s is the order of the system, $x=\left(x^1,\cdots,x^n\right)$ are independent variables, $u=u(x)=\left(u^1(x),u^2(x),\cdots,u^m(x)\right)$ are the dependent variables, the f_j are continuous functions and the G_j are differentiable functions with respect to their arguments. Suppose that (5.1) can be written into the form

$$u^j_{(k_{j,0})}[h_{j,0}](x) + F_j\left(x, u^{\alpha_{j,1}}_{(k_{j,1})}[h_{j,1}](x), u^{\alpha_{j,2}}_{(k_{j,2})}[h_{j,2}](x), \cdots, u^{\alpha_{j,r_j}}_{(k_{j,r_j})}[h_{j,r_j}](x)\right) = f_j(x), \qquad (5.2)$$

where $r_j, k_{j,i}, h_{j,i} \in \mathbb{N}$ with $\max\left\{k_{j,i}\ j = 1,2,\cdots,m,\ i = 0,\cdots,r_j\right\} = s$, $h_{j,i} \in \left\{1,2,\cdots,p_{k_{j,i}}\right\}$, $\alpha_{j,i} \in \{1,2,\cdots,m\}$ and F_j is a differential function with respect to its r_j arguments. Let L_j be the linear differential operator such that $L_j v = v_{(k_{j,0})}[h_{j,0}]$ and

$$L_j = \frac{\partial^{k_{j,0}}}{\left(\partial x^{\beta_{j,1}}\right)^{l_{j,1}} \left(\partial x^{\beta_{j,2}}\right)^{l_{j,2}} \cdots \left(\partial x^{\beta_{j,s_j}}\right)^{l_{j,s_j}}}$$

with $1 \le \beta_{j,1} < \beta_{j,2} < \cdots < \beta_{j,s_j} \le n$, $l_{j,1} + l_{j,2} + \cdots + l_{j,s_j} = k_{j,0}$ and $l_{j,i} \neq 0$. As in the previous case, it is suitable that L_j be the smallest order linear differential operator in G_j. Set $I_j = \left\{\beta_{j,1}, \beta_{j,2}, \cdots, \beta_{j,s_j}\right\}$. Define L_j^{-1}, the inverse of the operator L_j, by

$$\begin{aligned} L_j^{-1} v(x) \quad = \quad & \left(\int_{x_1^{\beta_{j,1}}}^{x^{\beta_{j,1}}} \int_{x_2^{\beta_{j,1}}}^{z^{\beta_{j,1}}_{l_{j,1}-1}} \cdots \int_{x^{\beta_{j,1}}_{l_{j,1}}}^{z_1^{\beta_{j,1}}}\right) \cdot \left(\int_{x_1^{\beta_{j,2}}}^{x^{\beta_{j,2}}} \int_{x_2^{\beta_{j,2}}}^{z^{\beta_{j,2}}_{l_{j,2}-1}} \cdots \int_{x^{\beta_{j,2}}_{l_{j,2}}}^{z_1^{\beta_{j,2}}}\right) \\ \cdots \quad & \left(\int_{x_1^{\beta_{j,s_j}}}^{x^{\beta_{j,s_j}}} \int_{x_2^{\beta_{j,s_j}}}^{z^{\beta_{j,s_j}}_{l_{j,s_j}-1}} \cdots \int_{x^{\beta_{j,s_j}}_{l_{j,s_j}}}^{z_1^{\beta_{j,s_j}}}\right) v\left(y^1, y^2, \cdots, y^n\right) \left(\mathrm{d}z_0^{\beta_{j,1}}\, \mathrm{d}z_1^{\beta_{j,1}} \cdots \mathrm{d}z^{i_1}_{l_{j,1}-1}\right) \\ \cdot \quad & \left(\mathrm{d}z_0^{\beta_{j,2}}\, \mathrm{d}z_1^{\beta_{j,2}} \cdots \mathrm{d}z^{\beta_{j,2}}_{l_{j,2}-1}\right) \cdots \left(\mathrm{d}z_0^{\beta_{j,s_j}}\, \mathrm{d}z_1^{\beta_{j,s_j}} \cdots \mathrm{d}z^{\beta_{j,s_j}}_{l_{j,s_j}-1}\right), \end{aligned}$$

where $y^i = z_0^i$ if $i \in I_j$ and $y^i = x^i$ otherwise. The constants x_k^i and the order of integration are closely related to the choice of boundary conditions in such a way that $L_j^{-1}(L_j v) = v(x) + g_j(x)$. Here, g_j is a function completely determined by the boundary conditions. Thus, applying L_j^{-1} to both sides of (5.2) yields

$$u^j(x) = -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j. \qquad (5.3)$$

Now as required in this case, the functions u^j can be derived as an infinite series:

$$u^j(x) = \sum_{\tau=0}^{+\infty} u^j_\tau(x), \qquad (5.4)$$

where the u^j_τ are functions to determine. Express the differential functions F_j as series

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \qquad (5.5)$$

where the $A_{j,\tau}$ are multivariate Adomian polynomials:

$$A_{j,\tau} = \frac{1}{\tau!}\left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau} F_j\left(x, \sum_{i=0}^{\tau} \lambda^i u^{\alpha_{j,1}}_{i,(k_{j,1})}[h_{j,1}], \sum_{i=0}^{\tau} \lambda^i u^{\alpha_{j,2}}_{i,(k_{j,2})}[h_{j,2}], \cdots, \sum_{i=0}^{\tau} \lambda^i u^{\alpha_{j,r_j}}_{i,(k_{j,r_j})}[h_{j,r_j}]\right)\right]_{\lambda=0}. \qquad (5.6)$$

The substitution in (5.3) of the expressions (5.4) and (5.5) along with (5.6) yields

$$\sum_{\tau=0}^{+\infty} u_\tau^j = -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1}\left(\sum_{\tau=0}^{+\infty} A_{j,\tau}\right). \tag{5.7}$$

The functions u_τ^j can be then obtained using the following recurrence relation

$$u_0^j = -g_j(x) + L_j^{-1} f_j(x), \quad u_{\tau+1}^j = -L_j^{-1} A_{j,\tau}, \quad \tau \geq 0, \quad j = 1,2,\cdots,m \tag{5.8}$$

since the $A_{j,\tau}$ depend only on $u_0^\beta, u_1^\beta, \cdots, u_\tau^\beta, \beta = 1,2,\cdots,m$. If the functions u_τ^j are determined for all $\tau \in \mathbb{N}$, then a solution u of the system (5.2) is immediately formed using the series (5.4). Otherwise, if only a finite number, says ν, of the functions u_τ^j is found, one can use the partial sums

$$\phi_\nu^j(x) = \sum_{\tau=0}^{\nu-1} u_\tau^j(x) \tag{5.9}$$

and consider the function $\phi_\nu = \left(\phi_\nu^1, \phi_\nu^2, \cdots, \phi_\nu^m\right)$ as an analytical approximate solution to the system of differential equations (5.2).

5.2 Application to the Ginzburg-Landau equations

Consider the Ginzburg-Landau (GL) equation [24, 25]

$$\begin{cases} \eta_1\left(\frac{\partial\psi}{\partial t} + i\kappa\phi\psi\right) + \left(\frac{i}{\kappa}\nabla + A\right)^2\psi - \psi + |\psi|^2\psi = 0 \\ \eta_2\left(\frac{\partial A}{\partial t} + \nabla\phi\right) + \text{curl curl}A + \frac{i}{2\kappa}\left(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}\right) + |\psi|^2 A = 0, \end{cases} \tag{5.10}$$

where

$$\begin{array}{llll} A(t,\cdot): & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longmapsto & (A^1(t,x,y,z), A^2\left(t,x,y,z), A^3(t,x,y,z)\right) \end{array}$$

is the real vector-valued magnetic potential;

$$\begin{array}{llll} \psi(t,\cdot): & \mathbb{R}^3 & \longrightarrow & \mathbb{C} \\ & (x,y,z) & \longmapsto & \psi^1(t,x,y,z) + i\psi^2(t,x,y,z) \end{array}$$

is the complex scalar-valued order parameter;

$$\begin{array}{llll} \phi(t,\cdot): & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & \phi(t,x,y,z) \end{array}$$

is the real scalar-valued electric potential; η_1 and η_2 are some given relaxation parameters and κ is the Ginzburg-Landau parameter. Assume that $\phi = -\text{div}(A) = -\left(A_x^1 + A_y^2 + A_z^3\right)$. Adopting the following notation: $u^1 = A^1$, $u^2 = A^2$, $u^3 = A^3$, $u^4 = \psi^1$, $u^5 = \psi^2$, and after some computation and identification, equations (5.10) become a determined system

$$u_t^j = F_j\left(u, u_{(1)}, u_{(2)}\right), \quad j = 1,2,3,4,5,\ldots \tag{5.11}$$

where

$$\begin{aligned}
F_1 &= -\frac{1}{\eta_2}\Big[-\eta_2 u^1_{2x} - u^1_{2y} - u^1_{2z} + (1-\eta_2)\left(u^2_{xy} + u^3_{xz}\right) - \frac{1}{\kappa}\left(u^4 u^5_x - u^5 u^4_x\right) \\
&+ u^1\left(\left(u^4\right)^2 + \left(u^5\right)^2\right),
\end{aligned}$$

$$\begin{aligned}
F_2 &= -\frac{1}{\eta_2}\Big[-u^2_{2x} - \eta_2 u^2_{2y} - u^2_{2z} + (1-\eta_2)\left(u^1_{xy} + u^3_{yz}\right) - \frac{1}{\kappa}\left(u^4 u^5_y - u^5 u^4_y\right) \\
&+ u^2\left(\left(u^4\right)^2 + \left(u^5\right)^2\right)\Big],
\end{aligned}$$

$$\begin{aligned}
F_3 &= -\frac{1}{\eta_2}\Big[-u^3_{2x} - u^3_{2y} - \eta_2 u^3_{2z} + (1-\eta_2)\left(u^1_{xz} + u^2_{yz}\right) - \frac{1}{\kappa}\left(u^4 u^5_z - u^5 u^4_z\right) \\
&+ u^3\left(\left(u^4\right)^2 + \left(u^5\right)^2\right)\Big],
\end{aligned}$$

$$\begin{aligned}
F_4 &= -\frac{1}{\eta_1}\Big[-\frac{1}{\kappa^2}\left(u^4_{2x} + u^4_{2y} + u^4_{2z}\right) + \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^5\left(u^1_x + u^2_y + u^3_z\right) \\
&- \frac{1}{\kappa}\left(u^1 u^5_x + u^2 u^5_y + u^3 u^5_z\right) + u^4\left(\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 + \left(u^4\right)^2 + \left(u^5\right)^2 - 1\right)\Big],
\end{aligned}$$

$$\begin{aligned}
F_5 &= -\frac{1}{\eta_1}\Big[-\frac{1}{\kappa^2}\left(u^5_{2x} + u^5_{2y} + u^5_{2z}\right) + \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^4\left(u^1_x + u^2_y + u^3_z\right) \\
&- \frac{1}{\kappa}\left(u^1 u^4_x + u^2 u^4_y + u^3 u^4_z\right) + u^5\left(\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 + \left(u^4\right)^2 + \left(u^5\right)^2 - 1\right)\Big],
\end{aligned} \tag{5.12}$$

etc.

