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POSITIVE PSEUDO ALMOST AUTOMORPHIC SOLUTIONS FOR SOME NONLINEAR INFINITE DELAY INTEGRAL EQUATIONS

PHILIPPE CIEUTAT*
Laboratoire de Mathématiques de Versailles,
Université Versailles-Saint-Quentin-en-Yvelines,
45 avenue des États-Unis, 78035 Versailles cedex, France

KHALIL EZZINBI†
Département de Mathématiques, Université Cadi Ayyad,
Faculté des Sciences B.P. 2390 Marrakech, Morocco

Abstract

We state sufficient conditions for the existence of positive pseudo almost automorphic solutions of the following nonlinear infinite delay integral equation:

$$x(t)=\int_{-\infty}^{t}a(t,t-s)f(s,x(s))\,ds.$$

We deduce some corollaries on a finite delay integral equation and on a delay differential equation.

Keywords: Pseudo almost automorphic solutions, delay differential equation, delay integral equation, Hilbert's projective metric, fixed point.

1 Introduction

For a continuous map $f:\mathbb{R}\times\mathbb{R}^{+}\longrightarrow\mathbb{R}^{+}$, we consider the following nonlinear integral equation:

$$x(t)=\int_{-\infty}^{t}a(t,t-s)f(s,x(s))\,ds, \tag{1.1}$$

where $a:\mathbb{R}\times\mathbb{R}^{+}\longrightarrow\mathbb{R}^{+}$ is a map such that $a(t,.)$ is nonnegative integrable function on $\mathbb{R}^{+}$, for each $t\in\mathbb{R}$. In this paper we give sufficient conditions for the existence of positive bounded solutions of Equation (1.1). Then we deduce some corollaries of this last result on the following finite delay integral equation:

$$x(t)=\int_{t-\sigma(t)}^{t}f(s,x(s))\,ds, \tag{1.2}$$

*E-mail address: Philippe.Cieutat@math.uvsq.fr
†E-mail address: ezzinbi@ucam.ac.ma

when the delay is time-dependent and on the following delay differential equation:

$$x'(t)+\alpha(t)x(t)=f(t,x(t-\tau)) \tag{1.3}$$

where $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ and $\tau \geq 0$.

Ait Dads and Ezzinbi [1] state sufficient conditions for the existence of positive pseudo almost periodic solutions for the following infinite delay integral equation:

$$x(t)=\int_{-\infty}^{t} b(t-s)f(s,x(s))\,ds, \tag{1.4}$$

that is a particular case of Equation (1.1).

The paper is organized as follows: in Section 2 we recall some notations and definitions on pseudo almost automorphic, then we recall the main notions related on the Hilbert's projective metric. We also give the list of hypotheses which will be made in the whole of this work. In Section 3, we state our main result on the existence and the uniqueness of the positive pseudo almost automorphic solution for Equation (1.1). Section 4 is concerned with the application of the main result to Equation (1.2)-(1.4).

Recently in [2], we treated the almost automorphic case and here we propose to extend this last paper to the pseudo almost automorphic case. The asymptotically almost periodic case and the pseudo almost periodic case are studied in [3].

2 Notation and definitions

2.1 Pseudo almost automorphic functions

In the sequel, we give some properties about pseudo almost automorphic functions. Let $BC(\mathbb{R},X)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to a Banach space X, equipped with the uniform norm topology. Throughout the paper X will be $\mathbb{R}$ the set of real numbers or $L^1(\mathbb{R}^+)$ the Lebesgue space of order one in $\mathbb{R}^+$ endowed with the norm

$$\| u\|_{L^1(\mathbb{R}^+)}=\int_0^{+\infty} \mid u(t) \mid \; dt.$$

Let $x \in BC(\mathbb{R},X)$ and $\tau \in \mathbb{R}$. We define the function x_τ by

$$x_\tau = x(\tau+s) \text{ for } s \in \mathbb{R}.$$

Definition 2.1. [4] *A bounded continuous function $x : \mathbb{R} \longrightarrow X$ is said to be almost periodic if*

$$\{x_\tau \; ; \; \tau \in \mathbb{R}\}$$

is relatively compact in $BC(\mathbb{R},X)$.

Definition 2.2. [7] *A continuous function $x : \mathbb{R} \longrightarrow X$ is said to be almost automorphic if for every sequence of real numbers $(t_n')_n$, there exists a subsequence $(t_n)_n$ such that*

$$y(t)=\lim_{n\to+\infty} x(t+t_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to+\infty} y(t-t_n) = x(t)$$

for each $t \in \mathbb{R}$.

For the sequel, $AA(X)$ will denote the set of almost automorphic X-valued functions.

