African Diaspora Journal of Mathematics
Volume 11, Number 1, pp. 124–139 (2011)
ISSN 1539-854X
www.math-res-pub.org/adjm

EXACT CONTROLLABILITY OF SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS

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Abstract

In this paper we study the exact controllability of the following semilinear stochastic evolution equation in a Hilbert space X

$$dx(t) = \{Ax(t) + Bu(t) + f(t,\omega,x(t),u(t))\}dt + \{\Sigma(t) + \sigma(t,\omega,x(t),u(t))\}dw(t),$$

where the control u is a stochastic process in the Hilbert space U, $A : D(A) \subset X \to X$, is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X and $B \in L(U,X)$. To this end, we give necessary and sufficient conditions for the exact controllability of the linear part of this system

$$dx(t) = Ax(t)dt + Bu(t)dt + \Sigma(t)dw(t).$$

Then, under a Lipschitzian condition on the non linear terms f and σ we prove that the exact controllability of this linear system is preserved by the semilinear stochastic system. Moreover, we obtain explicit formulas for a control steering the system from the initial state ξ_0 to a final state ξ_1 on time $T > 0$, for both system, the linear and the nonlinear one. Finally, we apply this result to a semilinear damped stochastic wave equation .

AMS Subject Classification: 93E03; 93B05.

Keywords: Exact controllability, Semilinear stochastic equation, damped stochastic wave equation.

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1 Introduction

The theory of controllability for finite dimensional systems is well known and there is a broad literature ont this subject, worth mentioning are R.E. Kaman [10] and Lee-Markus [11]. However, for infinite dimensional systems the theory is not completed, there still exist several problems to formulate and solve. Excellent references are: R.F.Curtain and H.J.Zwart [5], Fattorini [9] and Russell [18]. The theory of controllability for linear stochastic evolution equations is relatively new and the formulation still questionable; in fact, Agamirsa E. bashirov and Nazin I.Mahmudov [1] give three concepts of controllability and necessary and sufficient conditions for exact, approximate and s-controllability, which are subsequently extended to infinite dimensional systems by Nazin I. Mahmudov [16]. Others references in this direction are A.Lindquist [15], R.F.Curtain and A.Ichikawa [6].

For semilinear evolution equations, integral equations, stochastic evolution equations and stochastic integral equations, the controllability was studied respectively by S.K. Ntouyas and D. O'Regan [17], K.Balachandran and J.P.Dauer [2], K.Balachandran and R.Sakthivel [3], P. Balasubramaniam and J.P. Dauer [4], J.P.Dauer and K.Balachandran [7], R. Subalakshmi, K. Balachandran and J.Y. Park [19]. In most of these studies a crucial hypothesis is that the controllability operator G given by (3.3) is invertible; this hypothesis is used to find an implicit formula for the control steering the system from an initial state to a final state in time T, which reduces the problem of controllability to the problem of finding the fixed points of an operator in a suitable functional space. But, we observe that in finite dimensional control systems the controllability operator G is never invertible since it is defined on the space of controls, which is a space of functions, and takes values in a finite dimensional space; so it only could be surjective.

Unlike these authors, in our study the nonlinear terms depend not only on the state variable x, but also on time t and control u; and we only assume that the linear system is exactly controllable, and as a consequence we prove that the operator G is surjective; which implies that G has a right inverse Γ ($G \circ \Gamma = I$, see corollary 3.4), that is enough to find an explicit formula for a control steering the system from the initial state ξ_0 to a final state ξ_1 on time $T > 0$, for both systems, the linear and the nonlinear one, which is very important from engineering point of view. Finally, we would like to mention that the results here are obtained by standard and basic functional analysis such as Cauchy-Schwarz inequality, Hahn-Banach theorem, the open mapping theorem, Banach fixed point theorem, etc.

This paper has been motivated by work done by Nazin I. Mahmudov [16] and the work done by H. Leiva [12]. Particularly, in H. Leiva [12] the author characterizes the exact controllability of the following semilinear evolution equation

$$z' = Az + Bu(t) + f(t, z, u(t)), \ \ t \geq 0, \ \ z \in Z, \ \ u \in U.$$

Basically, the author proves the following statement: If the linear system $z' = Az + Bu(t)$ is exactly controllable, then, under some conditions on the nonlinear term f, the nonlinear system is also exactly controllable. Specifically, the author finds explicit formulas for a control steering the system from the initial state ξ_0 to a final state ξ_1 on time $T > 0$, for both systems, the linear and the nonlinear one.

In this direction, our goal is to study the exact controllability of the following semilinear stochastic evolution equation

$$\begin{aligned} &dx(t) = \{Ax(t) + Bu(t) + f(t,\omega,x(t),u(t))\}dt + \{\Sigma(t) + \sigma(t,\omega,x(t),u(t))\}dw(t), \\ &x(0) = \xi_0 \end{aligned} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on a separable Hilbert space X, $B \in L(U,X)$, with U a separable Hilbert space.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t \uparrow\subset \mathcal{F}, t \geq 0\}$ an increasing sequence of subsigma algebras of $\mathcal{F}$, then the control function $u = \{u(t,\omega) : t \in [0,T], \omega \in \Omega\}$ is a random process with values in U, ξ_0 is an $X-$ valued random variable, $\mathcal{F}_0-$measurable and $\{w(t),\ t \geq 0\}$ denotes a Wiener process with values in a separable Hilbert space E.