Using Adomian decomposition method, we solve this system subject to the initial conditions

$$u^j(0,x,y,z) = x + y + z, \quad j = 1,2,3,4,5. \tag{5.13}$$

Consider the differential operator L

$$L = \frac{\partial}{\partial t} \tag{5.14}$$

and define its inverse operator L^{-1} by

$$L^{-1}\, v(t,x,y,z) = \int_0^t v(s,x,y,z)\,\mathrm{d}s. \tag{5.15}$$

Applying L^{-1} to both sides of (5.11) and using the initial condition (5.13), we find

$$u^j(t,x,y,z) = x + y + z + L^{-1}\, F_j\left(u, u_{(1)}, u_{(2)}\right), \quad j = 1,2,3,4,5. \tag{5.16}$$

The Adomian decomposition method suggests that the functions u^j can be determined as

$$u^j(t,x,y,z) = \sum_{\tau=0}^{+\infty} u_\tau^j(t,x,y,z) \tag{5.17}$$

while the differential functions F_j are also given as a series:

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \tag{5.18}$$

where the Adomian polynomials $A_{j,\tau}$ are obtained by the formula

$$A_{j,\tau} = \frac{1}{\tau!}\left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau} F_j\left(\sum_{i=0}^{\tau}\lambda^i u_i, \sum_{i=0}^{\tau}\lambda^i u_{i,(1)}, \sum_{i=0}^{\tau}\lambda^i u_{i,(2)}\right)\right]_{\lambda=0} \tag{5.19}$$

from which we get:

$$\begin{aligned}
A_{1,0} &= -\frac{1}{\eta_2}\Bigg[-\eta_2 u^1_{0,2x} - u^1_{0,2y} - u^1_{0,2z} + (1-\eta_2)\left(u^2_{0,xy} + u^3_{0,xz}\right) - \frac{1}{\kappa}\left(u^4_0 u^5_{0,x} - u^5_0 u^4_{0,x}\right)\\
&+ u^1_0\left(\left(u^4_0\right)^2 + \left(u^5_0\right)^2\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
A_{2,0} &= -\frac{1}{\eta_2}\Bigg[-u^2_{0,2x} - \eta_2 u^2_{0,2y} - u^2_{0,2z} + (1-\eta_2)\left(u^1_{0,xy} + u^3_{0,yz}\right) - \frac{1}{\kappa}\left(u^4_0 u^5_{0,y} - u^5_0 u^4_{0,y}\right)\\
&+ u^2_0\left(\left(u^4_0\right)^2 + \left(u^5_0\right)^2\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
A_{3,0} &= -\frac{1}{\eta_2}\Bigg[-u^3_{0,2x} - u^3_{0,2y} - \eta_2 u^3_{0,2z} + (1-\eta_2)\left(u^1_{0,xz} + u^2_{0,yz}\right) - \frac{1}{\kappa}\left(u^4_0 u^5_{0,z} - u^5_0 u^4_{0,z}\right)\\
&+ u^3_0\left(\left(u^4_0\right)^2 + \left(u^5_0\right)^2\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
A_{4,0} &= -\frac{1}{\eta_1}\Bigg[-\frac{1}{\kappa^2}\left(u^4_{0,2x} + u^4_{0,2y} + u^4_{0,2z}\right) + \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^5_0\left(u^1_{0,x} + u^2_{0,y} + u^3_{0,z}\right)\\
&- \frac{1}{\kappa}\left(u^1_0 u^5_{0,x} + u^2_0 u^5_{0,y} + u^3_0 u^5_{0,z}\right)\\
&+ u^4_0\left(\left(u^1_0\right)^2 + \left(u^2_0\right)^2 + \left(u^3_0\right)^2 + \left(u^4_0\right)^2 + \left(u^5_0\right)^2 - 1\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
A_{5,0} &= -\frac{1}{\eta_1}\Bigg[-\frac{1}{\kappa^2}\left(u^5_{0,2x} + u^5_{0,2y} + u^5_{0,2z}\right) + \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^4_0\left(u^1_{0,x} + u^2_{0,y} + u^3_{0,z}\right)\\
&- \frac{1}{\kappa}\left(u^1_0 u^4_{0,x} + u^2_0 u^4_{0,y} + u^3_0 u^4_{0,z}\right)\\
&+ u^5_0\left(\left(u^1_0\right)^2 + \left(u^2_0\right)^2 + \left(u^3_0\right)^2 + \left(u^4_0\right)^2 + \left(u^5_0\right)^2 - 1\right)\Bigg],
\end{aligned}$$

$$\begin{aligned} A_{1,1} &= -\frac{1}{\eta_2}\Big[-\eta_2 u^1_{1,2x} - u^1_{1,2y} - u^1_{1,2z} + (1-\eta_2)\left(u^2_{1,xy} + u^3_{1,xz}\right) + u^1_1\left(\left(u^4_0\right)^2 + \left(u^5_0\right)^2\right) \\ &- \frac{1}{\kappa}\left(u^4_1 u^5_{0,x} + u^4_0 u^5_{1,x} - u^5_1 u^4_{0,x} - u^5_0 u^4_{1,x}\right) + u^1_0\left(2u^4_0 u^4_1 + 2u^5_0 u^5_1\right)\Big], \end{aligned}$$

$$\begin{aligned} A_{2,1} &= -\frac{1}{\eta_2}\Big[-u^2_{1,2x} - \eta_2 u^2_{1,2y} - u^2_{1,2z} + (1-\eta_2)\left(u^1_{1,xy} + u^3_{1,yz}\right) + u^2_1\left(\left(u^4_0\right)^2 + \left(u^5_0\right)^2\right) \\ &- \frac{1}{\kappa}\left(u^4 1 u^5_{0,y} + u^4_0 u^5_{1,y} - u^5_1 u^4_{0,y} - u^5_0 u^4_{1,y}\right) + u^2_0\left(2u^4_0 u^4_1 + 2u^5_0 u^5_1\right)\Big], \end{aligned}$$

$$\begin{aligned} A_{3,1} &= -\frac{1}{\eta_2}\Big[-u^3_{1,2x} - u^3_{1,2y} - \eta_2 u^3_{1,2z} + (1-\eta_2)\left(u^1_{1,xz} + u^2_{1,yz}\right) + u^3_1\left(\left(u^4_0\right)^2 + \left(u^5_0\right)^2\right) \\ &- \frac{1}{\kappa}\left(u^4_1 u^5_{0,z} + u^4_0 u^5_{1,z} - u^5_1 u^4_{0,z} - u^5_0 u^4_{1,z}\right) + u^3_0\left(2u^4_0 u^4_1 + 2u^5_0 u^5_1\right)\Big], \end{aligned}$$

$$\begin{aligned} A_{4,1} &= -\frac{1}{\eta_1}\Big[-\frac{1}{\kappa^2}\left(u^4_{1,2x} + u^4_{1,2y} + u^4_{1,2z}\right) + \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^5_1\left(u^1_{0,x} + u^2_{0,y} + u^3_{0,z}\right) \\ &+ u^4_0\left(2u^1_0 u^1_1 + 2u^3_0 u^3_1 + 2u^2_0 u^2_1 + 2u^5_0 u^5_1 + 2u^4_0 u^4_1\right) \\ &+ \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^5_0\left(u^1_{1,x} + u^2_{1,y} + u^3 1,z\right) \\ &- \frac{1}{\kappa}\left(u^1_1 u^5_{0,x} + u^1_0 u^5_{1,x} + u^2_1 u^5_{0,y} + u^2_0 u^5_{1,y} + u^3_1 u^5_{0,z} + u^3_0 u^5_{1,z}\right) \\ &+ u^4_1\left(\left(u^1_0\right)^2 + \left(u^2_0\right)^2 + \left(u^3_0\right)^2 + \left(u^4_0\right)^2 + \left(u^5_0\right)^2 - 1\right)\Big], \end{aligned}$$

$$\begin{aligned} A_{5,1} &= -\frac{1}{\eta_1}\Big[-\frac{1}{\kappa^2}\left(u^5_{1,2x} + u^5_{1,2y} + u^5_{1,2z}\right) + \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^4_1\left(u^1_{0,x} + u^2_{0,y} + u^3_{0,z}\right) \\ &+ u^5_0\left(2u^1_0 u^1_1 + 2u^3_0 u^3_1 + 2u^2_0 u^2_1 + 2u^5_0 u^5_1 + 2u^4_0 u^4_1\right) \\ &+ \left(\kappa\eta_1 - \frac{1}{\kappa}\right)u^5_0\left(u^1_{1,x} + u^2_{1,y} + u^3 1,z\right) \\ &- \frac{1}{\kappa}\left(u^1_1 u^4_{0,x} + u^1_0 u^4_{1,x} + u^2_1 u^4_{0,y} + u^2_0 u^4_{1,y} + u^3_1 u^4_{0,z} + u^3_0 u^4_{1,z}\right) \\ &+ u^5_1\left(\left(u^1_0\right)^2 + \left(u^2_0\right)^2 + \left(u^3_0\right)^2 + \left(u^4_0\right)^2 + \left(u^5_0\right)^2 - 1\right)\Big]; \end{aligned}$$

and so on. Then, from

$$\sum_{\tau=0}^{+\infty} u^j_\tau = x + y + z + L^{-1}\left(\sum_{\tau=0}^{+\infty} A_{j,\tau}\right) \tag{5.20}$$

we find the recursive relation

$$u^j_0 = x + y + z, \quad u^j_{\tau+1} = L^{-1} A_{j,\tau}, \quad \tau \geq 0, \quad j = 1,2,3,4,5. \tag{5.21}$$

