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Novel Exponential Estimate for Nonlinear Systems with Mixed Interval Time-Varying Nondifferentiable Delays

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Abstract

This paper addresses exponential stability problem for a class of nonlinear systems with mixed time-varying delays. The time delays are non-differentiable functions belonging to a given interval, in which the lower bound of delay is not restricted to zero. By constructing a suitable augmented Lyapunovs functional, new criteria for the exponential stability of the system are established in terms of linear matrix inequalities. The result has been applied to robust stability problem of uncertain systems with interval time-varying delays. Numerical examples are given to show the effectiveness of the conditions.

AMS Subject Classification: 62G05; 62G20.

Keywords: Exponential stability, robust stability, interval delay, Lyapunov function, linear matrix inequalities.

1 Introduction

Functional differential equations are frequently encountered in various areas,including physical and chemical processes, biological and economical systems, mechanics as well as engineering sciences (e.g. [5, 7, 14] and the references therein). Since time delay effects are often a source of instability and poor performance of the systems, the problem of stability analysis of time delay systems has been become a popular and challenging research and has attracted a lot of attention in the past decades [4, 6, 8-10]. In delay-dependent stability and stabilization criteria, the main concern is to enlarge the feasible region of criteria for guaranteeing asymptotic stability of dynamical systems in a given time-delay interval. On the other hand, the stability analysis of dynamic systems with interval time-varying delays has been a focused topic of theoretical and practical importance [3, 12] in very recent years. Interval time-varying delay means that a time delay varies in an interval in which the lower bound is not restricted to be zero. However, few results have been investigated to the problem of the stability for nonlinear systems with interval time-varying delays. In [10, 11],

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some stability conditions for the time-varying delay system with nonlinear perturbation, which is a bounded differentiable function and the upper bound of the derivative was strict less than one. Recently, some improved stability conditions for the nonlinear system with interval time-varying delays were obtained in [13], where the assumption on the derivative strictly bounded by one is removed, but the time-delay function is still assumed to be differentiable.

In this paper, we consider exponential stability problem for a class of nonlinear systems with mixed interval time-varying delay. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the exponential stability of the system. Compared to the existing results, our result has its own advantages. First, the time-delay system is subjected to nonlinear uncertainties and mixed time-varying delays. Second, the time delay is assumed to be any continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not necessary to be differentiable. This allows the time-delay to be a fast time-varying function and the lower bound is not restricted to being zero. Third, our approach allows us to apply in robust stability of uncertain systems with interval time-varying delays. Therefore, our results are more general than the related previous results.

The paper is organized as follows. Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Delay-dependent exponential stability conditions and applications to robust stability are presented in Section 3. Numerical examples showing the feasibility and effectiveness of the conditions are given in Section 4. The paper ends with conclusions and cited references.

2 Preliminaries

The following notations will be used in this paper. $\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the $n-$dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\| \, . \, \|$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)-$dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda; \lambda \in \lambda(A)\}$; $x_t := \{x(t+s) : s \in [-h,0]\}, \| x_t \| = \sup_{s\in[-h,0]} \| x(t+s) \|$; $C^1([0,t],\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n-$valued continuously differentiable functions on $[0,t]$; $L_2([0,t],\mathbb{R}^m)$ denotes the set of all the $\mathbb{R}^m-$valued square integrable functions on $[0,t]$,

Matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax,x\rangle \geq 0$, for all $x \in \mathbb{R}^n$;A is positive definite $(A > 0)$ if $\langle Ax,x\rangle > 0$ for all $x \neq 0$;$A > B$ means $A - B > 0$. The notation $\mathrm{diag}\{\ldots\}$ stands for a block-diagonal matrix. The symmetric term in a matrix is denoted by $*$.