Remark. By the pointwise convergence, the function y is just measurable and not necessarily continuous. If the convergence in both limits is uniform, then x is almost periodic. The concept of almost automorphy is then larger than almost automorphy. If we denote by $AP(X)$ the space of all almost periodic X-valued functions, then we have

$$AP(X) \subset AA(X) \subset BC(\mathbb{R}, X).$$

If x is almost automorphic, then its range is relatively compact, thus bounded in norm.

Definition 2.3. [7] *A continuous function $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is said to be almost automorphic in t uniformly with respect to $x \in \mathbb{R}^+$ if for every bounded subset B of $\mathbb{R}^+$ and for every sequence of real numbers $(t'_n)_n$, there exists a subsequence $(t_n)_n$ such that for each $x \in B$,*

$$g(t,x) = \lim_{n\to+\infty} f(t+t_n, x)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to+\infty} g(t-t_n, x) = f(t,x)$$

for each $t \in \mathbb{R}$.

Definition 2.4. [6] *A bounded continuous function $x : \mathbb{R} \longrightarrow X$ is said to be pseudo almost automorphic if x is decomposed as follows:*

$$x = x_1 + x_2$$

where x_1 is almost automorphic and x_2 is ergodic:

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{+r} |\, x_2(t) \,| \, dt = 0.$$

For the sequel, $PAA(X)$ will denote the set of pseudo almost automorphic functions. With these definitions, we have

$$AP(X) \subset AA(X) \subset PAA(X) \subset BC(\mathbb{R}, X).$$

Theorem 2.5. [9] *If we equip $PAA(X)$ with the sup norm, then $PAA(X)$ turns out to be a Banach spaces.*

Lemma 2.6. [9] *Let x be a pseudo almost automorphic function such that*

$$x = x_1 + x_2$$

where x_1 is almost automorphic and x_2 is ergodic. Then

$$\{x_1(t);\ t \in \mathbb{R}\} \subset \overline{\{x(t);\ t \in \mathbb{R}\}}.$$

Definition 2.7. [6] *A continuous function $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is said to be pseudo almost automorphic in t uniformly with respect to $x \in \mathbb{R}^+$ if and only if*

$$f(t,x) = f_1(t,x) + f_2(t,x) \text{ for } t \in \mathbb{R} \text{ and } x \geq 0,$$

where $f_1 : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is almost automorphic in t uniformly with respect to $x \in \mathbb{R}^+$ and

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{+r} |\ f_2(t,x)\ |\ \ dt = 0.$$

uniformly for x in any bounded subset of $\mathbb{R}^+$.

Theorem 2.8. [6] *Let $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a pseudo almost automorphic in t uniformly with respect to $x \in \mathbb{R}^+$. Denote respectively by f_1 and f_2 the almost automorphic part and the ergodic of the function f. Assume that f_1 and f_2 are uniformly continuous in any bounded set $K \subset \mathbb{R}^+$ uniformly in t. If $x \in PAA(\mathbb{R})$, then $f(.,x(.)) \in PAA(\mathbb{R})$. Moreover the almost automorphic part of the function $f(.,x(.))$ is given by $f_1(.,x_1(.))$ where x_1 and f_1 are respectively the almost automorphic part of x and f.*

By Lemma 2.6 and Theorem 2.8, we deduce the following result

Lemma 2.9. *Let $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a pseudo almost automorphic in t uniformly with respect to $x \in \mathbb{R}^+$. Assume that f is Lipschitzian function with respect to the second argument. If $x \in PAA(\mathbb{R})$, then $f(.,x(.)) \in PAA(\mathbb{R})$. Moreover the almost automorphic part of the function $f(.,x(.))$ is given by $f_1(.,x_1(.))$ where x_1 and f_1 are respectively the almost automorphic part of x and f.*

2.2 Hilbert's projective metric

Let X be a real Banach space. A closed convex set K in X is called a convex cone if the following conditions are satisfied:

$$(i) \qquad \text{if} \quad x \in K, \text{ then } \lambda x \in K \qquad \text{for } \lambda \geq 0$$

$$(ii) \qquad \text{if} \quad x \in K \text{ and } -x \in K, \text{ then } x = 0.$$

A cone K induces a partial ordering $\leq$ in X by

$$x \leq y \qquad \text{if and only if} \qquad y - x \in K.$$

A cone K is called *normal* if there exists a constant k such that

$$0 \le x \le y \qquad \text{implies that} \quad \| x \| \le k \| y \|$$

where $\| . \|$ is the norm on X. If K is now a general cone in a Banach space X and x and y are elements of $K^* = K - \{0\}$, we say that x and y are *comparable* if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha x \le y \le \beta x.$$

This define an equivalence relation on K^* and divides K^* into disjoint subsets which we call *components of* K. If x and y are comparable, we define the numbers $m(y/x)$ and $M(y/x)$ by

$$m(y/x) := \sup\{\alpha > 0; \alpha x \le y\} \tag{2.1}$$

$$M(y/x) := \inf\{\beta > 0; y \le \beta x\}. \tag{2.2}$$

We define a metric which was introduced by Thompson [8]. If x and $y \in K^*$ are comparable, define $d(x,y)$ by

$$\begin{aligned} d(x,y) &:= \max(\log M(y/x), \log M(x/y)) \\ &= \max(\log(M(y/x), -\log m(y/x)). \end{aligned} \tag{2.3}$$

If C is a component of K, one can easily prove (c.f. [8]) that d gives a metric on C. Moreover Thompson proves the following result.