The first components of the latter are determined in the form, setting $X = x + y + z$:

$$u_1^1 = u_1^2 = u_1^3 = -\frac{2X^3}{\eta_2} t,$$

$$u_1^4 = u_1^5 = \left[-\frac{5X^3}{\eta_1} + \left(-3\kappa + \frac{6}{\kappa\eta_1} + \frac{1}{\eta_1} \right) X \right] t,$$

$$u_2^1 = u_2^2 = u_2^3 = \left[\left(\frac{10}{\eta_2\eta_1} + \frac{2}{\eta_2^2} \right) X^5 + \left(-\frac{2}{\eta_2\eta_1} - \frac{12}{\eta_2\eta_1\kappa} + \frac{6\kappa}{\eta_2} \right) X^3 - \frac{18X}{\eta_2} \right] t^2,$$

$$\begin{aligned} u_2^4 = u_2^5 \quad = \quad & \left[\left(\frac{45}{2\eta_1^2} + \frac{6}{\eta_2\eta_1} \right) X^5 + \left(-\frac{12}{\eta_2\eta_1\kappa} - \frac{7}{\eta_1^2} + \frac{21\kappa}{\eta_1} - \frac{57}{\eta_1^2\kappa} + \frac{9\kappa}{\eta_2} \right) X^3 \right. \\ + \quad & \left. \left(-\frac{27}{\eta_1^2\kappa^2} + \frac{6}{\eta_1^2\kappa} - \frac{3\kappa}{\eta_1} + \frac{9\kappa^2}{2} + \frac{1}{2\eta_1^2} - \frac{18}{\eta_1} \right) X \right] t^2, \end{aligned}$$

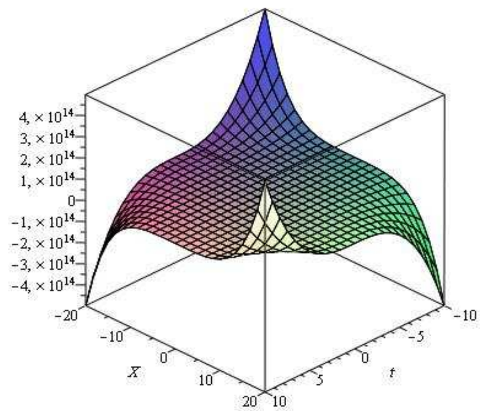
$$\begin{aligned} u_3^1 = u_3^2 = u_3^3 \quad = \quad & \left[\left(-\frac{12}{\eta_2\eta_1} + \frac{36\kappa}{\eta_2} - \frac{72}{\eta_2\eta_1\kappa} \right) X + \left(-\frac{140}{3\eta_2\eta_1^2} - \frac{4}{3\eta_2^3} - \frac{28}{\eta_2^2\eta_1} \right) X^7 \right. \\ + \quad & \left(\frac{8\kappa}{\eta_2\eta_1} - \frac{16}{\eta_2\kappa\eta_1^2} + \frac{248}{\eta_2\eta_1} - \frac{4}{3\eta_2\eta_1^2} - \frac{12\kappa^2}{\eta_2} + \frac{52}{\eta_2^2} + \frac{12}{\eta_2\kappa^2\eta_1^2} \right) X^3 \\ + \quad & \left. \left(\frac{16}{\eta_2\eta_1^2} + \frac{116}{\eta_2\kappa\eta_1^2} - \frac{48\kappa}{\eta_2\eta_1} - \frac{24\kappa}{\eta_2^2} + \frac{4}{\eta_2^2\eta_1} + \frac{40}{\eta_2^2\kappa\eta_1} \right) X^5 \right] t^3, \end{aligned}$$

$$\begin{aligned} u_3^4 = u_3^5 \quad = \quad & \left[\left(-\frac{58}{\eta_2\eta_1^2} - \frac{8}{\eta_2^2\eta_1} - \frac{235}{2\eta_1^3} \right) X^7 \right. \\ + \quad & \left(-\frac{10\kappa}{\eta_2^2} - \frac{291\kappa}{2\eta_1^2} + \frac{12}{\eta_2^2\kappa\eta_1} + \frac{10}{\eta_2\eta_1^2} + \frac{97}{2\eta_1^3} - \frac{137\kappa}{\eta_2\eta_1} + \frac{240}{\eta_2\kappa\eta_1^2} + \frac{426}{\eta_1^3\kappa} \right) X^5 \\ + \quad & \left(\frac{267}{\eta_1^2} - \frac{45\kappa^2}{\eta_2} - \frac{35}{6\eta_1^3} - \frac{20}{\eta_2\kappa\eta_1^2} - \frac{89}{\eta_1^3\kappa} + \frac{231}{\eta_1^3\kappa^2} + \frac{204}{\eta_2\eta_1} + \frac{35\kappa}{\eta_1^2} - \frac{105\kappa^2}{2\eta_1} \right. \\ - \quad & \left. \frac{24}{\eta_2\kappa^2\eta_1^2} + \frac{15\kappa}{\eta_2\eta_1} \right) X^3 + \left(\frac{27\kappa}{\eta_1} + \frac{117}{\eta_1^2\kappa} - \frac{72}{\eta_1^2\kappa^3\eta_2} + \frac{3}{\eta_1^3\kappa} + \frac{18}{\eta_2\eta_1\kappa} - \frac{9\kappa^3}{2} \right. \\ + \quad & \left. \left. \frac{1}{6\eta_1^3} - \frac{18}{\eta_1^2} - \frac{3\kappa}{2\eta_1^2} - \frac{396}{\eta_1^3\kappa^3} + \frac{9\kappa^2}{2\eta_1} - \frac{39}{\eta_1^3\kappa^2} + \frac{18\kappa}{\eta_2} \right) X \right] t^3. \end{aligned}$$

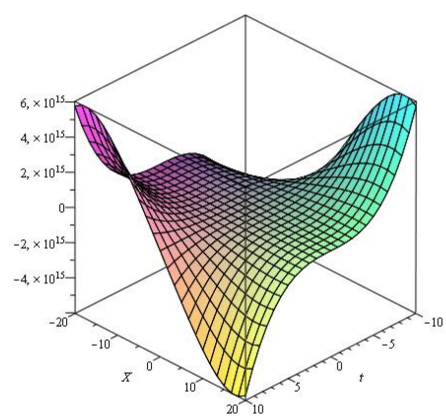
Therefore, an analytical approximate solution to the Ginzburg-Landau equations (5.10) under the assumption $\phi = -\mathrm{div}(A) = -\left(A_x^1 + A_y^2 + A_z^3\right)$ with the initial condition (5.13) is expressed as

$$A^1 = A^2 = A^3 \simeq u_0^1 + u_1^1 + u_2^1 + u_3^1, \quad \psi^1 = \psi^2 \simeq u_0^4 + u_1^4 + u_2^4 + u_3^4.$$

For fixed parameters $\eta_1 = \eta_2 = 1$ and $\kappa = 10^{-3}$, the graphic representation is shown in the Figure 3.



Function $A^1 = A^2 = A^3$ versus X and t`.`



Function $\psi^1 = \psi^2$ versus X and t.

Figure 3. Approximate solution of the Ginzburg-Landau system (5.11) versus $X = x + y + z$ and t.

5.3 Extension of the Adomian decomposition method to under-determined systems

Consider the system of partial differential equations

$$G_j\left(x, u^{(s)}(x), \widetilde{u}^{(s)}(x)\right) = f_j(x), \quad j = 1, 2, \cdots, m \tag{5.22}$$

where the nonzero positive integer s is the order of the system, $x = \left(x^1, \cdots, x^n\right)$ are independent variables, $u = u(x) = \left(u^1(x), u^2(x), \cdots, u^m(x)\right)$, $\widetilde{u} = \widetilde{u}(x) = \left(\widetilde{u}^1(x), \widetilde{u}^2(x), \cdots, \widetilde{u}^{\widetilde{m}}(x)\right)$ are the dependent variables, the f_j are continuous functions and the G_j are differentiable functions with respect to their arguments. Suppose that (5.22) can be written into the form (5.23) for $\widetilde{m} < m$

$$\begin{aligned} f_j &= u^j_{(k_{j,0})}[h_{j,0}] + \widetilde{u}^j_{(k_{j,0})}[h_{j,0}] \\ &+ F_j\left(x, u^{\alpha_{j,1}}_{(k_{j,1})}[h_{j,1}], \cdots, u^{\alpha_{j,r_j}}_{(k_{j,r_j})}[h_{j,r_j}], \widetilde{u}^{\widetilde{\alpha}_{j,1}}_{(\widetilde{k}_{j,1})}[\widetilde{h}_{j,1}], \cdots, \widetilde{u}^{\widetilde{\alpha}_{j,\widetilde{r}_j}}_{(\widetilde{k}_{j,\widetilde{r}_j})}[\widetilde{h}_{j,\widetilde{r}_j}]\right), \\ & j = 1, 2, \cdots, \widetilde{m}, \\ f_j &= u^j_{(k_{j,0})}[h_{j,0}] + F_j\left(x, u^{\alpha_{j,1}}_{(k_{j,1})}[h_{j,1}], \cdots, u^{\alpha_{j,r_j}}_{(k_{j,r_j})}[h_{j,r_j}], \widetilde{u}^{\widetilde{\alpha}_{j,1}}_{(\widetilde{k}_{j,1})}[\widetilde{h}_{j,1}], \cdots, \widetilde{u}^{\widetilde{\alpha}_{j,\widetilde{r}_j}}_{(\widetilde{k}_{j,\widetilde{r}_j})}[\widetilde{h}_{j,\widetilde{r}_j}]\right), \\ & j = \widetilde{m}+1, \widetilde{m}+2, \cdots, m \end{aligned} \tag{5.23}$$

or into the form (5.24) for $\widetilde{m} \geq m$

$$\begin{aligned} f_j &= u^j_{(k_{j,0})}[h_{j,0}] + \widetilde{u}^j_{(k_{j,0})}[h_{j,0}] \\ &+ F_j\left(x, u^{\alpha_{j,1}}_{(k_{j,1})}[h_{j,1}], \cdots, u^{\alpha_{j,r_j}}_{(k_{j,r_j})}[h_{j,r_j}], \widetilde{u}^{\widetilde{\alpha}_{j,1}}_{(\widetilde{k}_{j,1})}[\widetilde{h}_{j,1}], \cdots, \widetilde{u}^{\widetilde{\alpha}_{j,\widetilde{r}_j}}_{(\widetilde{k}_{j,\widetilde{r}_j})}[\widetilde{h}_{j,\widetilde{r}_j}]\right), \\ & j = 1, 2, \cdots, m-1, \\ f_m &= u^m_{(k_{m,0})}[h_{m,0}] + \sum_{\widetilde{j}=m}^{\widetilde{m}} \widetilde{u}^{\widetilde{j}}_{(k_{m,0})}[h_{m,0}] \\ &+ F_m\left(x, u^{\alpha_{j,1}}_{(k_{j,1})}[h_{j,1}], \cdots, u^{\alpha_{j,r_j}}_{(k_{j,r_j})}[h_{j,r_j}], \widetilde{u}^{\widetilde{\alpha}_{j,1}}_{(\widetilde{k}_{j,1})}[\widetilde{h}_{j,1}], \cdots, \widetilde{u}^{\widetilde{\alpha}_{j,\widetilde{r}_j}}_{(\widetilde{k}_{j,\widetilde{r}_j})}[\widetilde{h}_{j,\widetilde{r}_j}]\right), \end{aligned} \tag{5.24}$$