Consider a nonlinear system with mixed time-varying delays of the form

$$\begin{aligned}
&\dot{x}(t) = A_0x(t) + A_1x(t-h(t)) + A_2\int_{t-k(t)}^{t} x(s)\,ds + f(t,x(t),x(t-h(t)),\int_{t-k(t)}^{t} x(s)\,ds)\\
&x(t) = \phi(t), t \in [-d,0], \quad d = \max\{h_2,k_2\},
\end{aligned} \tag{2.1}$$

where the time-varying delay functions $h(t), k(t)$ satisfying

$$0 \le h_1 \le h(t) \le h_2, \quad \forall t \in \mathbb{R}^+,$$

$$0 \le k_1 \le k(t) \le k_2, \quad \forall t \in \mathbb{R}^+,$$

and $x(t) \in \mathbb{R}^n$ is the state; A_0, A_1, A_2 are given matrices of appropriate dimensions and $\phi(t) \in C^1([-d,0],\mathbb{R}^n)$ with the norm $\|\phi\| = \sup_{-d\le t\le 0}\sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}$. As in papers [8, 9, 10, 12, 13], we assume that the nonlinear function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|f(t,x,y,z)\|^2 \le a_0^2\|x\|^2 + a_1^2\|y\|^2 + a_2^2\|z\|^2, \quad \forall (x,y,z), \quad \forall t \in \mathbb{R}^+, \tag{2.2}$$

where a_0, a_1, a_2 are given nonnegative constants.

Remark 2.1. In contrast to most of previous results [8, 11-13], the time delay is a non-differentiable function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. Given $\alpha > 0$. The system (2.1) is $\alpha-$exponentially stable if there exist a positive number $\beta > 0$ such that every solution $x(t,\phi)$ satisfies the following condition:

$$\|x(t,\phi)\| \le \beta e^{-\alpha t}\|\phi\|, \quad \forall t \in \mathbb{R}^+.$$

We introduce the following technical well-known proposition, which will be used in the proof of the main result.

Proposition 2.1. [2] *For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds*

$$\left(\int_0^\gamma \omega(s)\,ds\right)^T M \left(\int_0^\gamma \omega(s)\,ds\right) \le \gamma\left(\int_0^\gamma \omega^T(s) M \omega(s)\,ds\right)$$

3 Main result

Let us denote $\lambda_1 = \lambda_{\min}(P)$, and

$$\begin{aligned}\lambda_2 = {} & \lambda_{\max}(P) + h_1\lambda_{\max}(Q) + \frac{1}{2}(h_2-h_1)^2(h_2+h_1)\lambda_{\max}(S) + \frac{1}{2}h_2^3\lambda_{\max}(R) \\ & + \frac{k_1^3}{2}\lambda_{\max}(T) + \frac{1}{2}(k_2-k_1)^2(k_2+k_1)\lambda_{\max}(U).\end{aligned}$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *Given positive number α. System (2.1) is $\alpha-$exponentially stable if there exist matrices $N_i, i = 1,\dots,7$, symmetric positive definite matrices P, Q, R, S, T, U and a pos-*

itive number ε *such that the following LMI holds*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & A_0^T N_3 & -N_1^T + A_0^T N_4 & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \Xi_{22} & e^{-2\alpha h_2} S + A_1^T N_3 & A_1^T N_4 - N_2^T & N_2^T A_2 + A_1^T N_5 & \Xi_{26} & \Xi_{27} \\ * & * & \Xi_{33} & -N_3^T & N_3^T A_2 & N_3^T A_2 & N_3^T \\ * & * & * & \Xi_{44} & N_4^T A_2 - N_5 & N_4^T A_2 - N_6 & N_4^T - N_7 \\ * & * & * & * & \Xi_{55} & \Xi_{56} & \Xi_{57} \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix} < 0, \tag{3.1}$$