Theorem 2.10. [8] *Let K be a normal cone in a Banach space X and let C be a component of K. Then C is a complete metric space with respect to the metric d.*

Proposition 2.11. [8] *Let K be a normal cone in a Banach space X with nonempty interior $\overset{\circ}{K}$. Then $\overset{\circ}{K}$ is a component of K.*

It follows that if K is a normal cone with nonempty interior $\overset{\circ}{K}$, then $\overset{\circ}{K}$ is a complete metric space with respect to the metric d.

Theorem 2.12. [5] *Let E be a complete space with respect to the metric d. If f be a mapping from E into E satisfying*

$$d(f(x), f(y)) \le \Phi(d(x,y)) \qquad \text{for all } x \text{ and } y \in E,$$

where Φ is a positive nondecreasing function continuous on $[0,+\infty[$, verifying $\Phi(r) < r$ for every $r > 0$ and $\Phi(0) = 0$, then f has exactly one fixed point in E.

2.3 Hypotheses

Now we give a list of hypotheses which are used.

From $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ and $a : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we formulate the following hypotheses.

(H1) *There exists a continuous map $\phi : (0,1) \longrightarrow \mathbb{R}^+$ satisfying $\phi(\lambda) > \lambda$ and for each x and $y > 0$, $t \in \mathbb{R}$ and $\lambda \in (0,1)$, one has*

$$\lambda x \le y \le \lambda^{-1} x \Longrightarrow f(t,y) \ge \phi(\lambda) f(t,x).$$

(H2) *For each $t \in \mathbb{R}$, $a(t,.) \in L^1(\mathbb{R}^+)$ and there exists $x_0 > 0$ such that*

$$\inf_{t\in\mathbb{R}} \int_0^{+\infty} a(t,s)f(t-s,x_0)\, ds > 0.$$

(H3) *$f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a pseudo almost automorphic function in t uniformly with respect to $x \in \mathbb{R}^+$.*

(H4) *The function $t \to a(t,.)$ is in $PAA(L^1(\mathbb{R}^+))$.*

(H5) *There exists $b \in L^1(\mathbb{R}^+)$ such that $| a_1(t,s) | \leq b(s)$ for all $t \in \mathbb{R}$ and almost everywhere for s in $\mathbb{R}^+$, where $t \to a_1(t,.)$ is the almost automorphic part of the pseudo almost automorphic function $t \to a(t,.)$.*

3 Main result

In this section, we state a result of the existence and the uniqueness of the pseudo almost automorphic solution of Equation (1.1) with a positive infinimum.

Theorem 3.1. *Suppose that (H1)-(H5) hold. Then Equation (1.1) has a unique pseudo almost automorphic solution x with a positive infinimum. Furthermore, the almost automorphic part x_1 of x is the unique almost automorphic solution of the equation:*

$$x_1(t) = \int_{-\infty}^{t} a_1(t,t-s)f_1(s,x_1(s))\, ds \tag{3.1}$$

with a positive infinimum, where f_1 is the almost automorphic part of f.

For the proof of Theorem 3.1 we use the following lemmas

Lemma 3.2. ([2], Lemma 3.3) *Let $f : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function. Suppose that (H1) holds and there exists $x_1 > 0$ such that $f(.,x_1) \in BC(\mathbb{R},\mathbb{R})$. Then one has*

i) $\forall x,y > 0$, $\forall t \in \mathbb{R}$, $f(t,y) \geq \min\left(\frac{x}{y},\frac{y}{x}\right) f(t,x)$.

ii) For each $[a,b] \subset]0,+\infty[$, f is bounded on $\mathbb{R}\times[a,b]$.

iii) For each $[a,b] \subset]0,+\infty[$, $\exists L \geq 0$, $\forall x,y \in [a,b]$, $\forall t \in \mathbb{R}$,

$$| f(t,x) - f(t,y) | \leq L | x-y | .$$

Lemma 3.3. ([2], Lemma 3.5) *Let $a : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ such that the function $t \to a(t,.)$ is in $BC(\mathbb{R},L^1(\mathbb{R}^+))$. If $f \in BC(\mathbb{R},\mathbb{R})$, then the function*

$$h(t) = \int_{-\infty}^{t} a(t,t-s)f(s)\, ds \tag{3.2}$$

is also continuous and bounded on $\mathbb{R}$.

