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ITERATED OPERATOR INEQUALITIES ON ORDERED LINEAR SPACES

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Abstract

An operator version of the Young's result [Proc. Amer. Math. Soc., 94 (1985), 636-640] is obtained, via non-differential techniques.

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1 Introduction

A basic tool of the differential/integral equations theory is the Gronwall-Bellman inequality; it asserts that, if the function $u : R_+ \to R_+$ is continuous and

(a01) $u(t) \le b(t) + \int_0^t k(s)u(s)ds,\ t \in R_+$

(where $b, k : R_+ \to R_+$ are continuous functions) then

$$u(t) \le b(t) + \int_0^t \exp[\int_0^s k(r)dr]b(s)ds,\ t \in R_+; \tag{1.1}$$

see, for instance, Corduneanu [4, Ch 1, Sect 1.5]. This evaluation is the best possible one; because the right member of (1.1) is just the solution of the integral equation attached to (a01). However, in many concrete situations, these solutions are difficult to be handled. So, it would be useful to substitute the exact evaluation (1.1) with an approximate one, having a "simpler" structure. For example, if b is in addition increasing, an approximate solution to (a01) is

$$u(t) \le b(t)\exp[\int_0^t k(s)ds],\ t \in R_+; \tag{1.2}$$

referred to as the Wendroff inequality; cf. Lakshmikantham and Leela [5, Ch 1, Sect 1.9]. A further extension of this result was performed in 1973 by Pachpatte [8]; which established that, if $u : R_+ \to R_+$ is continuous and

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(a02) $u(t) \le b(t) + \int_0^t k(s)u(s)ds + \int_0^t k(s) \int_0^s h(r)u(r)drds,\ t \in R_+$

(with $b : R_+ \to R_+$, continuous increasing and $k, h : R_+ \to R_+$, continuous) then

$$u(t) \le b(t)[1 + \int_0^t k(s) \exp[\int_0^s (k(r) + h(r))dr]b(s)ds],\ t \in R_+. \tag{1.3}$$

Finally, in his 1985 paper, Young [13] extended these results to the case of

(a03) $u(t) \le b(t) + \int_0^t k^1(t_1)u(t_1)dt_1 + \int_0^t \int_0^{t_1} k^1(t_1)k^2(t_2)u(t_2)dt_2dt_1 + ...$
$+ \int_0^t ... \int_0^{t_{p-1}} k^1(t_1)...k^p(t_p)u(t_p)dt_p...dt_1,\ t \in R_+;$

where $b, k^1, ..., k^p$ (for $p \ge 3$) are continuous functions from R_+ to itself. Precisely, let us introduce the (p, p)-matrix function and the p-vector function

$$A(t) = \begin{pmatrix} k^1 & k^1 & 0 & ... & 0 & 0 \\ k^2 & 0 & k^2 & ... & 0 & 0 \\ . & . & . & . & . & . \\ k^{p-1} & 0 & 0 & ... & 0 & k^{p-1} \\ k^p & 0 & 0 & ... & 0 & 0 \end{pmatrix};\ B(t) = b(t) \begin{pmatrix} k^1 \\ k^2 \\ ... \\ k^{p-1} \\ k^p \end{pmatrix}.$$

Then, each solution of (a03) is majorized as

$$u(t) \le b(t) + v_1(t),\quad t \in R_+; \tag{1.4}$$

where v_1 is the first component of the vector function $V(t) = \int_0^t Y(t)Y^{-1}(s)B(s)ds,\ t \in R_+$, and $Y(.)$ is a fundamental (p, p)-matrix of the linear differential system $Z'(t) = A(t)Z(t)$, $t \in R_+$.

The methods used by these authors are strongly related to the differential inequalities theory. An alternate operator way of proving Pachpatte's result was provided in the 1982 paper by Turinici [11]. It is our aim in this exposition to show that these techniques allow as well a non-differential proof of Young's result; details will be given in Section 4. The basic tool for deducing it is an "abstract" comparison principle in linear spaces ordered by cones; cf. Section 3. Finally, Section 2 is devoted to some preliminary facts.

2 Normal maps

Let X be a (real) linear space; and X_+, a (convex) *cone* of it [$\alpha X_+ + \beta X_+ \subseteq X_+$, for each $\alpha, \beta \ge 0$]. Its associated relation ($\le$) on X [defined as: $x \le y$ iff $y - x \in X_+$] is reflexive and transitive; hence a *quasi-order* (on X). If in addition

(b01) X_+ is pointed ($x, y \in X_+$, $x + y = 0$ imply $x = y = 0$)

this quasi-order is antisymmetric as well; hence, it becomes a (partial) order. Further, take the family of seminorms $\mathcal{G} = \{|.|_i; i \in I\}$ according to (b02) $\mathcal{G}$ is sufficient ($|x|_i = 0$ for all $i \in I$ imply $x = 0$).