where $r_j, k_{j,i}, h_{j,i}, \widetilde{r}_j, \widetilde{k}_{j,i}, \widetilde{h}_{j,i} \in \mathbb{N}$ with $\max\left\{k_{j,i}, \widetilde{k}_{j,i}\right\} = s$, $h_{j,i} \in \left\{1, 2, \cdots, p_{k_{j,i}}\right\}$, $\widetilde{h}_{j,i} \in \left\{1, 2, \cdots, p_{\widetilde{k}_{j,i}}\right\}$, $\alpha_{j,i} \in \{1, 2, \cdots, m\}$, $\widetilde{\alpha}_{j,i} \in \{1, 2, \cdots, \widetilde{m}\}$ and F_j is a differential function with respect to its r_j arguments. Let L_j be the linear differential operator such that $L_j v = v_{(k_{j,0})}[h_{j,0}]$ and

$$L_j = \frac{\partial^{k_{j,0}}}{\left(\partial x^{\beta_{j,1}}\right)^{l_{j,1}} \left(\partial x^{\beta_{j,2}}\right)^{l_{j,2}} \cdots \left(\partial x^{\beta_{j,s_j}}\right)^{l_{j,s_j}}}$$

with $1 \leq \beta_{j,1} < \beta_{j,2} < \cdots < \beta_{j,s_j} \leq n$, $l_{j,1} + l_{j,2} + \cdots + l_{j,s_j} = k_{j,0}$ and $l_{j,i} \neq 0$. As required, L_j has to be the smallest order linear differential operator in G_j. Set $I_j = \left\{\beta_{j,1}, \beta_{j,2}, \cdots, \beta_{j,s_j}\right\}$.

Define as previously L_j^{-1}, the inverse of the operator L_j by

$$\begin{aligned} L_j^{-1} v(x) &= \left(\int_{x_1^{\beta_{j,1}}}^{x^{\beta_{j,1}}} \int_{x_2^{\beta_{j,1}}}^{z_{l_{j,1}-1}^{\beta_{j,1}}} \cdots \int_{x_{l_{j,1}}^{\beta_{j,1}}}^{z_1^{\beta_{j,1}}} \right) \cdot \left(\int_{x_1^{\beta_{j,2}}}^{x^{\beta_{j,2}}} \int_{x_2^{\beta_{j,2}}}^{z_{l_{j,2}-1}^{\beta_{j,2}}} \cdots \int_{x_{l_{j,2}}^{\beta_{j,2}}}^{z_1^{\beta_{j,2}}} \right) \\ &\cdots \left(\int_{x_1^{\beta_{j,s_j}}}^{x^{\beta_{j,s_j}}} \int_{x_2^{\beta_{j,s_j}}}^{z_{l_{j,s_j}-1}^{\beta_{j,s_j}}} \cdots \int_{x_{l_{j,s_j}}^{\beta_{j,s_j}}}^{z_1^{\beta_{j,s_j}}} \right) v\left(y^1, y^2, \cdots, y^n\right) \left(\mathrm{d}z_0^{\beta_{j,1}} \, \mathrm{d}z_1^{\beta_{j,1}} \cdots \mathrm{d}z_{l_{j,1}-1}^{i_1} \right) \\ &\cdot \left(\mathrm{d}z_0^{\beta_{j,2}} \, \mathrm{d}z_1^{\beta_{j,2}} \cdots \mathrm{d}z_{l_{j,2}-1}^{\beta_{j,2}} \right) \cdots \left(\mathrm{d}z_0^{\beta_{j,s_j}} \, \mathrm{d}z_1^{\beta_{j,s_j}} \cdots \mathrm{d}z_{l_{j,s_j}-1}^{\beta_{j,s_j}} \right), \end{aligned}$$

where $y^i = z_0^i$ if $i \in I_j$ and $y^i = x^i$ otherwise. The constants x_k^i together with the order of integration are closely related to the choice of boundary conditions in such a way that $L_j^{-1}(L_j v) = v(x) + g_j(x)$. Here, g_j is a function completely determined by the boundary conditions. Thus, applying L_j^{-1} to both sides of (5.23) yields

$$\begin{aligned} u^j(x) + \widetilde{u}^j(x) &= -g_j(x) - \widetilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j, \quad j = 1, 2, \cdots, \widetilde{m}, \\ u^j(x) &= -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j, \quad j = \widetilde{m}+1, \widetilde{m}+2, \cdots, m; \end{aligned} \tag{5.25}$$

and applying L_j^{-1} to both sides of (5.24) gives

$$\begin{aligned} u^j(x) + \widetilde{u}^j(x) &= -g_j(x) - \widetilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1} F_j, \quad j = 1, 2, \cdots, m-1, \\ u^m(x) + \sum_{\widetilde{j}=m}^{\widetilde{m}} \widetilde{u}^{\widetilde{j}}(x) &= -g_m(x) - \sum_{\widetilde{j}=m}^{\widetilde{m}} \widetilde{g}_{\widetilde{j}}(x) + L_m^{-1} f_m(x) - L_m^{-1} F_m. \end{aligned} \tag{5.26}$$

The Adomian decomposition method allows to represent the functions u^j and $\widetilde{u}^{\widetilde{j}}$ as infinite series, i.e.

$$u^j(x) = \sum_{\tau=0}^{+\infty} u_\tau^j(x), \quad \widetilde{u}^{\widetilde{j}}(x) = \sum_{\tau=0}^{+\infty} \widetilde{u}_\tau^{\widetilde{j}}(x), \tag{5.27}$$

where the u_τ^j and $\widetilde{u}_\tau^{\widetilde{j}}$ are functions to determine. Express the differential functions F_j as series

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \tag{5.28}$$

where the $A_{j,\tau}$ are multivariate polynomials also called Adomian polynomials

$$\begin{aligned} A_{j,\tau} &= \frac{1}{\tau!} \left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau} F_j \left(x, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,1})}^{\alpha_{j,1}}[h_{j,1}], \cdots, \sum_{i=0}^{\tau} \lambda^i u_{i,(k_{j,r_j})}^{\alpha_{j,r_j}}[h_{j,r_j}], \right.\right. \\ &\left.\left. \sum_{i=0}^{\tau} \lambda^i \widetilde{u}_{i,(\widetilde{k}_{j,1})}^{\widetilde{\alpha}_{j,1}}[\widetilde{h}_{j,1}], \cdots, \sum_{i=0}^{\tau} \lambda^i \widetilde{u}_{i,(\widetilde{k}_{j,\widetilde{r}_j})}^{\widetilde{\alpha}_{j,\widetilde{r}_j}}[\widetilde{h}_{j,\widetilde{r}_j}] \right) \right]_{\lambda=0}. \end{aligned} \tag{5.29}$$

The substitution in (5.25) of the expressions (5.27) and (5.28) along with (5.29) yields

$$\begin{aligned}\sum_{\tau=0}^{+\infty} u_\tau^j + \sum_{\tau=0}^{+\infty} \widetilde{u}_\tau^j &= -g_j(x) - \widetilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1}\left(\sum_{\tau=0}^{+\infty} A_{j,\tau}\right), \quad j = 1,2,\cdots,\widetilde{m},\\ \sum_{\tau=0}^{+\infty} u_\tau^j &= -g_j(x) + L_j^{-1} f_j(x) - L_j^{-1}\left(\sum_{\tau=0}^{+\infty} A_{j,\tau}\right), \quad j = \widetilde{m}+1,\widetilde{m}+2,\cdots,m.\end{aligned} \tag{5.30}$$

Now substituting in (5.26) the expressions (5.27) and (5.28) along with (5.29) gives

$$\begin{aligned}\sum_{\tau=0}^{+\infty} u_\tau^j + \sum_{\tau=0}^{+\infty} \widetilde{u}_\tau^j &= -g_j(x) - \widetilde{g}_j(x) + L_j^{-1} f_j(x) - L_j^{-1}\left(\sum_{\tau=0}^{+\infty} A_{j,\tau}\right),\\ & \qquad j = 1,2,\cdots,m-1,\\ \sum_{\tau=0}^{+\infty} u_\tau^m + \sum_{\tau=0}^{+\infty}\left(\sum_{\widetilde{j}=m}^{\widetilde{m}} \widetilde{u}_\tau^{\widetilde{j}}\right) &= -g_m(x) - \sum_{\widetilde{j}=m}^{\widetilde{m}} \widetilde{g}_{\widetilde{j}}(x) + L_m^{-1} f_m(x) - L_m^{-1}\left(\sum_{\tau=0}^{+\infty} A_{j,\tau}\right).\end{aligned} \tag{5.31}$$

The functions u_τ^j and $\widetilde{u}_\tau^{\widetilde{j}}$ can be obtained using the following recursive relations.