Lemma 3.4. ([2], Lemma 4.4) *Let* $a : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that the function* $t \to a(t,.)$ *is in* $AA(L^1(\mathbb{R}^+))$. *If* $f \in AA(\mathbb{R})$, *then the function* h *defined by (3.2) is also almost automorphic.*

Lemma 3.5. *Let* $a : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ *such that the function* $t \to a(t,.)$ *is in* $PAA(L^1(\mathbb{R}^+))$. *Denote by* $t \to a_1(t,.)$ *its almost automorphic part. We assume that there exists* $b \in L^1(\mathbb{R}^+)$ *such that* $\mid a_1(t,s)\mid \leq b(s)$ *for all* $t \in \mathbb{R}$ *and almost everywhere* s *in* $\mathbb{R}^+$. *If* $f \in PAA(\mathbb{R})$, *then the function* h *defined by (3.2) is also pseudo almost automorphic. Furthermore the almost automorphic part of* h *is given by*

$$h_1(t) = \int_{-\infty}^{t} a_1(t,t-s)f_1(s)\,ds, \tag{3.3}$$

where f_1 *is the almost automorphic part of* f.

Proof. By Lemma 3.3, h is continuous and bounded and by Lemma 3.4, h_1 is almost automorphic. Let

$$I(r) := \frac{1}{2r}\int_{-r}^{r} \mid \int_{-\infty}^{t} a(t,t-s)f(s) - a_1(t,t-s)f_1(s)\,ds \mid dt.$$

To check that h is in $PAA(\mathbb{R})$ and that the almost automorphic part of h is given by (3.3), we must prove that $\lim_{r\to+\infty} I(r) = 0$. But

$$I(r) = \frac{1}{2r}\int_{-r}^{r} \mid \int_{-\infty}^{t} a_2(t,t-s)f(s) + a_1(t,t-s)f_2(s)\,ds \mid dt,$$

we obtain

$$\begin{aligned} I(r) &\leq \parallel f \parallel_\infty \frac{1}{2r}\int_{-r}^{r} \parallel a_2(t,.)\parallel_{L^1(\mathbb{R}^+)}\;dt \\ &+\frac{1}{2r}\int_{-r}^{r}\left(\int_0^{+\infty} \mid a_1(t,s)f_2(t-s)\mid\;ds\right)dt. \end{aligned} \tag{3.4}$$

By the hypothesis, one has

$$\frac{1}{2r}\int_{-r}^{r} \parallel a_2(t,.)\parallel_{L^1(\mathbb{R}^+)}\;dt \to 0 \quad as \quad r\to+\infty. \tag{3.5}$$

On the other hand, by the Fubini theorem, one has

$$\begin{aligned} &\frac{1}{2r}\int_{-r}^{r}\left(\int_0^{+\infty} \mid a_1(t,s)f_2(t-s)\mid\;ds\right)dt \\ &\leq \int_0^{+\infty} b(s)\left(\frac{1}{2r}\int_{-r}^{r} \mid f_2(t-s)\mid\;dt\right)ds \\ &= \int_0^{+\infty} b(s)F_r(s)\,ds, \end{aligned}$$

where

$$F_r(s) = \frac{1}{2r}\int_{-r-s}^{r-s} \mid f_2(t) \mid \, dt.$$

We have $\lim_{r\to+\infty} F_r(s) = 0$ and F_r is bounded, by the Lebesgue dominated convergence theorem, we have

$$\frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{+\infty} \mid a_1(t,s)f_2(t-s) \mid \, ds\right)dt \to 0 \quad as \quad r \to +\infty, \tag{3.6}$$

so by (3.4)-(3.6), we obtain $\lim_{r\to+\infty} I(r) = 0$. This ends the proof of Lemma.∎

Lemma 3.6. *Suppose that (H1) and (H3)-(H5) hold. If $x \in PAA(\mathbb{R})$ and x has a positive infinimum, then the function*

$$F(t) = \int_{-\infty}^{t} a(t,t-s)f(s,x(s))\, ds. \tag{3.7}$$

is also pseudo almost automorphic. Furthermore the almost automorphic part of F is given by

$$F_1(t) = \int_{-\infty}^{t} a_1(t,t-s)f_1(s,x_1(s))\, ds, \tag{3.8}$$

where x_1 and f_1 are respectively the almost automorphic parts of x and f.