Sometimes, one may impose upon X_+ a regularity condition like

(b03) X_+ is sequentially $\mathcal{G}$-closed (the $\mathcal{G}$-limit of each sequence in X_+ belongs to X_+).

The remaining conventions are standard.

(A) Let $\mathcal{M}(X)$ stand for the class of all maps $T : X \to X$. We say that $T \in \mathcal{M}(X)$ is *positive* if $x \geq 0$ implies $Tx \geq 0$ (i.e., $T(X_+) \subseteq X_+$). This property is naturally connected with the increasing one: $x \leq y$ implies $Tx \leq Ty$. Precisely, increasing $\Longrightarrow$ positive under $T(0) \geq 0$. On the other hand, increasing $\Longleftrightarrow$ positive if T is linear. Denote $\mathcal{IP}(X)$={$T \in \mathcal{M}(X)$; T is increasing positive}. This class is a cone in $\mathcal{M}(X)$; invariant to the functional product ($T, S \in \mathcal{IP}(X)$ implies $TS \in \mathcal{IP}(X)$). Moreover, under (b03), this cone is also invariant to the pointwise sequential $\mathcal{G}$-convergence. Finally, put $\mathcal{LIP}(X)$={$T \in \mathcal{IP}(X)$; T is linear}. This is a subcone in $\mathcal{IP}(X)$, invariant to the functional product (see above).

We are now passing to some basic facts involving these objects. Call the mapping T in $\mathcal{IP}(X)$, *normal* on X_+ when

i) T has a unique fixed point $z = z(T)$ in X_+

ii) $u \in X_+$, $u \leq Tu$ imply $u \leq z$.

A basic example of such maps is to be given in the context of (b02)+(b03) and

(b04) X is sequentially $\mathcal{G}$-complete: each $\mathcal{G}$-Cauchy sequence is $\mathcal{G}$-convergent.

Precisely, for each $i \in I$, let $f_i(.)$ stand for the (extended) real valued function

$$f_i(t) = \sup\{|Tx - Ty|_i; x, y \in X, |x - y|_i \leq t\},\ t \in R_+.$$

Call the mapping T, *$\mathcal{G}$-contractive* on X provided for each $i \in I$,

(b05) $f_i(t) < \infty$ and $f_i^n(t) \to 0$ as $n \to \infty$, for all $t > 0$.

(Here, f_i^n stands for the n-th iterate of f_i). Note that, as f_i is increasing on R_+, this gives $f_i(t) < t$, for all $t > 0$ and all $i \in I$; cf. Matkowski [6]. In particular, (b05) holds provided $f_i(t) \leq \lambda_i t$, $t > 0$, $i \in I$; where $\Lambda = (\lambda_i; i \in I)$ is such that $\lambda_i \in]0, 1[$, $i \in I$; we then say that T is *$(\Lambda, \mathcal{G})$-contractive*.

Proposition 2.1. *For each $T \in \mathcal{IP}(X)$ we have: $\mathcal{G}$-contractive $\Longrightarrow$ normal (on X_+).*

Proof. By Turinici [12], T has a unique fixed point $z \in X_+$; and

$$T^n x \to z \text{ (modulo } \mathcal{G}) \text{ as } n \to \infty, \text{ for each } x \in X_+. \tag{2.1}$$

Let $u \in X_+$ be such that $u \leq Tu$. From the increasing property, $u \leq T^n u$, $\forall n$; and this, along with (2.1), yields $u \leq z$ if one takes (b03) into account. □

Remark 2.2. The 1986 author's result we just quoted was partially re-obtained in 2005 by Nieto and Lopez [7]; further aspects will be discussed elsewhere.