- For $\widetilde{m} < m$, if $j = 1,2,\cdots,\widetilde{m}$

$$\begin{aligned}u_0^j &= -g_j(x) + L_j^{-1} f_j(x), \quad \widetilde{u}_0^j = -\widetilde{g}_j(x),\\ u_{\tau+1}^j &= -L_j^{-1}\left(B_{j,\tau}^0\right), \quad \widetilde{u}_{\tau+1}^j = -L_j^{-1}\left(B_{j,\tau}^1\right),\\ & \tau \geq 0, \quad B_{j,\tau}^0 + B_{j,\tau}^1 = A_{j,\tau}\end{aligned} \tag{5.32}$$

  and if $j = \widetilde{m}+1,\widetilde{m}+2,\cdots,m$

$$u_0^j = -g_j(x) + L_j^{-1} f_j(x), \quad u_{\tau+1}^j = -L_j^{-1} A_{j,\tau}, \quad \tau \geq 0. \tag{5.33}$$

- For $\widetilde{m} \geq m$, if $j = 1,2,\cdots,m-1$

$$\begin{aligned}u_0^j &= -g_j(x) + L_j^{-1} f_j(x), \quad \widetilde{u}_0^j = -\widetilde{g}_j(x),\\ u_{\tau+1}^j &= -L_j^{-1}\left(B_{j,\tau}^0\right), \quad \widetilde{u}_{\tau+1}^j = -L_j^{-1}\left(B_{j,\tau}^1\right),\\ & \tau \geq 0, \quad B_{j,\tau}^0 + B_{j,\tau}^1 = A_{j,\tau}\end{aligned} \tag{5.34}$$

  and if $\widetilde{j} = m,m+1,\cdots,\widetilde{m}$

$$\begin{aligned}u_0^m &= -g_m(x) + L_m^{-1} f_m(x), \quad \widetilde{u}_0^{\widetilde{j}} = -\widetilde{g}_{\widetilde{j}}(x),\\ u_{\tau+1}^m &= -L_m^{-1}\left(B_{m,\tau}^0\right), \quad \widetilde{u}_{\tau+1}^{\widetilde{j}} = -L_m^{-1}\left(B_{m,\tau}^{\widetilde{j}-m+1}\right),\\ & \tau \geq 0, \quad B_{m,\tau}^0 + \sum_{i=1}^{\widetilde{m}-m+1} B_{m,\tau}^i = A_{m,\tau}\end{aligned} \tag{5.35}$$

since the $A_{j,\tau}$ depend only on $u_0^\beta, \widetilde{u}_0^{\widetilde{\beta}}, u_1^\beta, \widetilde{u}_1^{\widetilde{\beta}}, \cdots, u_\tau^\beta, \widetilde{u}_\tau^{\widetilde{\beta}}, \beta = 1,2,\cdots,m, \widetilde{\beta} = 1,2,\cdots,\widetilde{m}$.

If the functions u_τ^j and $\widetilde{u}_\tau^{\widetilde{j}}$ are determined for all $\tau \in \mathbb{N}$, then a solution $(u,\widetilde{u})$ of the system (5.23) or (5.24) is immediately formed using the series (5.27). Otherwise, if only a finite number, says ν, of the functions u_τ^j and $\widetilde{u}_\tau^{\widetilde{j}}$ is found, one can use the partial sums

$$\phi_\nu^j(x) = \sum_{\tau=0}^{\nu-1} u_\tau^j(x), \quad \widetilde{\phi}_\nu^{\widetilde{j}}(x) = \sum_{\tau=0}^{\nu-1} \widetilde{u}_\tau^{\widetilde{j}}(x) \tag{5.36}$$

and consider the function $\phi_\nu = \left(\phi_\nu^1, \phi_\nu^2, \cdots, \phi_\nu^m, \widetilde{\phi}_\nu^1, \widetilde{\phi}_\nu^2, \cdots, \widetilde{\phi}_\nu^{\widetilde{m}}\right)$ as an approximate solution to the system of differential equations (5.23) or (5.24).

5.4 Application to Ginzburg-Landau equations

Consider again the Ginzburg-Landau equations (5.10) but without any assumption on the function ϕ. Adopting the following notation: $u^1 = A^1$, $u^2 = A^2$, $u^3 = A^3$, $u^4 = \psi^1$, $u^5 = \psi^2$, $\widetilde{u} = \phi$. After some computation and identification, the equations (5.10) generate the under determined system

$$\begin{aligned} u_t^4 + \widetilde{u}_t &= F_4^0\left(u, u_{(1)}, \widetilde{u}_{(1)}, u_{(2)}\right) + F_4^1\left(u, \widetilde{u}, u_{(1)}, u_{(2)}\right), \\ u_t^5 &= F_5\left(u, \widetilde{u}, u_{(1)}, u_{(2)}\right), \\ u_t^j &= F_j\left(u, u_{(1)}, \widetilde{u}_{(1)}, u_{(2)}\right), \quad j = 1,2,3, \end{aligned} \tag{5.37}$$

where

$$\begin{aligned} F_4^0 &= -\frac{1}{\eta_1}\left[-\eta_1 \widetilde{u}_t - \frac{1}{\kappa^2}\left(u_{2x}^4 + u_{2y}^4 + u_{2z}^4\right)\right. \\ &+ \left. u^4\left(\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 + \left(u^4\right)^2 + \left(u^5\right)^2 - 1\right)\right], \end{aligned}$$

$$F_4^1 = -\frac{1}{\eta_1}\left[-\eta_1 \kappa \widetilde{u} u^5 - \frac{1}{\kappa} u^5\left(u_x^1 + u_y^2 + u_z^3\right) - \frac{1}{\kappa}\left(u^1 u_x^5 + u^2 u_y^5 + u^3 u_z^5\right)\right],$$

$$\begin{aligned} F_5 &= -\frac{1}{\eta_1}\left[-\frac{1}{\kappa^2}\left(u_{2x}^5 + u_{2y}^5 + u_{2z}^5\right) - \frac{1}{\kappa} u^4\left(u_x^1 + u_y^2 + u_z^3\right) - \eta_1 \kappa \widetilde{u} u^4\right. \\ &- \left. \frac{1}{\kappa}\left(u^1 u_x^4 + u^2 u_y^4 + u^3 u_z^4\right) + u^5\left(\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 + \left(u^4\right)^2 + \left(u^5\right)^2 - 1\right)\right], \end{aligned}$$

$$F_1 = -\frac{1}{\eta_2}\left[u_{xy}^2 - u_{2y}^1 - u_{2z}^1 + u_{xz}^3 - \frac{1}{\kappa}\left(u^4 u_x^5 - u^5 u_x^4\right) + \eta_2 \widetilde{u}_x + u^1\left(\left(u^4\right)^2 + \left(u^5\right)^2\right)\right],$$

$$F_2 = -\frac{1}{\eta_2}\left[u_{yz}^3 - u_{2z}^2 - u_{2x}^2 + u_{xy}^1 - \frac{1}{\kappa}\left(u^4 u_y^5 - u_0^5 u_y^4\right) + \eta_2 \widetilde{u}_y + u^2\left(\left(u^4\right)^2 + \left(u^5\right)^2\right)\right],$$

$$F_3 = -\frac{1}{\eta_2}\left[u^1_{xz} - u^3_{2x} - u^3_{2y} + u^2_{yz} - \frac{1}{\kappa}\left(u^4 u^5_z - u^5 u^4_z\right) + \eta_2 \widetilde{u}_z + u^3\left(\left(u^4\right)^2 + \left(u^5\right)^2\right)\right].$$

Using the Adomian decomposition method, we solve this system with the initial conditions

$$\widetilde{u}(0,x,y,z) = u^j(0,x,y,z) = x+y+z, \quad j = 1,2,3,4,5. \tag{5.38}$$

For that, consider the differential operator L

$$L = \frac{\partial}{\partial t} \tag{5.39}$$

and define its inverse operator L^{-1} by

$$L^{-1}\, v(t,x,y,z) = \int_0^t v(s,x,y,z)\,\mathrm{d}s. \tag{5.40}$$

Applying L^{-1} to both sides of (5.37) and using the initial conditions (5.38), we find

$$\begin{aligned} u^4 + \widetilde{u} &= 2(x+y+z) + L^{-1}\, F_4^0\left(u, u_{(1)}, \widetilde{u}_{(1)}, u_{(2)}\right) + L^{-1}\, F_4^1\left(u, \widetilde{u}, u_{(1)}, u_{(2)}\right), \\ u^5 &= x+y+z+L^{-1}\, F_5\left(u, \widetilde{u}, u_{(1)}, u_{(2)}\right), \\ u^j &= x+y+z+L^{-1}\, F_j\left(u, u_{(1)}, \widetilde{u}_{(1)}, u_{(2)}\right), \quad j = 1,2,3. \end{aligned} \tag{5.41}$$

The Adomian decomposition method suggests that the functions $u^j, \widetilde{u}$ can be sought as

$$u^j(t,x,y,z) = \sum_{\tau=0}^{+\infty} u^j_\tau(t,x,y,z), \quad \widetilde{u}(t,x,y,z) = \sum_{\tau=0}^{+\infty} \widetilde{u}_\tau(t,x,y,z) \tag{5.42}$$

while the differential functions F_j, F_4^0, F_4^1 can be developed in a series:

$$F_j = \sum_{\tau=0}^{+\infty} A_{j,\tau}, \quad F_4^0 = \sum_{\tau=0}^{+\infty} B^0_{4,\tau}, \quad F_4^1 = \sum_{\tau=0}^{+\infty} B^1_{4,\tau}, \tag{5.43}$$

where the $A_{j,\tau}$, $B^0_{4,\tau}$, $B^1_{4,\tau}$ are Adomian polynomials obtained by the formula

$$A_{j,\tau} = \frac{1}{\tau!}\left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau} F_j\left(\sum_{i=0}^{\tau}\lambda^i u_i, \sum_{i=0}^{\tau}\lambda^i \widetilde{u}_i, \sum_{i=0}^{\tau}\lambda^i u_{i,(1)}, \sum_{i=0}^{\tau}\lambda^i \widetilde{u}_{i,(1)}, \sum_{i=0}^{\tau}\lambda^i u_{i,(2)}\right)\right]_{\lambda=0}, \tag{5.44}$$

$$B^k_{4,\tau} = \frac{1}{\tau!}\left[\frac{\mathrm{d}^\tau}{\mathrm{d}\lambda^\tau} F_4^k\left(\sum_{i=0}^{\tau}\lambda^i u_i, \sum_{i=0}^{\tau}\lambda^i \widetilde{u}_i, \sum_{i=0}^{\tau}\lambda^i u_{i,(1)}, \sum_{i=0}^{\tau}\lambda^i \widetilde{u}_{i,(1)}, \sum_{i=0}^{\tau}\lambda^i u_{i,(2)}\right)\right]_{\lambda=0}. \tag{5.45}$$