Proof. There exist a and $b \in \mathbb{R}$ such that $0 < a \le x(t) \le b$, for all $t \in \mathbb{R}$. By Lemma 3.2, we obtain $\mid f(t,x_1) - f(t,x_2) \mid \le L \mid x_1 - x_2 \mid$ for all $t \in \mathbb{R}$, x_1 and $x_2 \in [a,b]$. Since x is in $PAA(\mathbb{R})$ and f, by Lemma 2.9, we deduce that $t \to f(t,x(t))$ is pseudo almost automorphic and $(f(t,x(t)))_1 = f_1(t,x_1(t))$. The hypotheses of Lemma 3.5 are satisfied, then $F \in PAA(\mathbb{R})$ and the almost automorphic part of F is given by (3.8).∎

Proof of Theorem 3.1. We apply the results of Section 2 in order to prove the existence and uniqueness of the pseudo almost automorphic solution of Equation (1.1) with a positive infinimum. Let $X = PAA(\mathbb{R})$ be the Banach space of pseudo almost automorphic functions endowed with the norm defined by $\| f\|_\infty = \sup_{t\in\mathbb{R}} \mid f(t) \mid$ (c.f. Theorem 2.5). Let K be the cone of nonnegative functions in $PAA(\mathbb{R})$. Then K is a normal convex cone. Furthermore, one has

$$0 \le x \le y \qquad \Longrightarrow \quad \| x\|_\infty \le \| y\|_\infty.$$

The interior of K is given by $\overset{\circ}{K} = \{x \in PAA(\mathbb{R}) \; ; \; \inf_{t\in\mathbb{R}} x(t) > 0\}$. We denote by T the operator associated with the right-hand side of Equation (1.1), namely

$$(Tx)(t) = \int_{-\infty}^{t} a(t,t-s)f(s,x(s))\, ds. \tag{3.9}$$

Note that the pseudo almost automorphic solutions of Equation (1.1) with a positive infinimum are fixed points of T.

Now, we prove that T maps $\overset{\circ}{K}$ into itself. Let $x \in \overset{\circ}{K}$. Then there exists $\varepsilon > 0$ such that $\varepsilon \leq x(t) \leq \varepsilon^{-1}$, for each $t \in \mathbb{R}$. By Lemma 3.2, one has

$$(Tx)(t) \geq \int_{-\infty}^{t} a(t,t-s)\min\left(\frac{x(s)}{x_0},\frac{x_0}{x(s)}\right) f(s,x_0)\, ds$$

$$\geq \varepsilon \min\left(\frac{1}{x_0},x_0\right)\int_{-\infty}^{t} a(t,t-s)f(s,x_0)\, ds.$$

So

$$(Tx)(t) \geq \varepsilon \min\left(\frac{1}{x_0},x_0\right)\inf_{t\in\mathbb{R}}\int_{0}^{+\infty} a(t,s)f(t-s,x_0)\, ds > 0.$$

Furthermore, by Lemma 3.6, $Tx \in PAA(\mathbb{R})$. Then $Tx \in \overset{\circ}{K}$ for all $x \in \overset{\circ}{K}$.

To have a fixed point of T in $\overset{\circ}{K}$, we use Theorem 2.12. We know that $(\overset{\circ}{K},d)$ is a complete metric space with d defined by (2.3), (c.f. Proposition 2.11). By (H2), there exists $t_0 \in \mathbb{R}$ such that $f(t_0,x_0) > 0$ and by (H1), one has $f(t_0,x_0) \geq \phi(\lambda)f(t_0,x_0)$ and $\phi(\lambda) > \lambda$ for all $\lambda \in (0,1)$, then $\lim_{\lambda\to 1}\phi(\lambda) = 1$. Now we consider that the function ϕ is defined and continuous on $]0,1]$. We can assume that ϕ is nondecreasing (for that change ϕ by $\phi_1(\lambda) = \inf\{\phi(\mu)\ ;\ \lambda \leq \mu \leq 1\}$). Let x and $y \in \overset{\circ}{K}$, $\lambda \in (0,1)$ such that $\lambda x \leq y \leq \lambda^{-1}x$. By (H1), one has

$$\forall t \in \mathbb{R},\ f(t,y(t)) \geq \phi(\lambda)f(t,x(t)).$$

We also have $\lambda y \leq x \leq \lambda^{-1}y$, then

$$\forall t \in \mathbb{R},\ \phi(\lambda)f(t,x(t)) \leq f(t,y(t)) \leq (\phi(\lambda))^{-1} f(t,x(t)),$$

thus

$$\phi(\lambda)Tx \leq Ty \leq (\phi(\lambda))^{-1}Tx,$$

therefore

$$d(Tx,Ty) \leq \ln\left(\frac{1}{\phi(\lambda)}\right).$$

For $\lambda = \left(\max\left(M(\frac{y}{x}),M(\frac{x}{y})\right)\right)^{-1}$, we have $d(x,y) = \ln\left(\lambda^{-1}\right)$. If we choose the function $\Phi(r) := -\ln(\phi(e^{-r}))$ for $r \geq 0$, we deduce that

$$d(Tx,Ty) \leq \Phi\left(d(x,y)\right).$$

Furthermore Φ is a positive, continuous and nondecreasing function on $[0,+\infty[$ satisfying $\Phi(r) < r$ for all $r > 0$ and $\Phi(0) = 0$, then T has exactly one fixed point in $\overset{\circ}{K}$ which is an pseudo almost automorphic solution of Equation (1.1) with a positive infinimum.