(B) A basic model of these developments is the one below. Let m be a positive integer; and R^m stand for the m-dimensional vector space endowed with some usual norm $\|.\|$ and the ordering $(\leq)$ given by the standard positive cone R^m_+ [$x \geq 0$ iff $x \in R^m_+$]. Given the (m,m)-matrix $A = (a_{ij})$ over the reals, denote $Q(A) = \{x \in R^m; Ax \geq 0\}$; it is a closed (convex) cone in R^m. [In general, $Q(A)$ is not pointed; however, when A is non-singular ($Ax = 0 \Longleftrightarrow x = 0$) this happens]. The quasi-order associated to $Q(A)$ will be denoted as $(\preceq)$; hence: $x \in Q(A)$ $\Longleftrightarrow x \succeq 0 \Longleftrightarrow Ax \geq 0$. In particular, when $A = I$ (the identity (m,m)-matrix) the associated cone $Q(A)$ is just R^m_+.

Let C^m (resp., C^0) indicate the class of all continuous functions from R_+ to R^m (resp., R_+). A useful Frechet structure on C^m is that indicated as below. Let $(\alpha(i); i \in N)$ be a sequence in $R^0_+ :=]0,\infty[$ with

(b06) $(\alpha(i))$ is strictly ascending and divergent ($\alpha(i) \to \infty$ as $i \to \infty$).

We associate it a sequence $(g_i; i \in N)$ in C^0 with

(b07) g_i is strictly positive on $[0,\alpha(i)]$, $i \in N$.

Let the family of seminorms $\mathcal{G} = \{|.|_i; i \in N\}$ on C^m be defined as:

$$|x|_i = \sup\{\|x(t)\|/g_i(t); t \in [0,\alpha(i)]\},\ x \in C^m,\ i \in N.$$

The associated topology is nothing but the local uniform one; because

$$x_n \to x \text{ (modulo } \mathcal{G}) \text{ iff } x_n \to x \text{ uniformly on each compact of } R_+. \tag{2.2}$$

This shows that $\mathcal{G}$ is sufficient and C^m is sequentially $\mathcal{G}$-complete.

A basic (convex) cone in C^m is to be constructed as below. Let $A = (a_{ij})$ be a (m,m)-matrix over the reals; and $Q := Q(A)$ denote its associated cone (see above). Denote by $C^m[Q]$ the class of all continuous functions from R_+ to Q. Clearly, it is a cone in C^m (by the choice of Q). Its associated quasi-order in C^m will be also denoted as $(\preceq)$; hence $x \in C^m(Q)$ means $x \succeq 0$; and

$$x \succeq 0 \text{ iff } x(t) \succeq 0,\ \forall t \in R_+ \text{ (or, equivalently: } Ax(t) \geq 0,\ \forall t \in R_+).$$

Moreover, by the closeness of Q (in R^m), $C^m[Q]$ is $\mathcal{G}$-closed.

The following auxiliary fact will be useful in the sequel.

Lemma 2.3. *Let the (m,m)-matrix $H = (h_{ij})$ over C^m be such that A commutes with $H(t)$, for each $t \geq 0$. Further, let $x \in C^m$ be arbitrary fixed. Then,*

$$A\int_p^q H(s)x(s)ds = \int_p^q H(s)Ax(s)ds,\ \ \forall p,q \in R_+,\ p < q.$$

Proof. By definition, we have (with $r_{i,n} = p + i(q-p)/n$, $i \geq 0$)

$$\int_p^q H(s)x(s)ds = \lim_n \sum_{i=0}^{n-1}((q-p)/n)H(r_{i,n})x(r_{i,n}).$$

This gives at once (by the posed hypothesis)

$$A\int_p^q H(s)x(s)ds = A\lim_n \sum_{i=0}^{n-1}((q-p)/n)H(r_{i,n})x(r_{i,n}) =$$
$$\lim_n \sum_{i=0}^{n-1}((q-p)/n)AH(r_{i,n})x(r_{i,n}) =$$
$$\lim_n \sum_{i=0}^{n-1}((q-p)/n)H(r_{i,n})Ax(r_{i,n}) = \int_p^q H(s)Ax(s)ds;$$

and the conclusion follows. □

Now, let D^0 stand for the class of all continuous functions from $R_+^{(2)}$:=$\{(t,s) \in R_+^2; t \geq s\}$ to R_+. Denote by $\mathcal{M}_m(D^0)$ the class of all (m,m)-matrices with elements in D^0. Fix such an object, $K = (k_{ij})$, with

(b08) A commutes with $K(t,s)$, $\forall (t,s) \in R_+^{(2)}$.