The explicit expressions of the Adomian polynomials $A_{j,\tau}$, $B^0_{4,\tau}$, $B^1_{4,\tau}$ for the nonlinear functions F_j, F_4^0, F_4^1 can be now calculated using the formula (5.44)-(5.45). There results the following:

$$\begin{aligned} B^0_{4,0} &= -\frac{1}{\eta_1}\left[-\eta_1 \widetilde{u}_{0,t} - \frac{1}{\kappa^2}\left(u^4_{0,2x} + u^4_{0,2y} + u^4_{0,2z}\right)\right. \\ &+ \left. u^4_0\left(\left(u^1_0\right)^2 + \left(u^2_0\right)^2 + \left(u^3_0\right)^2 + \left(u^4_0\right)^2 + \left(u^5_0\right)^2 - 1\right)\right], \end{aligned}$$

$$B^1_{4,0} = -\frac{1}{\eta_1}\left[-\eta_1\kappa\widetilde{u}_0 u^5_0 - \frac{1}{\kappa}u^5_0\left(u^1_{0,x}+u^2_{0,y}+u^3_{0,z}\right) - \frac{1}{\kappa}\left(u^1_0 u^5_{0,x}+u^2_0 u^5_{0,y}+u^3_0 u^5_{0,z}\right)\right],$$

$$\begin{aligned} A_{5,0} &= -\frac{1}{\eta_1}\left[-\frac{1}{\kappa^2}\left(u^5_{0,2x}+u^5_{0,2y}+u^5_{0,2z}\right)-\frac{1}{\kappa}u^4_0\left(u^1_{0,x}+u^2_{0,y}+u^3_{0,z}\right)-\eta_1\kappa\widetilde{u}_0 u^4_0\right.\\ &- \left.\frac{1}{\kappa}\left(u^1_0u^4_{0,x}+u^2_0u^4_{0,y}+u^3_0u^4_{0,z}\right)+u^5_0\left(\left(u^1_0\right)^2+\left(u^2_0\right)^2+\left(u^3_0\right)^2+\left(u^4_0\right)^2+\left(u^5_0\right)^2-1\right)\right], \end{aligned}$$

$$\begin{aligned} A_{1,0} &= -\frac{1}{\eta_2}\left[u^2_{0,xy}-u^1_{0,2y}-u^1_{0,2z}+u^3_{0,xz}-\frac{1}{\kappa}\left(u^4_0u^5_{0,x}-u^5_0u^4_{0,x}\right)+\eta_2\widetilde{u}_{0,x}\right.\\ &+ \left.u^1_0\left(\left(u^4_0\right)^2+\left(u^5_0\right)^2\right)\right], \end{aligned}$$

$$\begin{aligned} A_{2,0} &= -\frac{1}{\eta_2}\left[u^3_{0,yz}-u^2_{0,2z}-u^2_{0,2x}+u^1_{0,xy}-\frac{1}{\kappa}\left(u^4_0u^5_{0,y}-u^5_0u^4_{0,y}\right)+\eta_2\widetilde{u}_{0,y}\right.\\ &+ \left.u^2_0\left(\left(u^4_0\right)^2+\left(u^5_0\right)^2\right)\right], \end{aligned}$$

$$\begin{aligned} A_{3,0} &= -\frac{1}{\eta_2}\left[u^1_{0,xz}-u^3_{0,2x}-u^3_{0,2y}+u^2_{0,yz}-\frac{1}{\kappa}\left(u^4_0u^5_{0,z}-u^5_0u^4_{0,z}\right)+\eta_2\widetilde{u}_{0,z}\right.\\ &+ \left.u^3_0\left(\left(u^4_0\right)^2+\left(u^5_0\right)^2\right)\right], \end{aligned}$$

$$\begin{aligned} B^0_{4,1} &= -\frac{1}{\eta_1}\left[-\eta_1\widetilde{u}_{1,t}+u^4_1\left(\left(u^1_0\right)^2+\left(u^2_0\right)^2+\left(u^3_0\right)^2+\left(u^4_0\right)^2+\left(u^5_0\right)^2-1\right)\right.\\ &- \left.\frac{1}{\kappa^2}\left(u^4_{1,2x}+u^4_{1,2y}+u^4_{1,2z}\right)+u^4_0\left(2u^1_0u^1_1+2u^3_0u^3_1+2u^2_0u^2_1+2u^5_0u^5_1+2u^4_0u^4_1\right)\right], \end{aligned}$$

$$\begin{aligned} B^1_{4,1} &= -\frac{1}{\eta_1}\left[-\frac{1}{\kappa}u^5_1\left(u^1_{0,x}+u^2_{0,y}+u^3_{0,z}\right)-\frac{1}{\kappa}u^5_0\left(u^1_{1,x}+u^2_{1,y}+u^3_{1,z}\right)-\eta_1\kappa\widetilde{u}_1u^5_0-\eta_1\kappa\widetilde{u}_0u^5_1\right.\\ &- \left.\frac{1}{\kappa}\left(u^1_1u^5_{0,x}+u^1_0u^5_{1,x}+u^2_1u^5_{0,y}+u^2_0u^5_{1,y}+u^3_1u^5_{0,z}+u^3_0u^5_{1,z}\right)\right], \end{aligned}$$

$$\begin{aligned} A_{5,1} &= -\frac{1}{\eta_1}\left[-\eta_1\kappa\widetilde{u}_1u^4 0-\frac{1}{\kappa^2}\left(u^5_{1,2x}+u^5_{1,2y}+u^5_{1,2z}\right)-\frac{1}{\kappa}u^4_1\left(u^1_{0,x}+u^2_{0,y}+u^3_{0,z}\right)\right.\\ &- \eta_1\kappa\widetilde{u}_0u^4_1+u^5_0\left(2u^1_0u^1_1+2u^3_0u^3_1+2u^2_0u^2_1+2u^5_0u^5_1+2u^4_0u^4_1\right)-\frac{1}{\kappa}u^4_0\left(u^1_{1,x}+u^2_{1,y}+u^3_{1,z}\right)\\ &- \frac{1}{\kappa}\left(u^1_1u^4_{0,x}+u^1_0u^4_{1,x}+u^2_1u^4_{0,y}+u^2_0u^4_{1,y}+u^3_1u^4_{0,z}+u^3_0u^4_{1,z}\right)\\ &+ \left.u^5_1\left(\left(u^1_0\right)^2+\left(u^2_0\right)^2+\left(u^3_0\right)^2+\left(u^4_0\right)^2+\left(u^5_0\right)^2-1\right)\right], \end{aligned}$$

$$\begin{aligned}
A_{1,1} &= -\frac{1}{\eta_2}\Bigg[u^2_{1,xy}-u^1_{1,2y}-u^1_{1,2z}+u^3_{1,xz}-\frac{1}{\kappa}\left(u^4_1u^5_{0,x}+u^4_0u^5_{1,x}-u^5_1u^4_{0,x}-u^5_0u^4_{1,x}\right)\\
&+ u^1_0\left(2u^4_0u^4_1+2u^5_0u^5_1\right)+\eta_2\widetilde{u}_{1,x}+u^1_1\left(\left(u^4_0\right)^2+\left(u^5_0\right)^2\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
A_{2,1} &= -\frac{1}{\eta_2}\Bigg[u^3_{1,yz}-u^2_{1,2z}-u^2_{1,2x}+u^1_{1,xy}-\frac{1}{\kappa}\left(u^4_1u^5_{0,y}+u^4_0u^5_{1,y}-u^5_1u^4_{0,y}-u^5_0u^4_{1,y}\right)\\
&+ u^2_0\left(2u^4_0u^4_1+2u^5_0u^5_1\right)+\eta_2\widetilde{u}_{1,y}+u^2_1\left(\left(u^4_0\right)^2+\left(u^5_0\right)^2\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
A_{3,1} &= -\frac{1}{\eta_2}\Bigg[u^1_{1,xz}-u^3_{1,2x}-u^3_{1,2y}+u^2_{1,yz}-\frac{1}{\kappa}\left(u^4_1u^5_{0,z}+u^4_0u^5_{1,z}-u^5_1u^4_{0,z}-u^5_0u^4_{1,z}\right)\\
&+ u^3_0\left(2u^4_0u^4_1+2u^5_0u^5_1\right)+\eta_2\widetilde{u}_{1,z}+u^3_1\left(\left(u^4_0\right)^2+\left(u^5_0\right)^2\right)\Bigg],
\end{aligned}$$

and so on. Using the above assumptions, we have

$$\begin{aligned}
\sum_{\tau=0}^{+\infty}u^4_\tau+\sum_{\tau=0}^{+\infty}\widetilde{u}_\tau &= 2(x+y+z)+L^{-1}\left(\sum_{\tau=0}^{+\infty}B^0_{j,\tau}\right)+L^{-1}\left(\sum_{\tau=0}^{+\infty}B^1_{j,\tau}\right),\\
\sum_{\tau=0}^{+\infty}u^j_\tau &= x+y+z+L^{-1}\left(\sum_{\tau=0}^{+\infty}A_{j,\tau}\right),\quad j=1,2,3,5
\end{aligned}$$

which induces the recursive relation

$$u^4_0=x+y+z,\quad u^4_{\tau+1}=L^{-1}\left(B^0_{4,\tau}\right),\quad \tau\geq 0, \tag{5.46}$$

$$\widetilde{u}_0=x+y+z,\quad \widetilde{u}_{\tau+1}=L^{-1}\left(B^1_{4,\tau}\right),\quad \tau\geq 0, \tag{5.47}$$

$$u^j_0=x+y+z,\quad u^j_{\tau+1}=L^{-1}\left(A_{j,\tau}\right),\quad \tau\geq 0,\quad j=1,2,3,5. \tag{5.48}$$

The first components are then determined (we have set $x+y+z=X$) as:

$$u^4_1=\left(-\frac{5X^3}{\eta_1}+\frac{X}{\eta_1}\right)t,\quad u^5_1=\left[\left(\frac{6}{\eta_1\kappa}+\frac{1}{\eta_1}\right)X+\kappa X^2-\frac{5X^3}{\eta_1}\right]t,$$

$$u^1_1=u^2_1=u^3_1=\left(-1-\frac{2X^3}{\eta_2}\right)t,\quad \widetilde{u}_1=\left(\frac{6X}{\eta_1\kappa}+\kappa X^2\right)t,$$

$$u^3_2=\left[\frac{6X^5}{\eta_2^2}+\frac{3X^2}{\eta_2}+\left(\frac{30}{\eta_2\eta_1}-\kappa-\frac{12}{\eta_2^2}\right)X-\frac{3}{\eta_1\kappa}-\frac{\kappa}{\eta_2}\right]t^2,$$

$$u^1_2=\left[\left(\frac{4}{\eta_2^2}+\frac{5}{\eta_2\eta_1}\right)X^5-\frac{X^3}{\eta_2\eta_1}+\frac{2X^2}{\eta_2}+\left(\frac{6}{\eta_2^2}-\frac{15}{\eta_2\eta_1}-\kappa\right)X-\frac{\kappa}{\eta_2}-\frac{3}{\eta_1\kappa}\right]t^2,$$

$$\widetilde{u}_2=\left[-\frac{\kappa X^4}{\eta_2}+\left(-\frac{16}{\kappa\eta_2\eta_1}+\frac{\kappa^2}{2}-\frac{20}{\eta_1^2\kappa}\right)X^3+\frac{9X^2}{2\eta_1}+\left(\frac{6}{\eta_1^2\kappa^2}-\frac{\kappa}{2}+\frac{2}{\eta_1^2\kappa}\right)X-\frac{2}{\eta_1\kappa}\right]t^2,$$