Our result can be described as follows: Suppose the linear system

$$dx(t) = Ax(t)dt + Bu(t)dt + \Sigma(t)dw(t) \tag{1.2}$$

is exactly controllable. Then, the control u steering an initial state ξ_0 to a final state $x(T) = \xi_1$, on time $T \geq 0$, is given by the following formula

$$u(t) = B^*S^*(T-t)\mathbf{E}\left[\Pi^{-1}\left(\xi_1 - S(T)\xi_0 - \int_0^T S(T-s)\Sigma(s)dw(s)\right)\middle|\mathcal{F}_t\right], \tag{1.3}$$

where Π is given by

$$\Pi\{\cdot\} = \int_0^T S(T-s)BB^*S^*(T-s)\mathbf{E}\{\cdot \mid \mathcal{F}_t\}\,ds.$$

Moreover, the system (1.2) is exactly controllable if, and only if, Π is invertible.

Next, under some suitable conditions on f y σ, the exact controllability of the linear system (1.2) is preserved by the semilinear system (1.1), and the control steering an initial state ξ_0 to a final state $\xi_1 = x(T)$, on time $T \geq 0$, is given by

$$u(t) = B^*S^*(T-t)\mathbf{E}\left[\Pi^{-1}(\xi_1 - S(T)\xi_0 - \int_0^T S(T-s)\Sigma(s)dw(s))\middle|\mathcal{F}_t\right](I+K)^{-1},$$

where K is a nonlinear operator given by

$$\begin{aligned} K(\xi)(T) = &\int_0^T S(T-r)f(r,x_\xi(r),(\Gamma\xi)(r))dr \\ &+ \int_0^T S(T-r)\sigma(r,x_\xi(r),(\Gamma\xi)(r))d\omega(r), \end{aligned} \tag{1.4}$$

with x_ξ the solution of the equation (1.2), corresponding to the control defined by (1.3).

We have taken the following convenient notation:

$$f(t) = f(t,\omega,x(t),u(t)) = f(t,x(t),u(t)) \quad \text{and} \quad \sigma(t) = \sigma(t,\omega,x(t),u(t)) = \sigma(t,x(t),u(t)).$$

Finally, as an application we study the stochastic damped wave equation

$$\begin{cases} d[\frac{\partial}{\partial t}z(t,x) + cz] = [\theta z_{xx} + u(t,x) + f(t,z,z_t,u(t,x))]dt + \sigma(t,z,z_t,u(t,x))dw(t), \\ z(t,0) = z(t,1) = 0,\ 0 < x < 1,\ t \in \mathbb{R}. \end{cases}$$

where $c \geq 0$ and $\theta > 0$, distributed control $u \in L_2^{\mathcal{F}}([0,T],U)$, the terms f and σ satisfy the Lipschitz condition.

$$|f(t,x_2,v_2,u_2) - f(t,x_1,v_1,u_1)|^2 + |\sigma(t,x_2,v_2,u_2) - \sigma(t,x_1,v_1,u_1)|^2 \\ \leq k\{|x_2 - x_1|^2 + |v_2 - v_1|^2 + |u_2 - u_1|^2\}.$$

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space, together with this space we consider an increasing sequence of subsigma algebras $\{\mathcal{F}_t \uparrow\subset \mathcal{F}, t \geq 0\}$. Let $\mathbf{E}\{\cdot\}$ denote the expectation of a random variable or the Lebesgue integral with respect to the probability measure P.

Let E be a separable Hilbert space and $\{w(t), t \geq 0\}$ a Wiener process with values in E and covariance operator Q, with Q a positive nuclear operator on E. We will assume that there is a complete orthonormal basis $\{e_k\}$ in E, $\{\lambda_k\}$ a bounded sequence of nonnegative real numbers such that $Qe_k = \lambda_k e_k$, $k = 1,2,\cdots$, and β_k $(k = 1,2,\dots)$ be the sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, P)$ such that

$$w(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \ \ t \geq 0.$$

We assume that $\mathcal{F}_t$ is generated by $\{w(s) : 0 \leq s \leq \ t\}$.

Let $L_2^0 := L_2(Q^{1/2}E, X)$ be the space of all Hilbert-Schmidt operators:

$$\Psi : Q^{1/2}E \to X.$$

The space L_2^0 endowed with the norm $\| \Psi \|_{L_2^0}^2 = tr[\Psi Q \Psi^*] = \sum_{k=1}^{\infty} \| \sqrt{\lambda_k}\Psi e_k\|^2$ is a separable Hilbert space.

All random processes considered in the paper will be assumed to be strongly $\mathcal{F}_t$- progressively measurable processes unless otherwise stated.

$L_2(\Omega, \mathcal{F}, P, X) = L_2(\mathcal{F}, X)$ denotes the Hilbert space of strongly $\mathcal{F}$-measurable, X-valued random variables satisfying

$$\mathbf{E} \parallel x \parallel_X^2 < \infty.$$

Since for each $t \geq 0$ the sub-σ-algebra $\mathcal{F}_t$ is complete, $L_2(\mathcal{F}_t, X)$ is a closed subspace of $L_2(\mathcal{F}, X)$, and hence $L_2(\mathcal{F}_t, X)$ is a Hilbert space.