The associated mapping $L := L[K]$ from $X := C^m$ to itself

(b09) $L(x)(t) = \displaystyle\int_0^t K(t,s)x(s)ds, \quad x \in X$

is linear and leaves invariant the cone $X_+ := C^m[Q]$; since (cf. Lemma 2.3)

$$AL(x)(t) = A\int_0^t K(t,s)x(s)ds = \int_0^t K(t,s)Ax(s)ds \geq 0,\ \forall t \in R_+,\ \forall x \in X_+.$$

As a consequence, L is increasing with respect to the associated quasi-order $(\leq)$; i.e.: $x \leq y$ implies $L(x) \leq L(y)$. wherefrom, $L \in \mathcal{LIP}(X)$ (see above). This also tells us that, for the fixed $b \in X_+$, the translated operator $T = b + L$ is increasing and positive (modulo $(\leq)$); so, it is an element of $\mathcal{IP}(X)$. Concerning its contractive properties, put

(b10) $\omega_i = \sup\{\|K(t,s)\|; 0 \leq s \leq t \leq \alpha(i)\}$, $i \in N$;

and let $\Lambda = (\lambda_i; i \in N)$ be taken so as $\lambda_i \in]0,1[$, $i \in N$. Define the functions

(b11) $g_i(t) = \exp[(\omega_i/\lambda_i)t]$, $t \in R_+$, $i \in N$.

Clearly, (b07) holds; moreover, we have

$$\omega_i \int_0^t g_i(s)ds \leq \lambda_i g_i(t),\ t \in R_+,\ i \in N. \tag{2.3}$$

Proposition 2.4. *Let these conventions be accepted. Then, T is $(\Lambda, \mathcal{G})$-contractive; hence, all the more normal.*

Proof. Let $x, y \in C^m$ be such that $|x-y|_i \leq \tau$, for some $\tau \geq 0$, $i \in N$. By definition, $\|x(t) - y(t)\| \leq \tau g_i(t)$, $t \in [0,\alpha(i)]$; so that (if one takes (2.3) into account)

$$\|Tx(t) - Ty(t)\| \leq \int_0^t \|K(t,s)\|.\|(x(s) - y(s)\|ds$$
$$\leq \tau\omega_i \int_0^t g_i(s)ds \leq \lambda_i \tau g_i(t),\ t \in [0,\alpha(i)].$$

As a consequence, $|Tx - Ty|_i \leq \lambda_i \tau$; and from this, we are done. □

(C) Let D^m stand for the class of all continuous functions from $R_+^{(2)}$ to R^m. Denote by $\mathcal{M}_m(D)$ the class of all (m,m)-matrices with elements in $D := D^1$. A useful Frechet structure on $\mathcal{M}_m(D)$ is that indicated under the model above. Precisely, let $(\alpha(i); i \in N)$ be a sequence in R_+^0 with the property (b06); and $(g_i; i \in N)$ be a sequence in C^0 as in (b07). Let the family of seminorms $\mathcal{G} = \{|.|_i; i \in N\}$ on $\mathcal{M}_m(D)$ be defined as:

$$|z|_i = \sup\{\|z(t,s)\|/g_i(t); 0 \le s \le t \le \alpha(i)\},\ z \in \mathcal{M}_m(D),\ i \in N.$$

The associated topology is nothing but the local uniform one. This results at once from the matrix analog of (2.2); we do not give details. Hence, $\mathcal{G}$ is sufficient and $\mathcal{M}_m(D)$ is sequentially $\mathcal{G}$-complete (see above). Fix an object, $K = (k_{ij})$ in $\mathcal{M}_m(D^0)$. The mapping $M := M[K]$ from $X := \mathcal{M}_m(D)$ to itself

(b12) $$M(Z)(t,s) = \int_s^t K(t,r)Z(r,s)ds,\ \ (t,s) \in R_+^{(2)}, Z \in \mathcal{M}_m(D)$$

is linear and leaves invariant the cone $X_+ := \mathcal{M}_m(D^0)$; so, it is element of $\mathcal{LIP}(X)$. Moreover, given $B = (b_{ij})$ in $\mathcal{M}_m(D^0)$, the translated operator $U = B + M$ is again increasing and positive (modulo X_+); so, it is an element of $\mathcal{IP}(X)$. Concerning its contractive properties, let $(\omega_i; i \in N)$ be the one of (b10); and let $\Lambda = (\lambda_i; i \in N)$ be taken so as $\lambda_i \in]0,1[$, $i \in N$. Further, define the functions $(g_i; i \in N)$ in C^0 via (b11).

Proposition 2.5. *Let these conventions be accepted. Then, U is $(\Lambda,\mathcal{G})$-contractive; hence, all the more normal.*

The proof is very similar to the one of Proposition 2.4; so, we omit it.