$$u_2^2=\left[\left(\frac{4}{\eta_2^2}+\frac{5}{\eta_2\eta_1}\right)X^5-\frac{\kappa X^4}{\eta_2}+\left(-\frac{6}{\kappa\eta_2\eta_1}-\frac{1}{\eta_2\eta_1}\right)X^3+\frac{2X^2}{\eta_2}+\left(\frac{6}{\eta_2^2}-\frac{15}{\eta_2\eta_1}-\kappa\right)X+\frac{2\kappa}{\eta_2}-\frac{3}{\eta_1\kappa}\right]t^2,$$

$$\begin{aligned}u_2^4 &= \left[\left(\frac{11}{\eta_2\eta_1}+\frac{10}{\eta_1^2}\right)X^5-\frac{\kappa X^4}{\eta_1}+\left(-\frac{1}{\eta_2\eta_1}-\frac{2}{\eta_1^2}-\frac{6}{\eta_1^2\kappa}\right)X^3+\frac{11X^2}{2\eta_1}-\frac{18X}{\kappa^2\eta_1\eta_2}-\frac{1}{2\eta_1}\right]t^2\\ &+ \left(\frac{6X}{\eta_1\kappa}+\kappa X^2\right)t,\end{aligned}$$

$$\begin{aligned}u_2^5 &= \left[\left(\frac{11}{\eta_2\eta_1}+\frac{10}{\eta_1^2}\right)X^5+\left(-\frac{\kappa}{\eta_2}-\frac{\kappa}{\eta_1}\right)X^4+\left(-\frac{16}{\kappa\eta_2\eta_1}-\frac{2}{\eta_1^2}-\frac{1}{\eta_2\eta_1}+\frac{\kappa^2}{2}-\frac{26}{\eta_1^2\kappa}\right)X^3+\frac{10X^2}{\eta_1}\right.\\ &+ \left.\left(-\frac{18}{\kappa^2\eta_1\eta_2}+\frac{6}{\eta_1^2\kappa^2}+\frac{2}{\eta_1^2\kappa}-\frac{\kappa}{2}\right)X-\frac{2}{\eta_1\kappa}-\frac{1}{2\eta_1}\right]t^2,\end{aligned}$$

$$\begin{aligned}u_3^1 &= \left[-\frac{\kappa X^4}{\eta_2}+\frac{2\kappa}{\eta_2}-\frac{6X^3}{\kappa\eta_2\eta_1}\right]t^2+\left[\frac{\kappa}{6}-\frac{2}{3\eta_1^2\kappa}-\frac{1}{3\eta_2^2}+\frac{2}{3\eta_2\eta_1}-\frac{2}{\eta_1^2\kappa^2}+\left(\frac{2\kappa}{3\eta_2\eta_1}+\frac{2\kappa}{3\eta_2^2}\right)X^6\right.\\ &+ \left(-\frac{20}{3\eta_2\eta_1^2}-\frac{28}{3\eta_2^3}-\frac{24}{\eta_2^2\eta_1}\right)X^7+\left(\frac{4}{\kappa\eta_2\eta_1^2}+\frac{8}{3\kappa\eta_2^3}+\frac{4}{3\eta_2\eta_1^2}+\frac{4}{\eta_2^2\eta_1}-\frac{8}{3\kappa\eta_2^2\eta_1}\right)X^5\\ &+ \left(-\frac{5}{\eta_2^2}-\frac{31}{3\eta_2\eta_1}\right)X^4+\left(\frac{30}{\eta_2^2\eta_1}+\frac{2}{3\kappa\eta_2^2\eta_1}+\frac{16}{\kappa^2\eta_2^2\eta_1}+\frac{200}{3\eta_2\eta_1^2}-\frac{68}{3\eta_2^3}+\frac{8\kappa}{3\eta_2}\right)X^3\\ &- \left(\frac{\kappa^2}{\eta_2}+\frac{2}{3\eta_2}-\frac{32}{\kappa\eta_2^2\eta_1}+\frac{4}{\eta_2\eta_1^2}+\frac{3}{\eta_1}-\frac{28}{\kappa\eta_2\eta_1^2}\right)X\\ &+ \left.\left(-\frac{\kappa^2}{2}+\frac{1}{3\kappa\eta_2^2}+\frac{20}{\eta_1^2\kappa}+\frac{5}{3\eta_2\eta_1}+\frac{10\kappa}{3\eta_2^2}-\frac{4\kappa}{\eta_2\eta_1}+\frac{20}{\kappa\eta_2\eta_1}\right)X^2\right]t^3,\end{aligned}$$

$$\begin{aligned}u_3^2 &= -\frac{\kappa}{\eta_2}t^2+\left[\frac{\kappa}{6}-\frac{2}{\eta_1^2\kappa^2}-\frac{2}{3\eta_1^2\kappa}-\frac{1}{3\eta_2^2}+\frac{29}{3\eta_2\eta_1}+\left(-\frac{20}{3\eta_2\eta_1^2}-\frac{28}{3\eta_2^3}-\frac{24}{\eta_2^2\eta_1}\right)X^7\right.\\ &+ \left(\frac{4\kappa}{\eta_2^2}+\frac{2\kappa}{3\eta_2\eta_1}\right)X^6+\left(\frac{52}{3\kappa\eta_2\eta_1^2}+\frac{8}{3\kappa\eta_2^3}+\frac{4}{3\eta_2\eta_1^2}+\frac{24}{\kappa\eta_2^2\eta_1}+\frac{4}{\eta_2^2\eta_1}-\frac{\kappa^2}{3\eta_2}\right)X^5\\ &- \left(\frac{92}{\kappa\eta_2\eta_1^2}+\frac{64}{\kappa\eta_2^2\eta_1}+\frac{4}{\eta_2\eta_1^2}+\frac{2}{3\eta_2}-\frac{2\kappa^2}{\eta_2}+\frac{3}{\eta_1}\right)X+\left(-\frac{5}{\eta_2^2}-\frac{40}{3\eta_2\eta_1}\right)X^4\\ &+ \left(\frac{30}{\eta_2^2\eta_1}-\frac{68}{3\eta_2^3}+\frac{2}{3\kappa\eta_2^2\eta_1}+\frac{200}{3\eta_2\eta_1^2}-\frac{4}{\eta_2\eta_1^2\kappa^2}-\frac{4}{3\kappa\eta_2\eta_1^2}+\frac{13\kappa}{3\eta_2}+\frac{16}{\kappa^2\eta_2^2\eta_1}\right)X^3\\ &+ \left.\left(\frac{5}{3\eta_2\eta_1}+\frac{20}{\eta_1^2\kappa}+\frac{1}{3\kappa\eta_2^2}-\frac{26\kappa}{3\eta_2^2}-\frac{\kappa^2}{2}+\frac{88}{3\kappa\eta_2\eta_1}-\frac{4\kappa}{\eta_2\eta_1}\right)X^2\right]t^3,\end{aligned}$$

$$
\begin{aligned}
u_3^3 &= -\frac{\kappa}{\eta_2}t^2 + \left[\frac{\kappa}{6} + \frac{11}{3\eta_2^2} - \frac{31}{3\eta_2\eta_1} - \frac{2}{3\eta_1^2\kappa} - \frac{2}{\eta_1^2\kappa^2} + \left(-\frac{20}{3\eta_2^2\eta_1} - \frac{52}{3\eta_2^3}\right)X^7 + \frac{2\kappa X^6}{3\eta_2^2}\right. \\
&+ \left(-\frac{8}{3\kappa\eta_2^2\eta_1} + \frac{8}{3\kappa\eta_2^3} + \frac{4}{3\eta_2^2\eta_1}\right)X^5 - \frac{35X^4}{3\eta_2^2} \\
&+ \left(\frac{4}{\kappa^2\eta_2^2\eta_1} - \frac{400}{3\eta_2\eta_1^2} + \frac{2}{3\kappa\eta_2^2\eta_1} + \frac{160}{3\eta_2^3} + \frac{10\kappa}{3\eta_2} - \frac{80}{\eta_2^2\eta_1}\right)X^3 \\
&+ \left(\frac{20}{\eta_1^2\kappa} - \frac{\kappa^2}{2} + \frac{4\kappa}{\eta_2^2} + \frac{1}{3\kappa\eta_2^2} + \frac{22}{\kappa\eta_2\eta_1} + \frac{8\kappa}{\eta_2\eta_1}\right)X^2 \\
&+ \left.\left(-\frac{2}{\eta_2} + \frac{32}{\kappa\eta_2^2\eta_1} - \frac{\kappa^2}{\eta_2} + \frac{8}{\eta_2\eta_1^2} + \frac{64}{\kappa\eta_2\eta_1^2} - \frac{3}{\eta_1}\right)X\right]t^3,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{u}_3 &= \left[\left(-\frac{\kappa^2}{3\eta_2} - \frac{24}{\kappa^3\eta_1^2\eta_2} + \frac{4}{3\kappa^2\eta_1^3} + \frac{50}{\kappa\eta_2\eta_1^2} - \frac{20}{\kappa\eta_2^2\eta_1} - \frac{22}{3\eta_1} + \frac{4}{\eta_1^3\kappa^3}\right)X - \frac{4}{3\eta_2\eta_1} - \frac{20}{3\eta_1^2\kappa^2} - \frac{1}{\eta_1^2\kappa}\right. \\
&+ \frac{2\kappa X^6}{\eta_2^2} + \left(\frac{94}{\kappa\eta_2\eta_1^2} + \frac{40}{\kappa\eta_1^3} + \frac{52}{\kappa\eta_2^2\eta_1} - \frac{\kappa^2}{\eta_2}\right)X^5 + \left(\frac{1}{6\kappa^3} - \frac{43}{3\eta_2\eta_1} - \frac{10}{\eta_1^2}\right)X^4 \\
&- \left(\frac{28}{3\kappa\eta_2\eta_1^2} + \frac{112}{3\eta_2\eta_1^2\kappa^2} - \frac{13\kappa}{6\eta_1} - \frac{\kappa}{\eta_2} + \frac{128}{3\kappa^2\eta_1^3} + \frac{16}{3\kappa\eta_1^3}\right)X^3 \\
&+ \left.\left(\frac{15}{\kappa\eta_2\eta_1} + \frac{55}{2\eta_1^2\kappa} + \frac{2}{3\eta_1^2} - \frac{5\kappa^2}{6} + \frac{10\kappa}{\eta_2\eta_1} - \frac{4\kappa}{\eta_2^2}\right)X^2\right]t^3 + \left[\frac{6X}{\eta_1^2\kappa^2} + \frac{3X^2}{2\eta_1}\right]t^2.
\end{aligned}
$$