where

$$\begin{aligned}
\Xi_{11} &= P(A_0 + \alpha I) + (A_0 + \alpha I)^T P + \varepsilon a_0^2 I + Q + k_1^2 T + (k_2 - k_1)^2 U - e^{-2\alpha h_2} R \\
&\quad + N_1^T A_0 + A_0^T N_1, \\
\Xi_{12} &= PA_1 + N_1^T A_1 + A_0^T N_2 + e^{-2\alpha h_2} R, \\
\Xi_{15} &= (N_1^T + P)A_2 + A_0^T N_5, \\
\Xi_{16} &= (N_1^T + P)A_2 + A_0^T N_6, \\
\Xi_{17} &= N_1^T + P + A_0^T N_7, \\
\Xi_{22} &= \varepsilon a_1^2 I - e^{-2\alpha h_2} R - e^{-2\alpha h_2} S + N_2^T A_1 + A_1^T N_2, \\
\Xi_{26} &= N_2^T A_2 + A_1^T N_6, \\
\Xi_{27} &= N_2^T + A_1^T N_7, \\
\Xi_{33} &= -e^{-2\alpha h_2} S - e^{-2\alpha h_1} Q, \\
\Xi_{44} &= h_2^2 R + (h_2 - h_1)^2 S - N_4 - N_4^T, \\
\Xi_{55} &= 2\varepsilon a_2^2 I - e^{-2\alpha k_2} U + N_5^T A_2 + A_2^T N_5, \\
\Xi_{56} &= N_5^T A_2 + A_2^T N_6, \\
\Xi_{57} &= N_5^T + A_2^T N_7, \\
\Xi_{66} &= 2\varepsilon a_2^2 I - e^{-2\alpha k_1} T + N_6^T A_2 + A_2^T N_6, \\
\Xi_{67} &= N_6^T + A_2^T N_7, \\
\Xi_{77} &= -\varepsilon I + N_7 + N_7^T.
\end{aligned}$$

Moreover, the solution $x(t,\phi)$ *of the system satisfies*

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\| e^{-\alpha t}, \quad \forall t \ge 0.$$

Proof. We consider the following Lyapunov-Krasovskii functional for the system (2.1)

$$V(t,x_t) = \sum_{i=1}^{6} V_i(t,x_t),$$

$$
\begin{aligned}
V_1 &= x^T(t)Px(t),\\
V_2 &= \int_{t-h_1}^{t} e^{2\alpha(s-t)} x^T(s)Qx(s)\,ds,\\
V_3 &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)R\dot{x}(\tau)\,d\tau\,ds,\\
V_4 &= (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)S\dot{x}(\tau)\,d\tau\,ds,\\
V_5 &= k_1\int_{-k_1}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}x^T(\tau)Tx(\tau)\,d\tau\,ds,\\
V_6 &= (k_2-k_1)\int_{-k_2}^{-k_1}\int_{t+s}^{t} e^{2\alpha(\tau-t)}x^T(\tau)Ux(\tau)\,d\tau\,ds.
\end{aligned}
$$

It easy to check that

$$
\lambda_1 \parallel x(t) \parallel^2 \le V(t,x_t) \le \lambda_2 \parallel x_t \parallel^2, \quad \forall t \ge 0. \tag{3.2}
$$

Taking the derivative of V_1 along the solution of system (2.1) we have

$$
\begin{aligned}
\dot{V}_1 &= 2x^T(t)P\dot{x}(t)\\
&= 2x^T(t)P[A_0x(t)+A_1x(t-h(t))+A_2\int_{t-k(t)}^{t}x(s)\,ds+f(t)]\\
&= x^T(t)[PA_0+A_0^TP]x(t)+2x^T(t)PA_1x(t-h(t))+2x^T(t)PA_2\int_{t-k(t)}^{t-k_1}x(s)\,ds\\
&\quad +2x^T(t)PA_2\int_{t-k_1}^{t}x(s)\,ds+2x^T(t)Pf(t),
\end{aligned}
$$

where, for convenient, we denote $f(t) := f(t,x(t),x(t-h(t)),\int_{t-k(t)}^{t}x(s)\,ds)$.

From (2.2) we have

$$
\begin{aligned}
\|f(t)\|^2 &\le a_0^2\|x(t)\|^2+a_1^2\|x(t-h(t))\|^2+a_2^2\|\int_{t-k(t)}^{t}x(s)\,ds\|^2\\
&= a_0^2\|x(t)\|^2+a_1^2\|x(t-h(t))\|^2+a_2^2\|\int_{t-k(t)}^{t-k_1}x(s)\,ds+\int_{t-k_1}^{t}x(s)\,ds\|^2\\
&\le a_0^2\|x(t)\|^2+a_1^2\|x(t-h(t))\|^2+2a_2^2\|\int_{t-k(t)}^{t-k_1}x(s)\,ds\|^2+2a_2^2\|\int_{t-k_1}^{t}x(s)\,ds\|^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\varepsilon\Bigg[a_0^2x^T(t)x(t)+a_1^2x^T(t-h(t))x(t-h(t))+2a_2^2\left(\int_{t-k(t)}^{t-k_1}x(s)\,ds\right)^T\left(\int_{t-k(t)}^{t-k_1}x(s)\,ds\right)\\
&+2a_2^2\left(\int_{t-k_1}^{t}x(s)\,ds\right)^T\left(\int_{t-k_1}^{t}x(s)\,ds\right)-f^T(t)f(t)\Bigg]\ge 0,
\end{aligned}
$$