$L_2^{\mathcal{F}}([0,T],X)$ will denote the Hilbert space of all random processes $\mathcal{F}_t$-progressively measurable defined on $[0,T]$, taking values from X satisfying

$$\mathbf{E}\int_0^T \parallel x(t) \parallel_X^2 dt < \infty.$$

The norm in $L_2(\mathcal{F}_t, X)$ is given by

$$\|x\|_{L_2(\mathcal{F}_t,X)} = (\mathbf{E}\|x\|_X^2)^{1/2},$$

and the norm for space $L_2^{\mathcal{F}}([0,T],U)$ is given by

$$\|u\|_{L_2^{\mathcal{F}}([0,T],U)} = (\sup_{t\in[0,T]} \mathbf{E}\|u\|_U^2)^{1/2}.$$

As for the nonlinear terms f and σ, we assume the followings conditions:

(A) $f : [0,T]\times\Omega\times X\times U \to X$, $(t,\omega,x,u)\to f(t,\omega,x,u)$ is continuous,

(B) $\sigma : [0,T]\times\Omega\times X\times U \to L_2^0, (t,\omega,x,u)\to\sigma(t,\omega,x,u)$ is continuous,

(C) There exist a constant $k>0$ such that

$$\|f(t,x,u_1)-f(t,y,u_2)\|_X^2+\|\sigma(t,x,u_1)-\sigma(t,y,u_2)\|_{L_2^0}^2$$
$$\le k\{\|x(t)-y(t)\|_X^2+\|u_1(t)-u_2(t)\|_U^2\}\ \text{ for all }\ t\in[0,T],\ \ x,y\in X,\ \ u\in U,$$

(D)
$$\|\, f(t,x,u)\,\|_X^2+\|\,\sigma(t,x,u)\,\|_{L_2^0}^2\le k\ \text{ for all }\ t\in[0,T],\ \ x,y\in X,\ \ u\in U.$$

Under these conditions, the system (1.1) admits a mild solution $x\in\mathcal{H}_2$ for any $\xi_0\in X$, and y $u(\cdot)\in L_2^{\mathcal{F}}([0,T],U)$.

The following theorem from nonlinear analysis theory will be used further in this work.

Theorem 2.1. *Let Z be a Banach space and $K : Z\to Z$ a Lipschitz function with a Lipschitz constant $k<1$ and consider $G(z)=z+Kz$. Then, G is a homeomorphism whose inverse is a Lipschitz function with Lipschitz constant $(1-k)^{-1}$.*

3 Controllability of a Linear Stochastic System

Let us start with some results about the exact controllability of the linear stochastic system

$$dx(t)=Ax(t)dt+Bu(t)dt+\Sigma(t)dw(t)$$

that will be used in the next section. To this end, we shall use the fact that

$$\begin{aligned}&dx(t)=Ax(t)dt+Bu(t)dt+\Sigma(t)dw(t),\\&x(0)=\xi_0,\end{aligned}\tag{3.1}$$

admits only one mild solution given by

$$x(t)=S(t)\xi_0+\int_0^t S(t-s)Bu(s)ds+\int_0^t S(t-s)\Sigma(s)dw(s),\tag{3.2}$$

for all $\Sigma\in L_2([0,T],L_2^0)$, $u\in L_2^{\mathcal{F}}([0,T],U)$ and ξ_0 a random variable $\mathcal{F}_0$-measurable.

Definition 3.1. (Exact controllability) The system (3.1) is exactly controllable on $[0,T]$ if for all $\xi_0,\ \xi_1\in L_2(\mathcal{F}_t,X)$ there is a control $u\in L_2^{\mathcal{F}}([0,T],U)$ such that the solution (3.2), corresponding to u, verifies $x(T)=\xi_1$.

Consider the following operators:

a) The controllability operator $G : L_2^{\mathcal{F}}([0,T],U)\to L_2(\mathcal{F}_T,X)$

$$Gu=\int_0^T S(T-s)Bu(s)ds.\tag{3.3}$$

b) The controllability operator of (3.2), $\Pi : L_2(\mathcal{F}_T,X) \to L_2(\mathcal{F}_T,X)$

$$\Pi\{\cdot\} = \int_0^T S(T-s)BB^*S^*(T-s)\mathbf{E}[\cdot \mid \mathcal{F}_t]ds. \tag{3.4}$$

The Operators G, Π are linear and bounded, and the adjoint G^* of G, $G^* : L_2(\mathcal{F}_T,X) \to L_2^{\mathcal{F}}([0,T],U)$ is given by

$$G^*x = B^*S^*(T-t)\mathbf{E}[x|\mathcal{F}_t].$$

We can see that the operator (3.4) is equal to:

$$\Pi = GG^*.$$

The following theorem is found in [16].

Theorem 3.2. *The control system (3.1) is exactly controllable on $[0,T]$ if, and only if, any one of the following condition holds.*

1. $\mathbf{E}\langle \Pi x.x\rangle \geq \gamma \mathbf{E}\|x\|^2$,
2. $R(\lambda,\Pi)$ *converges as* $\lambda \to o^+$ *in uniform topology,*
3. $\lambda R(\lambda,\Pi)$ *converges to the zero operator as* $\lambda \to o^+$ *in uniform topology*

where $R(\lambda,\Pi) = (\lambda I + \Pi)^{-1}$.

Now, we are ready to formulate and prove a new result on exact controllability of the system (3.1).