By using Lemma 3.6, we can assert that to end the proof, it suffices to state that Equation (3.1) has a unique almost automorphic solution with a positive infinimum. For that we use ([2], Theorem 4.1) on f_1 and a_1. By Lemma 2.6, we deduce that the function $f_1 : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfies $f_1(t,x) \geq 0$ for each $t \in \mathbb{R}$ and $x \geq 0$, since $f \geq 0$. Since f satisfy (H1) and (H3), then for x and $y > 0$, $\lambda \in (0,1)$ and $\lambda x \leq y \leq \lambda^{-1}x$, the function $t \to f(t,y) - \phi(\lambda)f(t,x)$ is pseudo almost automorphic and nonnegative, thus by Lemma 2.6, we deduce that its almost automorphic part is nonnegative: $f_1(t,y)-\phi(\lambda)f_1(t,x)\geq 0$, consequently f_1 verifies (H1). For (H2), by using Lemma 3.5, we deduce that the almost automorphic part of $t\to\int_0^{+\infty}a(t,s)f(t-s,x_0)\;ds$ is $t\to\int_0^{+\infty}a_1(t,s)f_1(t-s,x_0)\;ds$, then a_1 and f_1 satisfy (H2), since a and f also satisfy (H2). By helps of ([2], Theorem 4.1), we obtain the existence and uniqueness of the positive almost automorphic solution of Equation (3.1). This ends the proof of Theorem 3.1.■

4 Consequences on the main result

In this Section, we apply our main result to Equation (1.2)-(1.4).

Corollary 4.1. *Suppose that (H1) and (H3) hold. In addition, we assume that*

i) σ is a positive pseudo almost automorphic function,

ii) there exists $x_0>0$ such that

$$\inf_{t\in\mathbb{R}}\int_{t-\sigma(t)}^{t}f(s,x_0)\;ds>0. \tag{4.1}$$

Then Equation (1.2) has a unique pseudo almost automorphic solution with a positive infinimum. Furthermore, the almost automorphic part x_1 of x, is the unique almost automorphic solution of the equation:

$$x_1(t)=\int_{t-\sigma_1(t)}^{t}f_1(s,x_1(s))\;ds$$

with a positive infinimum, where σ_1 and f_1 are respectively the almost automorphic parts of σ and f.

Proof. We use Theorem 3.1 with the function $a(t,s):=1_{[0,\sigma(t)]}(s)$. (where $1_{[0,\sigma(t)]}(s)=1$ if $0\leq s\leq\sigma(t)$ and 0 elsewhere). Obviously (4.1) implies (H2). Since $t\to 1_{[0,\sigma_1(t)]}(.)\in AA(L^1(\mathbb{R}^+))$ (c.f. proof of Corollary 4.2 in [2] and by using

$$\|\;1_{[0,\sigma(t)]}-1_{[0,\sigma_1(t)]}\;\|_{L^1(\mathbb{R}^+)}=|\;\sigma(t)-\sigma_1(t)\;|,$$

we deduce that (H4) is satisfied and $a_1(t,s)=1_{[0,\sigma_1(t)]}(s)$. By remarking the following inequality $|\;a_1(t,s)\;|\leq 1_{[0,\|\sigma_1\|_\infty]}(s)$, we obtain (H5).■

Corollary 4.2. *Let $b\in L^1(\mathbb{R}^+)$ Suppose that there exists $x_0>0$ such that*

$$\inf_{t\in\mathbb{R}}\int_0^{+\infty}b(s)f(t-s,x_0)\;ds>0.$$

Assume that (H1) and (H3) hold. Then Equation (1.4) has a unique pseudo almost automorphic solution x with a positive infinimum. Furthermore, the almost automorphic part x_1 of x is the unique almost automorphic solution of the equation:

$$x_1(t)=\int_{-\infty}^{t}b(t-s)f_1(s,x_1(s))\;ds$$

with a positive infinimum, where f_1 is the almost automorphic part of f.