$$
\begin{aligned}
u_3^5 &= \left[\frac{6X}{\eta_1^2\kappa^2} - \frac{6X^3}{\eta_1^2\kappa} - \frac{\kappa X^4}{\eta_1} + \frac{3X^2}{2\eta_1}\right]t^2 + \left[\frac{6}{\kappa^2\eta_1\eta_2} - \frac{2}{\eta_1^2\kappa} + \frac{5}{3\eta_2\eta_1} - \frac{\kappa}{3\eta_2\eta_1} - \frac{20}{3\eta_1^2\kappa^2}\right. \\
&+ \left(\frac{16}{3\eta_2\eta_1^2} + \frac{2}{\eta_2^2\eta_1} - \frac{4\kappa^2}{3\eta_2} + \frac{28}{3\eta_1^3} + \frac{40}{\kappa\eta_2^2\eta_1} + \frac{244}{3\kappa\eta_1^3} - \frac{2\kappa^2}{3\eta_1} + \frac{440}{3\kappa\eta_2\eta_1^2}\right)X^5 \\
&+ \left(-\frac{4}{\eta_2^2\eta_1} - \frac{\kappa}{3\eta_1} + \frac{4}{3\kappa^2\eta_1^3} - \frac{24}{\kappa^3\eta_1^2\eta_2} - \frac{31}{3\eta_1} + \frac{10}{\eta_2\eta_1^2} + \frac{16}{\kappa\eta_2^2\eta_1} + \frac{2\kappa^2}{3\eta_2} - \frac{40}{\kappa\eta_2\eta_1^2} + \frac{4}{\eta_1^3\kappa^3}\right)X \\
&+ \left(\frac{41\kappa}{6\eta_1} + \frac{120}{\kappa^2\eta_2^2\eta_1} - \frac{176}{3\kappa^2\eta_1^3} + \frac{2\kappa}{3\eta_2} - \frac{40}{3\kappa\eta_2\eta_1^2} - \frac{50}{\eta_2\eta_1^2} + \frac{20}{\eta_2^2\eta_1} - \frac{112}{3\eta_2\eta_1^2\kappa^2} - \frac{2}{3\eta_1^3} - \frac{32}{3\kappa\eta_1^3}\right)X^3 \\
&- \left(\frac{30}{\eta_1^3} + \frac{94}{3\eta_2^2\eta_1} + \frac{104}{3\eta_2\eta_1^2}\right)X^7 + \left(\frac{13\kappa}{3\eta_2\eta_1} + \frac{4\kappa}{3\eta_2^2} + \frac{14\kappa}{3\eta_1^2}\right)X^6 \\
&- \left(\frac{\kappa}{3\eta_2\eta_1} + \frac{31}{\eta_1^2} - \frac{\kappa^3}{6} + \frac{2\kappa}{3\eta_1^2} + \frac{43}{\eta_2\eta_1}\right)X^4 \\
&+ \left.\left(-\frac{10\kappa}{3\eta_2\eta_1} + \frac{8}{3\eta_1^2} + \frac{1}{\eta_2\eta_1} - \frac{5\kappa^2}{6} + \frac{12}{\kappa\eta_2\eta_1} + \frac{2\kappa}{\eta_2^2} + \frac{263}{6\eta_1^2\kappa}\right)X^2\right]t^3,
\end{aligned}
$$

$$
\begin{aligned}
u_3^4 \;=\; & \Bigg[\left(-\frac{\kappa}{\eta_2}-\frac{\kappa}{\eta_1}\right)X^4+\left(-\frac{16}{\kappa\eta_2\eta_1}-\frac{26}{\eta_1^2\kappa}+\frac{\kappa^2}{2}\right)X^3+\frac{9X^2}{2\eta_1}+\left(\frac{6}{\eta_1^2\kappa^2}-\frac{\kappa}{2}+\frac{2}{\eta_1^2\kappa}\right)X-\frac{2}{\eta_1\kappa}\Bigg]t^2 \\
+ & \Bigg[\left(-\frac{43}{\eta_2\eta_1^2}-\frac{30}{\eta_1^3}-\frac{28}{\eta_2^2\eta_1}\right)X^7+\left(\frac{14\kappa}{3\eta_1^2}+\frac{13\kappa}{3\eta_2\eta_1}\right)X^6+\frac{4}{\kappa^2\eta_1\eta_2}-\frac{1}{\eta_1^2\kappa}+\frac{2\kappa}{3\eta_2\eta_1} \\
+ & \left(\frac{28}{3\eta_1^3}+\frac{4}{3\eta_2^2\eta_1}+\frac{124}{3\kappa\eta_1^3}-\frac{2\kappa^2}{3\eta_1}+\frac{98}{3\kappa\eta_2\eta_1^2}+\frac{26}{3\eta_2\eta_1^2}\right)X^5+\left(-\frac{2\kappa}{3\eta_1^2}-\frac{20}{\eta_2\eta_1}-\frac{21}{\eta_1^2}-\frac{\kappa}{3\eta_2\eta_1}\right)X^4 \\
- & \left(\frac{12}{\kappa\eta_2\eta_1}+\frac{10\kappa}{3\eta_2\eta_1}-\frac{2}{\eta_1^2}-\frac{2}{3\eta_2\eta_1}-\frac{49}{3\eta_1^2\kappa}\right)X^2 \\
- & \left(\frac{36}{\kappa^3\eta_1^2\eta_2}+\frac{6}{\eta_2\eta_1^2\kappa^2}+\frac{3}{\eta_1}+\frac{5}{\eta_2\eta_1^2}+\frac{\kappa}{3\eta_1}-\frac{2}{\eta_2^2\eta_1}\right)X \\
+ & \left(-\frac{2}{\kappa\eta_2\eta_1^2}+\frac{80}{\kappa^2\eta_2^2\eta_1}-\frac{16}{3\kappa\eta_1^3}-\frac{10}{\eta_2^2\eta_1}+\frac{74}{3\eta_2\eta_1^2}+\frac{124}{\eta_2\eta_1^2\kappa^2}+\frac{14\kappa}{3\eta_1}-\frac{2}{3\eta_1^3}-\frac{16}{\kappa^2\eta_1^3}\right)X^3\Bigg]t^3.
\end{aligned}
$$

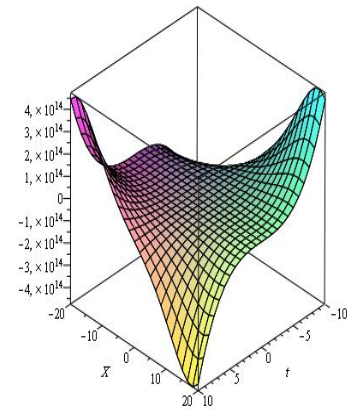
Therefore, we deduce an approximate solution to the Ginzburg-Landau equations (5.10) subject to the initial condition (5.38) as follows:

$$
A^1 = A^2 = A^3 \simeq u_0^1+u_1^1+u_2^1+u_3^1, \quad \phi \simeq \widetilde{u}_0+\widetilde{u}_1+\widetilde{u}_2+\widetilde{u}_3,
$$

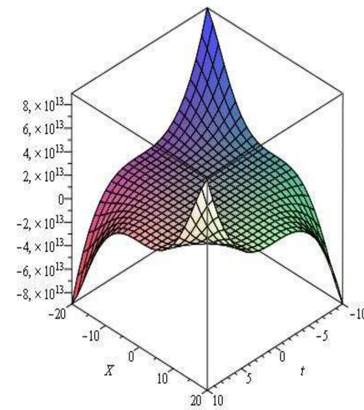
$$
\psi^1 \simeq u_0^4+u_1^4+u_2^4+u_3^4, \quad \psi^2 \simeq u_0^5+u_1^5+u_2^5+u_3^5.
$$

The corresponding graphic is shown in the Figure 4 for fixed parameters $\eta_1=\eta_2=1$ and $\kappa=10^{-3}$.

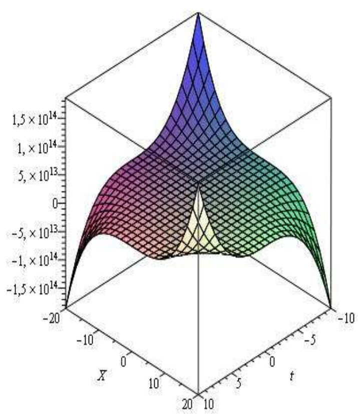
In this work, we have shown that the Adomian decomposition method is also adaptable to provide approximate solution to under-determined systems of NPDEs.



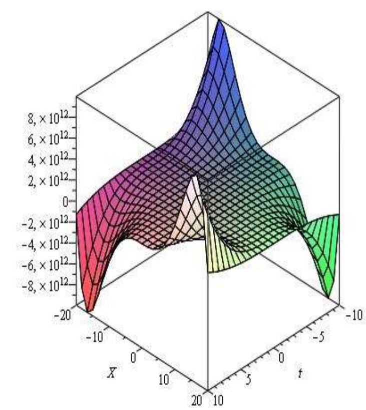
Function ϕ versus X and t.



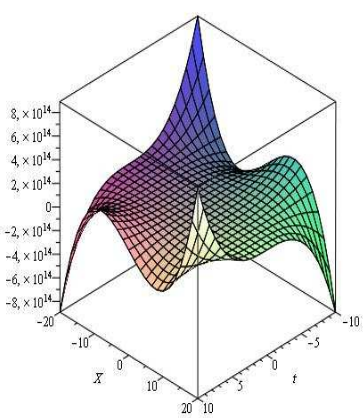
Function A^1 versus X and t.



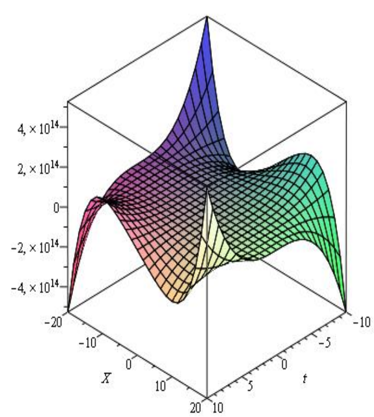
Function A^2 versus X and t.



Function A^3 versus X and t.



Function ψ^1 versus X and t.



Function ψ^2 versus X and t.

Figure 4. Approximate solution of the Ginzburg-Landau system (5.37) versus $X = x + y + z$ and t.

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