Theorem 3.3. *The system (3.1) is exactly controllable on $[0,T]$ if, and only if, the operator Π is invertible. Moreover, the control $u \in L_2^{\mathcal{F}}([0,T],U)$ steering an initial state ξ_0 to a final state $\xi_1 = x(T)$, on time $T > 0$, is given by the following formula*

$$u(t) = B^*S^*(T-t)\mathbf{E}\left[\Pi^{-1}\left(\xi_1 - S(T)\xi_0 - \int_0^T S(T-s)\Sigma(s)dw(s)\right)\Big|\mathcal{F}_t\right] \tag{3.5}$$

Proof Suppose the system (3.1) is exactly controllable. Then, from Theorem 3.2, we have that

$$\mathbf{E}\langle \Pi x,x\rangle_X \geq \gamma \mathbf{E}\|x\|_X^2, \quad \text{for some} \quad \gamma > 0 \text{ and all} \quad x \in L_2(\mathcal{F}_T,X). \tag{3.6}$$

This implies that operator Π is one to one. Now, we shall prove that Π is surjective,that is to say

$$\mathrm{Range}(\Pi) = L_2(\mathcal{F}_T,X).$$

For the purpose of contradiction, let us assume that Range(Π) is strictly contained in $L_2(\mathcal{F}_T,X)$. Using the Cauchy-Schwarz inequality and the equation (3.6), we obtain that

$$\gamma \mathbf{E}\|x\|_X^2 \leq \mathbf{E}\langle \Pi x,x\rangle_X \leq (\mathbf{E}\|\Pi x\|_X^2 \mathbf{E}\|x\|_X^2)^{1/2}.$$

So, it follows that $\mathbf{E}\|\Pi x\|_X^2 \geq \gamma^2 \mathbf{E}\|x\|_X^2$, i.e.,

$$\|\Pi x\|_{L_2(\mathcal{F}_T,X)} \geq \gamma \|x\|_{L_2(\mathcal{F}_T,X)}, \ \ x \in L_2(\mathcal{F}_T,X),$$

which implies that Range(Π) is closed. Then from the Hahn Banach's Theorem, there exits $x_0 \in L_2(\mathcal{F}_T, X)$ with $x_0 \neq 0$ such that

$$\langle \Pi x, x_0 \rangle_{L_2(\mathcal{F}_T,X)} = 0, \text{ for all } x \in L_2(\mathcal{F}_T, X).$$

In particular, putting $x = x_0$ we get from the equation (3.6)

$$0 = \langle \Pi x_0, x_0 \rangle_{L_2(\mathcal{F}_T,X)} \geq \gamma \|x_0\|^2_{L_2(\mathcal{F}_T,X)}.$$

Therefore, $x_0 = 0$, which is a contradiction. So, Π is surjective and hence Π is a bijection, and by the open mapping theorem Π^{-1} is a bounded linear operator.

Now, suppose that Π is invertible. Then, given $x \in L_2(\mathcal{F}_T, X)$ we shall prove the existence of a control $u \in L_2^{\mathcal{F}}([0,T], X)$ such that $Gu = x$. Moreover, this control u can be taken as follows

$$u(t) = B^* S^*(T-t)\mathbf{E}[\Pi^{-1}x|\mathcal{F}_t].$$

In fact

$$Gu = \int_0^T S(T-s)Bu(s)ds = \int_0^T S(T-s)BB^*S^*(T-s)\mathbf{E}[\Pi^{-1}x|\mathcal{F}_t]ds = \Pi(\Pi^{-1}(x)) = x.$$

Finally, if we put $x = \xi_1 - S(T)\xi_0 - \int_0^T S(T-s)\Sigma(s)dw(s)$ in the above formula for the control u, we obtain the formula (3.5) for the control steering the initial state ξ_0 to final state ξ_1 on time T.

Corollary 3.4. *If the system (3.1) is exactly controllable, then the operator*

$$\Gamma : L_2(\mathcal{F}_T, X) \to L_2^{\mathcal{F}}([0,T], U)$$

defined by

$$\Gamma\xi = G^*\Pi^{-1}\xi \ \text{ or } \ (\Gamma\xi)(t) = B^*S^*(T-t)\mathbf{E}(\Pi^{-1}\xi \mid \mathcal{F}_t) \tag{3.7}$$

is a right inverse of G, i.e. , $G \circ \Gamma = I$.

4 Controllability of nonlinear Stochastic Systems

Under the conditions imposed on the nonlinear terms f and σ, the equation (1.1) with the initial condition $x(0) = \xi_0$ and control $u \in L_2^{\mathcal{F}}([0,T], U)$ admits only one mild solution given by

$$\begin{aligned} x(t) = S(t)\xi_0 &+ \int_0^t S(t-s)[Bu(s) + f(s, x(s), u(s))]ds \\ &+ \int_0^t S(t-s)\{\Sigma(s) + \sigma(s, x(s), u(s))\}dw(s). \end{aligned} \tag{4.1}$$

Definition 4.1. The system (1.1) is said to be exactly controllable on $[0,T]$ if for all ξ_0 and $\xi_1 \in L_2(\mathcal{F}_T, X)$, there is control $u \in L_2^{\mathcal{F}}([0,T], U)$ such that the corresponding solution (4.1) satisfies $x(T) = \xi_1$.

Define the following operator $G_{f\sigma} : L_2^{\mathcal{F}}([0,T],U) \to L_2(\mathcal{F}_T,X)$ by

$$G_{f\sigma}u = Gu + \int_0^T S(T-r)f(r,x(r),u(r))dr + \int_0^T S(T-r)\sigma(r,x(r),u(r))dw(r), \quad (4.2)$$

where $x(r) = x(r,u)$ is the solution (4.1) corresponding to the control u. The following proposition is a characterization for the controllability of the system (1.1)

Proposition 4.2. *The system (1.1) is exact controllable on* $[0,T]$ *if, and only if,*

$$Range(G_{f\sigma}) = L_2(\mathcal{F}_T,X).$$

Lemma 4.3. *Let* u_1, $u_2 \in L_2^{\mathcal{F}}([0,T],U)$, ξ_0 *is a* $\mathcal{F}_0$*- measurable random variable and* x_1, x_2 *the corresponding solutions of (4.1). Then, the following estimate holds:*

$$\mathbf{E}\|x_1(t) - x_2(t)\|_X^2 \leq \Phi\|u_1 - u_2\|^2, \quad (4.3)$$

where

$$\Phi = \{4M^2\|B\|^2T + 16M^2kT + 16M^2k\}Te^{\{16M^2kT+16M^2k\}T},$$

$t \in [0,T]$ *and* $M = \sup_{0\leq r\leq t\leq T}\|S(t-r)\|$.