Proof. We use theorem 3.1 with the function $a(t,s) = b(s)$.■

Now, we apply our main result for the existence of the pseudo almost automorphic solutions with a positive infinimum to the first order semilinear differential Equation (1.3). Let $\alpha \in BC(\mathbb{R},\mathbb{R})$ and $\tau \geq 0$. Recall that the homogeneous linear equation

$$x'(t) + \alpha(t)x(t) = 0 \tag{4.2}$$

has an exponential dichotomy if there exist k and $c > 0$ such that

$$\exp\left(-\int_s^t \alpha(\xi)\, d\xi\right) \leq ke^{-c(t-s)}, \quad \forall t \geq s. \tag{4.3}$$

If Equation (4.2) has an exponential dichotomy, then for any $p \in BC(\mathbb{R},\mathbb{R})$, the linear equation

$$x'(t) + \alpha(t)x(t) = p(t)$$

has a unique bounded solution which is given by

$$x(t) = \int_{-\infty}^t \exp\left(-\int_s^t \alpha(\xi)\, d\xi\right) p(s)\, ds.$$

Similarly, if Equation (4.2) has an exponential dichotomy and if f is bounded on every $\mathbb{R} \times K$ where K is a compact subset of $\mathbb{R}^+$, then x is a bounded solution of Equation (1.3) if and only if x is a bounded solution of

$$x(t) = \int_{-\infty}^t \exp\left(-\int_s^t \alpha(\xi)\, d\xi\right) f(s,x(s-\tau))\, ds. \tag{4.4}$$

By making the change of variables of s to $s+\tau$, one can rewrite Equation (4.4) as

$$x(t) = \int_{-\infty}^t \exp\left(-\int_{s+\tau}^t \alpha(\xi)\, d\xi\right) 1_{[\tau,+\infty]}(t-s) f(s+\tau,x(s))\, ds. \tag{4.5}$$

To start, we give a result on the exponential dichotomy of Equation (4.2) in the bounded case.

Lemma 4.3. ([2], Lemma 6.1) *Let $\alpha \in BC(\mathbb{R},\mathbb{R})$. If there exists $r_0 > 0$ such that*

$$\inf_{t\in\mathbb{R}} \int_{t-r_0}^t \alpha(\xi)\, d\xi > 0, \tag{4.6}$$

then Equation (4.2) has an exponential dichotomy.

Remark. The converse of Lemma 4.3 is obviously true, but we will not use it.

Proposition 4.4. *Let $\alpha \in PAA(\mathbb{R})$. Assume that f satisfies (H1) and (H3). In addition we suppose that there exists $r_0 > 0$ such that (4.6) and*

$$\inf_{t\in\mathbb{R}} \int_{t-r_0}^t f(s,x_0)\, ds > 0 \tag{4.7}$$

hold. Then Equation (1.3) has a unique pseudo almost automorphic solution x with a positive infinimum. Furthermore, the almost automorphic part x_1 of x is the unique almost automorphic solution x of the equation:

$$\frac{d}{dt}x_1(t)+\alpha_1(t)x_1(t)=f_1(t,x_1(t)), \tag{4.8}$$

where α_1 and f_1 are respectively the almost automorphic parts of α and f.

For the proof of Proposition 4.4, we use the following lemmas.

Lemma 4.5. *Let $\tau \geq 0$. We assume that $\alpha \in BC(I\!R, I\!R)$ such that Equation (4.2) has an exponential dichotomy. We denote by a the function defined by*

$$a(t,s) := \exp\left(-\int_{t+\tau-s}^{t}\alpha(\xi)\,d\xi\right)1_{[\tau,+\infty[}(s). \tag{4.9}$$

i) If $\alpha \in AA(I\!R)$, then the function $t \to a(t,.)$ is in $AA(L^1(I\!R^+))$.

ii) If $\alpha \in PAA(I\!R)$, then the equation

$$x'(t)+\alpha_1(t)x(t)=0 \tag{4.10}$$

has an exponential dichotomy and the function $t \to a(t,.)$ is in $PAA(L^1(I\!R^+))$ and its almost automorphic part is given by $t \to b(t,.)$, where b is defined by

$$b(t,s) := \exp\left(-\int_{t+\tau-s}^{t}\alpha_1(\xi)\,d\xi\right)1_{[\tau,+\infty[}(s). \tag{4.11}$$

Proof. i) See the second part of the proof of ([2], Proposition 6.2)

ii) We denote by A and B the functions defined by

$$A(t,s) := \exp\left(-\int_{t-s}^{t}\alpha(\xi)\,d\xi\right), \quad t \in I\!R,\ s \geq 0, \tag{4.12}$$

$$B(t,s) := \exp\left(-\int_{t-s}^{t}\alpha_1(\xi)\,d\xi\right), \quad t \in I\!R,\ s \geq 0. \tag{4.13}$$

Let $\sigma \geq 0$. By using Lemma 3.5 with $a(t,s) = 1_{[0,\sigma]}(s)$ and $f(s) = \alpha(s)$, we deduce that

$$t \to \int_{t-\sigma}^{t}\alpha(\xi)\,d\xi \in PAA(I\!R) \tag{4.14}$$

and its almost automorphic part is

$$[t \to \int_{t-\sigma}^{t}\alpha(\xi)\,d\xi]_1 = t \to \int_{t-\sigma}^{t}\alpha_1(\xi)\,d\xi. \tag{4.15}$$