3 Main results

Let X be a (real) linear space; and X_+, some (convex) cone of it. Also, let $\mathcal{G} = \{|.|_i; i \in I\}$ be a family of seminorms on X; and assume that the regularity conditions (b02)-(b04) hold. Take a map S in $\mathcal{IP}(X)$ and consider the operator inequality on X_+

$$(u \in X_+ \text{ and})\ u \le S(u). \tag{3.1}$$

It is our aim in the following to give some upper bounds for the solutions of (3.1). First, as a direct consequence of Proposition 2.1, one has

Theorem 3.1. *Suppose that (in addition)*

(c01) S is contractive on X (cf. Section 2).

Then, necessarily,

$$u \le z\ (=\text{the unique fixed point of } S \text{ in } X_+). \tag{3.2}$$

This evaluation is the best possible one (see above). However, when S has a complicated structure, it is difficult to be handled. This e.g., is the case when $S = S_1$ is to be expressed in terms of a family $\{T_1, ..., T_n\}$ in $\mathcal{IP}(X)$ as

(c02) $S_j = T_j(I + S_{j+1})$, $j = 1, ..., n$ (where $S_{n+1} = 0$).

[As usually, I denotes the identity map]. And then, the question arises of determining an upper bound for a solution of (3.1) with the aid of these maps. An appropriate answer to this may be given along the following lines.

Theorem 3.2. *Assume that*

(c03) $P := T_1 + ... + T_n$ *is normal (see above).*

Then, for each solution $u \in X_+$ *of (3.1) we have the evaluation*

$$u \le T_1(T_1 + T_2)...(T_1 + ... + T_{n-1})(w) \tag{3.3}$$

where w *is the unique fixed point of* P *in* X_+.

Proof. Let $u = u_1$ be a solution in X_+ of (3.1); hence $u_1 \le S_1(u_1) = T_1(u_1 + S_2(u_1))$. Denote $u_2 = u_1 + S_2(u_1)$; hence $u_1 \le T_1(u_2)$. By the positivity of S_2, we have $u_1 \le u_2$; so that (by the above) $u_2 \le T_1(u_2) + S_2(u_2) \le T_1(u_2) + T_2(u_2 + S_3(u_2))$. Further, denote $u_3 = u_2 + S_3(u_2)$; hence $u_2 \le T_1(u_2) + T_2(u_3)$. By the positivity of S_3 one has $u_2 \le u_3$; so that $u_2 \le (T_1 + T_2)(u_3)$ as well as $u_3 \le (T_1 + T_2)(u_3) + S_3(u_3)$; and so on. Hence, after n steps, we construct a system of elements $\{u_1, u_2, ..., u_n\}$ in X_+ with $u = u_1 \le ... \le u_n$ and

$$u_i \le (T_1 + ... + T_i)(u_{i+1}), i = 1, ..., n-1;\ u_n \le (T_1 + ... + T_n)(u_n).$$

But, from these relations, (3.3) is clear. Hence the conclusion. □

Remark 3.3. Denote for simplicity $P_i = T_1 + ... + T_i$, $i = 1, ..., n$; hence $P_1 = T_1$, $P_n = P$. From $P_1 \le P_2 \le ... \le P_n(= P)$, we derive $P_i w \le Pw = w$, $i = 1, ..., n$; and the conclusion (3.3) of Theorem 3.2 yields: for each solution $u \in X_+$ of (3.1) we have

$$u \le w \text{ (=the unique fixed point of } P \text{ in } X_+). \tag{3.4}$$

Clearly, the evaluation (3.3) is finer than this one. However, in many concrete situations, it gives, practically, the same amount of information.

4 Particular aspects

(A) Let m be a positive integer. Given the (m, m)-matrix $A = (a_{ij})$ over the reals, denote by $Q := Q(A)$ the associated cone; and by $(\le)$, the induced quasi-order. Further, take an object, $K = (k_{ij})$ in $\mathcal{M}_m(D^0)$, according to (b08); as well as some $b \in C^m[Q]$. The mapping $T := b + L$ from $X := C^m$ to itself where $L := L[K]$ is given by (b09), is increasing and leaves invariant the cone $X_+ := C^m[Q]$; so, it is element of $\mathcal{IP}(X)$ (cf. Section 2). Sufficient conditions for the normality of T were established in Proposition 2.4 above. Suppose these are effective. It then follows, via Theorem 3.1, that each solution $u \in C^m[Q]$ of

(d01) $u \le T(u)$; i.e.: $u(t) \le b(t) + \int_0^t K(t, s)u(s)ds,\ t \in R_+$

is majorized (modulo $(\le)$) by the unique solution $w \in C^m[Q]$ of (d02) $w = T(w)$; i.e.: $w(t) = b(t) + \int_0^t K(t,s)w(s)ds,\ t \in R_+$.