Proof Let x_1, x_2 be solutions of (1.1) corresponding to u_1 u_2 respectively. Then, using proposition 4.5 pg 91, from [8], the Cauchy-Schwarz's inequality and the condition (C), we obtain that

$$\begin{aligned}
\mathbf{E}\|x_1(t) - x_2(t)\|_X^2 &= \mathbf{E}\Bigg\|\int_0^t S(t-r)B[u_1(r) - u_2(r)]dr \\
&+ \int_0^t S(t-r)[f(r,x_1(r),u_1(r)) - f(r,x_2(r),u_2(r))]dr \\
&+ \int_0^t S(t-r)[\sigma(r,x_1(r),u_1(r)) - \sigma(r,x_2(r),u_2(r))]dw\Bigg\|_X^2 \\
&\leq \mathbf{E}\Bigg\{\Bigg\|\int_0^t S(t-r)B[u_1(r) - u_2(r)]dr\Bigg\|_X \\
&+ \mathbf{E}\Bigg\|\int_0^t S(t-r)[f(r,x_1(r),u_1(r)) - f(r,x_2(r),u_2(r))]dr\Bigg\|_X \\
&+ \mathbf{E}\Bigg\|\int_0^t S(t-r)[\sigma(r,x_1(r),u_1(r)) - \sigma(r,x_2(r),u_2(r))]dw\Bigg\|_X\Bigg\}^2 \\
&\leq 4M^2\|B\|^2\mathbf{E}\left[\int_0^t \|u_1(r) - u_2(r)\|_U dr\right]^2 \\
&+ 16M^2\mathbf{E}\left[\int_0^t \|f(r,x_1(r),u_1(r)) - f(r,x_2(r),u_2(r))\|_X dr\right]^2 \\
&+ 16M^2\mathbf{E}\int_0^t \|\sigma(r,x_1(r),u_1(r)) - \sigma(r,x_2(r),u_2(r))\|_{L_2^0}^2 dr
\end{aligned}$$

$$\leq 4M^2\|B\|^2T\mathbf{E}\int_0^T \|u_1(r)-u_2(r)\|_U^2dr$$
$$+16M^2kT\mathbf{E}\int_0^T \{\|x_1(r)-x_2(r)\|_X^2+\|u_1(r)-u_2(r)\|_U^2\}dr$$
$$+16M^2k\mathbf{E}\int_0^T \{\|x_1(r)-x_2(r)\|_X^2+\|u_1(r)-u_2(r)\|_U^2\}dr.$$

Finally, we have that

$$\mathbf{E}\|x_1(t)-x_2(t)\|_X^2 \leq \{4M^2\|B\|^2T+16M^2kT+16M^2k\}\int_0^T \mathbf{E}\|u_1(r)-u_2(r)\|_U^2dr$$
$$+\{16M^2kT+16M^2k\}\int_0^T \mathbf{E}\|x_1(r)-x_2(r)\|_X^2dr,$$

that is,

$$\mathbf{E}\|x_1(t)-x_2(t)\|_X^2 \leq \{4M^2\|B\|^2T+16M^2kT+16M^2k\}T\|u_1-u_2\|^2$$
$$+\{16M^2kT+16M^2k\}\int_0^T \mathbf{E}\|x_1(r)-x_2(r)\|_X^2dr.$$

Now, using Gronwall's inequality we get that

$$\mathbf{E}\|x_1(t)-x_2(t)\|_X^2 \leq \{4M^2\|B\|^2T+16M^2kT+16M^2k\}Te^{\{16M^2kT+16M^2k\}T}\|u_1-u_2\|^2.$$

Theorem 4.4. *If the following estimate holds*

$$L=\left[4M^2kT\left\{\Phi T\|\Gamma\|^2+T\|\Gamma\|^2+\Phi+\|\Gamma\|^2\right\}\right]^{1/2}<1, \tag{4.4}$$

where $\Phi=\{4M^2\|B\|^2T+16M^2kT+16M^2k\}Te^{\{16M^2kT+16M^2k\}T}$, *then the nonlinear system (1.1) is exactly controllable on* $[0,T]$.

Proof To this end, it is enough to prove that the operator $G_{f\sigma}$ given by (4.2) is onto. That is to say,
$$G_{f\sigma}(L_2^{\mathcal{F}}([0,T],U))=Range(G_{f\sigma})=L_2(\mathcal{F}_T,X).$$
Since the linear stochastic system (1.2) is exactly controllable, by corollary 3.4, it follows that the operator Γ defined by (3.7) is a right inverse of G. It is enough to prove that the operator $\tilde{G}_{f\sigma}=G_{f\sigma}\circ\Gamma$ is surjective.