Since the function (4.14) and (4.15) are bounded on $\mathbb{R}$ and the function $x \to e^{-x}$ is Lipschitzian on each bounded interval, then there exists $c > 0$ (depends of σ and not of t) such that

$$\forall t \in \mathbb{R}, \qquad | A(t,\sigma) - B(t,\sigma) | \leq c \left| \int_{t-\sigma}^{t} \alpha(\xi)\, d\xi - \int_{t-\sigma}^{t} \alpha_1(\xi)\, d\xi \right|$$

By using (4.14) and (4.15), we deduce that

$$t \to A(t,\sigma) \in PAA(\mathbb{R}) \tag{4.16}$$

and its almost automorphic part is

$$[t \to A(t,\sigma)]_1 = t \to B(t,\sigma). \tag{4.17}$$

Moreover, Equation (4.2) has an exponential dichotomy, then there exist k and $c > 0$ such that

$$0 \leq A(t,s) \leq ke^{-cs}, \quad t \in \mathbb{R},\ s \geq 0, \tag{4.18}$$

then by Lemma 2.6, its almost automorphic part satisfies

$$0 \leq B(t,s) \leq ke^{-cs}, \quad t \in \mathbb{R},\ s \geq 0, \tag{4.19}$$

therefore Equation (4.10) has an exponential dichotomy. By (4.18), one has

$$0 \leq a(t,s) \leq ke^{-c(s-\tau)}, \quad t \in \mathbb{R},\ s \geq 0, \tag{4.20}$$

moreover, the function $t \to a(t,s)$ is continuous, so by the Lebesgue dominated convergence theorem, we deduce that $t \to a(t,.)$ is in $BC(\mathbb{R}, L^1(\mathbb{R}^+))$. To check that $t \to a(t,.)$ is in $PAA(L^1(\mathbb{R}^+))$ and its almost automorphic part is given by (4.11), we must prove that

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \| a(t,.) - b(t,.) \|_{L^1(\mathbb{R}^+)}\, dt = 0, \tag{4.21}$$

because $t \to b(t,.)$ is in $AA(L^1(\mathbb{R}^+))$ (cf. i) of this lemma). By making the change of variables of s to $s - \tau$ and by using the Fubini theorem, we deduce that (4.21) is equivalent to

$$\lim_{r \to +\infty} \int_0^{+\infty} \left(\frac{1}{2r} \int_{-r}^{r} | A(t,s) - B(t,s) |\ dt \right) ds = 0. \tag{4.22}$$

By (4.16) and (4.17), $t \to A(t,s) - B(t,s)$ is ergodic, that is to mean

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} | A(t,s) - B(t,s) |\ dt = 0$$

and by (4.18) and (4.19), one has

$$0 \leq \frac{1}{2r} \int_{-r}^{r} | A(t,s) - B(t,s) |\ dt \leq 2ke^{-cs},$$

so by the Lebesgue dominated convergence theorem, we deduce that (4.22) is satisfied, this ends the proof.■

Lemma 4.6. ([3], Lemma 7.5) *Let $c > 0$ and let $\phi \in BC(\mathbb{R},\mathbb{R})$ such that $\phi \geq 0$. If there exists $r_0 > 0$ such that*

$$\inf_{t\in\mathbb{R}} \int_{t-r_0}^{t} \phi(s)\, ds > 0, \tag{4.23}$$

then

$$\inf_{t\in\mathbb{R}} \int_{0}^{+\infty} e^{-cs}\phi(t-s)\, ds > 0. \tag{4.24}$$

Proof of Proposition 4.4. By Lemma 4.3, Equation (4.2) admits an exponential dichotomy, then a pseudo almost automorphic function x is a solution of Equation (1.3) if and only if x is a solution of Equation (4.5). To state Proposition 4.4, we use Theorem 3.1 with the function $(t,s) \to a(t,s)$ defined by (4.9) and the function $(t,x) \to f(t+\tau,x)$. It suffices to prove that hypotheses (H2), (H4) and (H5) are satisfied. Hypothesis (H4) results of Lemma 4.5. By this lemma, Equation (4.10) admits an exponential dichotomy and the almost automorphic part of $t \to a(t,.)$ is defined by (4.11), thus (H5) is satisfied with the function $b \in L^1(\mathbb{R}^+)$ defined by $b(s) := ke^{c(\tau-s)}$ for all $s \geq 0$. Let us verify (H2). By (4.6), one has $\|\alpha\|_\infty > 0$. By using (4.7) and Lemma 4.6, we obtain

$$\delta := \inf_{t\in\mathbb{R}} \int_{0}^{+\infty} e^{-s\|\alpha\|_\infty} f(t-s,x_0)\, ds > 0,$$

then (H2) is fulfilled because

$$\inf_{t\in\mathbb{R}} \int_{0}^{+\infty} a(t,s) f(t+\tau-s,x_0)\, ds \geq \delta > 0. \blacksquare$$

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