Now (cf. Tricomi [10, Ch 1, Sect 1.3]) this unique solution has the representation

$$w(t) = b(t) + \int_0^t H(t,s)b(s)ds, t \in R_+ \tag{4.1}$$

where $H = (h_{ij})$ (the resolvent kernel) is the solution in $\mathcal{M}_m(D)$ of the matrix Volterra integral equation:

(d03) $Z(t,s) = K(t,s) + \int_s^t K(t,r)Z(r,s)dr,\ (t,s) \in R_+^{(2)}$.

Note that the associated operator $U = K + M$, where $M := M[K]$ is the linear operator of (b12), was already shown to be normal; hence, the existence and uniqueness along $\mathcal{M}_m(D)$ of (d03) are assured. Summing up, we derived

Theorem 4.1. *Let $u \in C^m[Q]$ be a solution of the integral inequality (d01). Then, $u(t) \le w(t)$, $t \in R_+$, where $w \in C^m[Q]$ is the solution of the integral equation (d02); which, in addition, has the representation (4.1).*

In particular, when $Q = R_+^m$, this result is nothing but the one in Chu and Metcalf [3]; see also Chandra and Davis [2].

(B) Fix a finite family $\{K^r = (k_{ij}^r); r = 1,...,p\}$ (where $p \ge 3$), of elements in $\mathcal{M}_m(D^0)$ fulfilling (b08); as well as some $b \in C^m[Q]$. The mapping $T_1 := b + L[K^1]$ from $X := C^m$ to itself is a normal element of $\mathcal{IP}(X)$; and, for each $i \in \{2,...,p\}$, $T_i := L[K^i]$ is a normal element of $\mathcal{LIP}(X)$. Note that, as a consequence of this, $T := T_1 + T_2 + ... + T_p$ is a normal element of $\mathcal{IP}(X)$. And this, along with Theorem 3.2 gives the following practical result. Denote for simplicity $K^{[i]} = K^1 + ... + K^i$, $i = 1,2,...,p$ (hence $K^{[1]} = K^1$).

Theorem 4.2. *Let $u \in C^m[Q]$ be a solution of the (iterated Gronwall-Bellman) inequality*

$$\begin{aligned} & u(t) \le b(t) + \int_0^t K^1(t,t_1)u(t_1)dt_1 \\ (d04)\quad & + \int_0^t \int_0^{t_1} K^1(t,t_1)K^2(t_1,t_2)u(t_2)dt_2dt_1 \\ & + ... + \int_0^t ... \int_0^{t_{p-1}} K^1(t,t_1)...K^p(t_{p-1},t_p)u(t_p)dt_p...dt_1, t \in R_+. \end{aligned}$$

Then, necessarily,

$$\begin{aligned} & u(u) \le b(t) + \int_0^t K^{[1]}(t,t_1)b(t_1)dt_1 \\ & + \int_0^t \int_0^{t_1} K^{[1]}(t,t_1)K^{[2]}(t_1,t_2)b(t_2)dt_2dt_1 \\ & + ... + \int_0^t ... \int_0^{t_{p-2}} K^{[1]}(t,t_1)...K^{[p-1]}(t_{p-2},t_{p-1})w(t_{p-1})dt_{p-1}...dt_1,\ t \in R_+, \end{aligned} \tag{4.2}$$

where $w \in C^m[Q]$ is the unique solution of the integral equation

(d05) $w(t) = b(t) + \int_0^t K^{[p]}(t,s)w(s)ds,\ t \in R_+$.

Remark 4.3. By the developments in Section 3, one derives that for each solution $u \in X_+$ of (d04) we have $u \le w$ (=the unique solution in X_+ of (d05)). Clearly, the evaluation (4.2) is finer than this one. However, in many concrete situations, it gives, practically, the same amount of information.

In particular, when $K^i(t,s)$ is not depending on the first variable (for $i \in \{1,...,p\}$) Theorem 4.2 includes the Young's result [13] we already quoted. Note that, under the lines in Popenda [9], all these statements have a correspondent in the difference inequalities theory. Further aspects may be found in Bainov and Simeonov [1, Ch 1, Sect 1].

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