To this end, the operator $G_{f\sigma}$ can be written as follows

$$G_{f\sigma}u=Gu+F(u),$$

where $F:L_2^{\mathcal{F}}([0,T],U)\to L_2(\mathcal{F},X)$ is given by

$$F(u)=\int_0^T S(T-r)f(r,x_\xi(r),(\Gamma\xi)(r))dr+\int_0^T S(T-r)\sigma(r,x_\xi(r),(\Gamma\xi)(r))dw(r). \tag{4.5}$$

Then,
$$\tilde{G}_{f\sigma}=G_{f\sigma}\circ\Gamma=I+F\circ\Gamma=I+K, \tag{4.6}$$

where $K : L_2(\mathcal{F}_T, X) \to L_2(\mathcal{F}_T, X)$ is given by

$$K(\xi) = \int_0^T S(T-r) f(r, x_\xi(r), (\Gamma\xi)(r))dr + \int_0^T S(T-r)\sigma(r, x_\xi(r), (\Gamma\xi)(r))dw(r). \quad (4.7)$$

Hence, from Theorem 2.1 it is enough to prove that K is a Lipschitz function with a Lipschitz constant $k < 1$. In fact, let x_{ξ_1} and x_{ξ_2} be solutions of (1.1) corresponding to the control $\Gamma\xi_1$, $\Gamma\xi_2$ respectively. Consider the following estimate

$$\begin{aligned} K\xi_2 - K\xi_1 &= \int_0^T S(T-r) f(r, x_{\xi_2}(r), \Gamma\xi_2(r))dr + \int_0^T S(T-r)\sigma(r, x_{\xi_2}(r), \Gamma\xi_2(r))dw(r) \\ &- \Big(\int_0^T S(T-r) f(r, x_{\xi_1}(r), \Gamma\xi_1(r))dr + \int_0^T S(T-r)\sigma(r, x_{\xi_1}(r), \Gamma\xi_1(r))dw \Big). \end{aligned} \quad (4.8)$$

In the same way as before we obtain that

$$\begin{aligned} \mathbf{E}\Big\| K\xi_2 - K\xi_1 \Big\|_X^2 = \mathbf{E}\Big\| &\int_0^T S(T-r)\Big[f(r, x_{\xi_2}(r), \Gamma\xi_2) - f(r, x_{\xi_1}(r), \Gamma\xi_1)\Big] dr \Big\|_X^2 \\ &+ \int_0^T S(T-r)\Big[\sigma(r, x_{\xi_2}(r), \Gamma\xi_2) - \sigma(r, x_{\xi_1}(r), \Gamma\xi_1)\Big] dw \Big\|_X^2. \end{aligned}$$

Therefore, applying the formula $(a+b)^2 \le 4(a^2 + b^2)$ with $a > 0$ and $b > 0$ we obtain

$$\begin{aligned} \mathbf{E}\Big\| K\xi_2 - K\xi_1 \Big\|_X^2 &\le 4\mathbf{E}\Big\| \int_0^T S(T-r)\Big[f(r, x_{\xi_2}(r), \Gamma\xi_2) - f(r, x_{\xi_1}(r), \Gamma\xi_1)\Big] dr \Big\|_X^2 \\ &+ 4\mathbf{E}\Big\| \int_0^T S(T-r)\Big[\sigma(r, x_{\xi_2}(r), \Gamma\xi_2) - \sigma(r, x_{\xi_1}(r), \Gamma\xi_1)\Big] dw \Big\|_X^2. \end{aligned}$$

Let us now consider the following quantities

$$I_1 = \mathbf{E}\Big\| \int_0^T S(T-r)\Big[f(r, x_{\xi_2}(r), \Gamma\xi_2) - f(r, x_{\xi_1}(r), \Gamma\xi_1)\Big] dr \Big\|_X^2,$$

$$I_2 = \mathbf{E}\Big\| \int_0^T S(T-r)\Big[\sigma(r, x_{\xi_2}(r), \Gamma\xi_2) - \sigma(r, x_{\xi_1}(r), \Gamma\xi_1)\Big] dw(r) \Big\|_X^2.$$

Applying Cauchy-Schwarz's inequality, the condition (C) and the equation (4.3) we get that

$$\begin{aligned}
I_1 &\le \mathbf{E}\left(\int_0^T \left\|S(T-r)\right\|_X \left\|\left[f(r,x_{\xi_2}(r),\Gamma\xi_2)-f(r,x_{\xi_1}(r),\Gamma\xi_1)\right]\right\|_X dr\right)^2 \\
&\le M^2\mathbf{E}\left(\int_0^T \left\|f(r,x_{\xi_2}(r),\Gamma\xi_2)-f(r,x_{\xi_1}(r),\Gamma\xi_1)\right\|_X\right)^2 \\
&\le M^2\mathbf{E}\left(\left[\int_0^T dr\right]^{1/2}\left[\int_0^T \left\|f(r,x_{\xi_2}(r),\Gamma\xi_2)-f(r,x_{\xi_1}(r),\Gamma\xi_1)\right\|_X^2 dr\right]^{1/2}\right)^2 \\
&\le M^2T\mathbf{E}\left(\int_0^T \left\|f(r,x_{\xi_2}(r),\Gamma\xi_2)-f(r,x_{\xi_1}(r),\Gamma\xi_1)\right\|_X^2 dr\right) \\
&\le M^2T\mathbf{E}\left(\int_0^T k\left\|x_{\xi_2}(r)-x_{\xi_1}(r)\right\|_X^2 + k\left\|\Gamma\xi_2(r)-\Gamma\xi_1(r)\right\|_U^2 dr\right) \\
&\le M^2kT\int_0^T \mathbf{E}\left\|x_{\xi_2}(r)-x_{\xi_1}(r)\right\|_X^2 dr + M^2kT\int_0^T \mathbf{E}\left\|\Gamma\xi_2(r)-\Gamma\xi_1(r)\right\|_U^2 dr \\
&\le M^2k\Phi T^2\|\Gamma\xi_2-\Gamma\xi_1\|^2 + M^2kT^2\left\|\Gamma\xi_2-\Gamma\xi_1\right\|^2 \\
&= \{M^2k\Phi T^2 + M^2kT^2\}\|\Gamma\|^2\|\xi_2-\xi_1\|^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \mathbf{E}\left\|\int_0^T S(T-r)\left[\sigma(r,x_{\xi_2}(r),\Gamma\xi_2)-\sigma(r,x_{\xi_1}(r),\Gamma\xi_1)\right]dw(r)\right\|_X^2 \\
&= \mathbf{E}\left\|\int_0^T S(T-r)\left[\sigma(r,x_{\xi_2}(r),\Gamma\xi_2)-\sigma(r,x_{\xi_1}(r),\Gamma\xi_1)\right]dw(r)\right\|_X^2 \\
&\le M^2\mathbf{E}\left\|\int_0^T \left[\sigma(r,x_{\xi_2}(r),\Gamma\xi_2)-\sigma(r,x_{\xi_1}(r),\Gamma\xi_1)\right]dw(r)\right\|_X^2 \\
&\le M^2\mathbf{E}\int_0^T \left\|\sigma(r,x_{\xi_2}(r),\Gamma\xi_2)-\sigma(r,x_{\xi_1}(r),\Gamma\xi_1)\right\|_{L_2^0}^2 dr \\
&= M^2\mathbf{E}\int_0^T \left\|\sigma(r,x_{\xi_2}(r),\Gamma\xi_2)-\sigma(r,x_{\xi_1}(r),\Gamma\xi_1)\right\|_{L_2^0}^2 dr \\
&\le M^2\mathbf{E}\left\{\int_0^T k\parallel x_{\xi_2}(r)-x_{\xi_1}(r)\parallel_X^2 + k\parallel\Gamma\xi_2(r)-\Gamma\xi_1(r)\parallel_U^2 dr\right\}
\end{aligned}$$

$$\begin{aligned}
&= M^2k\int_0^T \mathbf{E}\|x_{\xi_2}(r)-x_{\xi_1}(r)\|_X^2 dr + M^2k\int_0^T \mathbf{E}\parallel\Gamma\xi_2(r)-\Gamma\xi_1(r)\parallel_U^2 dr \\
&\le M^2kT\Phi\|\Gamma\xi_2-\Gamma\xi_1\|^2 + M^2kT\|\Gamma\xi_2-\Gamma\xi_1\|^2 \\
&= \left\{M^2kT\Phi + M^2kT\right\}\|\Gamma\|^2\|\xi_2-\xi_1\|^2.
\end{aligned}$$

So, we obtain that

$$\mathbf{E}\parallel K\xi_2-K\xi_1\parallel_X^2 \le 4M^2kT\left\{\Phi T+T+\Phi+1\right\}\|\Gamma\|^2\|\xi_2-\xi_1\|^2,$$

i.e.

$$\|K\xi_2-K\xi_1\| \le \left[4M^2kT\left\{\Phi T+T+\Phi+1\right\}\right]^{1/2}\|\Gamma\|^2\|\xi_2-\xi_1\|.$$

Therefore, K is a Lipschitz's function with a Lipschitz's constant given by

$$k = \left[4M^2kT\left\{\Phi T + T + \Phi + 1\right\}\right]^{1/2}\|\Gamma\|^2 < 1.$$

Then, from Theorem 2.1 we obtain that $\tilde{G}_{f\sigma} = I + K$ is a homeomorphism and therefore the operator $G_{f\sigma}$ is surjective. i.e.,

$$G_{f\sigma}(L_2^{\mathcal{F}}([0,T],U)) = Range(G_{f\sigma}) = L_2(\mathcal{F}_T, X).$$

Corollary 4.5. *Under the assumptions of Theorem 4.4 the control steering an initial state ξ_0 to a final state ξ_1 is given by*

$$u(t) = B^*T^*(T-t)\mathbf{E}\left[\Pi^{-1}(\xi_1 - S(T)\xi_0 - \int_0^T S(T-s)\Sigma(s)dw(s))\middle|\mathcal{F}_t\right](I+K)^{-1}. \quad (4.9)$$

5 Application to the damped Stochastic wave equation

Consider the damped wave equation

$$\begin{cases} d[z_t + cz] = [\theta z_{yy} + u(t,y) + f(t,z,z_t,u(t,y))]dt + dw(t), \\ z(t,0) = z(t,1) = 0, \;\; 0 < x < 1 \;\; t \in \mathbb{R} \end{cases} \quad (5.1)$$

where $\theta > 0$, $c \geq 0$, the control $u \in L_2^{\mathcal{F}}([0,t_1], L_2(0,1))$, the nonlinear term $f(t,z,z_t,u(t,y))$ is a Lipschitz function, i.e., there is a constant $k > 0$ such that, for all (t,z_1,v_1,u_1) and $(t,z_2,v_2,u_2) \in [0,t_1] \times \mathbb{R}^3$ we have

$$|f(t,z_2,v_2,u_2) - f(t,z_1,v_1,u_1)|^2 \leq k\{|z_2 - z_1|^2 + |v_2 - v_1|^2 + |u_2 - u_1|^2\}. \quad (5.2)$$

Now we choose the space in which this system will be set as an abstract second order ordinary differential equation.

Let $Z = U = L_2[0,1]$ and consider the unbounded operator $A : D(A) \subset Z \to Z$ defined by $Az = -z_{yy}$, where

$$D(A) = \{z \in Z : z, z_y \text{ are absolutely continuous } z_{yy} \in Z; z(0) = z(1) = 0\} \quad (5.3)$$

Operator Properties:

i) The spectrum of A consists of only eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \to \infty$, each one with multiplicity one.

ii) There is a complete orthonormal set $\{\phi_n\}$ of eigenvectors of A.

iii) For all $z \in D(A)$

$$Az = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n z. \quad (5.4)$$

Here $\langle \cdot, \cdot \rangle$ is the inner product on Z, $\lambda_n = n^2\pi^2$ and $\phi_n(z) = \sqrt{2}\sin(n\pi z)$.

Thus, $\{E_n\}$ is a family of orthogonal projections complete in Z and $z = \sum_{n=1}^{\infty} E_n z$, $z \in Z$.

iv) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}z = \sum_{n=1}^{\infty} e^{-\lambda t} E_n z. \tag{5.5}$$

v) The fractional powered spaces Z^r are given by

$$Z^r = D(A^r) = \left\{ z \in Z : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n z\|^2 < \infty \right\},\ r \geq 0, \tag{5.6}$$

with the norm

$$\|z\|_r = \|A^r z\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n z\|^2 \right\}^{1/2},\ z \in Z,$$

and

$$A^r z = \sum_{n=1}^{\infty} \lambda_n^r E_n z. \tag{5.7}$$

Also, for $r \geq 0$ we define $X_r = Z^r \times Z$, which is a Hilbert space with the norm given by

$$\left\| \begin{bmatrix} z \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|z\|_r^2 + \|v\|^2.$$

Using the change of variable $z' = v$, the second-order equation (5.1) can be written as a first order system of ordinary differential equation in the Hilbert space $X_{1/2} = D(A^{1/2}) \times Z = Z^{1/2} \times Z$ as

$$d\eta = [\mathcal{A}\eta + Bu + F(t, \eta, u(t))]dt + \Sigma(t, \eta, u(t))dw,\ \eta \in X_{1/2},\ t \geq 0, \tag{5.8}$$

where

$$\eta = \begin{bmatrix} z \\ v \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I_Z \end{bmatrix},\ \mathcal{A} = \begin{bmatrix} 0 & I_Z \\ -lA & -cI_Z \end{bmatrix} \tag{5.9}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times Z$ and

$$F = \begin{bmatrix} 0 \\ f(t, w, v, u) \end{bmatrix},\ \Sigma = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{5.10}$$

The function F is defined as $F : [0, t_1] \times X_{1/2} \times U \to X_{1/2}$, and the function Σ is defined as follows $\Sigma : [0, t_1] \times X_{1/2} \times U \to L_2^0$, where $L_2^0 := L_2(Q^{1/2} X_{1/2}, X_{1/2})$

Since $Z^{1/2}$ is continuously included in Z we have for $x_1,\, x_2 \in X_{1/2}$ and $u_1,\ u_2 \in U = Z$

$$\|F(t, x_2, u_2) - F(t, x_1, u_1)\|_{X_{1/2}}^2 \leq K\|x_2 - x_1\|_{X_{1/2}}^2 + \|u_1 - u_2\|_{L_2^{\mathcal{F}}([0,t_1],U)}. \tag{5.11}$$

The following proposition is in [13] and [14]

Proposition 5.1. *The operator $\mathcal{A}$ given by (5.9), is the infinitesimal generator of strongly continuous semigroup $\{S(t)\}_{t\in\mathbb{R}}$ in $X_{1/2}$ given by*

$$S(t)x = \sum_{n=1}^{\infty} e^{A_n t} P_n x, \quad x \in X_{1/2},$$

where $\{P_n\}_{n\geq 0}$ is a family of orthogonal projections on the Hilbert space $X_{1/2}$ and is given by

$$P_n = diag(E_n, E_n), \quad n \geq 1,$$

and

$$A_n = B_n P_n, \quad B_n = \begin{pmatrix} 0 & 1 \\ -\theta\lambda_n & -c \end{pmatrix}, n \geq 1$$

This semigroup decays exponentially to zero. In fact, we have the following estimate

$$\|S(t)\| \leq M(c,\theta)e^{-\frac{c}{2}t}, \quad t \geq 0.$$

where

$$\frac{M(c,\theta)}{2\sqrt{2}} = \sup_n \left\{ 2\left|\frac{c \pm \sqrt{4\theta\lambda_n - c^2}}{\sqrt{c^2 - 4\theta\lambda_n}}\right|, (2+\theta)\left|\frac{\sqrt{\lambda_n}}{\sqrt{4\theta\lambda_n - c^2}}\right| \right\}.$$

Proof See [13] and [14].

Theorem 5.2. *The stochastic linear system*

$$\begin{cases} d\eta(t) = [\mathcal{A}\eta(t) + Bu]dt + \Sigma dw(t), \quad \eta \in X_{1/2}, \ t > 0, \\ \eta(0) = \eta_0. \end{cases} \tag{5.12}$$

is exactly controllable on $[0,T]$.

Proof From Leiva [13], the linear deterministic system associated to (5.12) is exactly controllable, and from Theorem 3.2, in Mahmudov [16], we get the controllability of (5.12).

Theorem 5.3. *If the following estimate holds*

$$(4(M(c,\theta))^2 kT(T+1))^{1/2} \exp\frac{1}{2}\{4(M(c,\theta))^2 k(T+1)\} < 1,$$

then the system (5.8) is exactly controllable on $[0,T]$.

Proof It follows from Theorem 4.4 one we note that in this case $\|B\| \leq 1$.

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