we have

$$\begin{aligned}\dot V_1 \le\ & x^T(t)[P(A_0+\alpha I)+(A_0+\alpha I)^T P+\varepsilon a_0^2 I]x(t)\\ &+2x^T(t)PA_1x(t-h(t))+2x^T(t)PA_2\int_{t-k(t)}^{t-k_1}x(s)\,ds+2x^T(t)PA_2\int_{t-k_1}^{t}x(s)\,ds\\ &+2x^T(t)Pf(t)+\varepsilon a_1^2x^T(t-h(t))x(t-h(t))+2\varepsilon a_2^2\left(\int_{t-k(t)}^{t-k_1}x(s)\,ds\right)^T\left(\int_{t-k(t)}^{t-k_1}x(s)\,ds\right)\\ &+2\varepsilon a_2^2\left(\int_{t-k_1}^{t}x(s)\,ds\right)^T\left(\int_{t-k_1}^{t}x(s)\,ds\right)-\varepsilon f^T(t)f(t)-2\alpha V_1.\end{aligned} \tag{3.3}$$

Next, the derivatives of $V_i(t,x_t), i=2,\dots,6$ give

$$\begin{aligned}&\dot V_2 = x^T(t)Qx(t)-e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1)-2\alpha V_2;\\ &\dot V_3 \le h_2^2\dot x^T(t)R\dot x(t)-h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot x^T(s)R\dot x(s)\,ds-2\alpha V_3;\\ &\dot V_4 \le (h_2-h_1)^2\dot x^T(t)S\dot x(t)-(h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\dot x^T(s)S\dot x(s)\,ds-2\alpha V_4;\\ &\dot V_5 \le k_1^2x^T(t)Tx(t)-k_1e^{-2\alpha k_1}\int_{t-k_1}^{t}x^T(s)Tx(s)\,ds-2\alpha V_5;\\ &\dot V_6 \le (k_2-k_1)^2x^T(t)Ux(t)-(k_2-k_1)e^{-2\alpha k_2}\int_{t-k_2}^{t-k_1}x^T(s)Ux(s)\,ds-2\alpha V_6.\end{aligned} \tag{3.4}$$

Applying Proposition 2.1 and the Leibniz - Newton formula, we have

$$\begin{aligned}-h_2\int_{t-h_2}^{t}\dot x^T(s)R\dot x(s)\,ds &\le -h(t)\int_{t-h(t)}^{t}\dot x^T(s)R\dot x(s)\,ds\\ &\le -\left[\int_{t-h(t)}^{t}\dot x(s)\,ds\right]^T R\left[\int_{t-h(t)}^{t}\dot x(s)\,ds\right]\\ &= -[x(t)-x(t-h(t))]^T R[x(t)-x(t-h(t))]\\ &= -x^T(t)Rx(t)+2x^T(t)Rx(t-h(t))-x^T(t-h(t))Rx(t-h(t))\\ &\quad -(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot x^T(s)S\dot x(s)\,ds\\ &\le -(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot x^T(s)S\dot x(s)\,ds\\ &\le -\left[\int_{t-h(t)}^{t-h_1}\dot x(s)\,ds\right]^T S\left[\int_{t-h(t)}^{t-h_1}\dot x(s)\,ds\right]\\ &= -[x(t-h_1)-x(t-h(t))]^T S[x(t-h_1)-x(t-h(t))]\\ &= -x^T(t-h_1)Sx(t-h_1)+2x^T(t-h_1)Sx(t-h(t))\\ &\quad -x^T(t-h(t))Sx(t-h(t)).\end{aligned} \tag{3.5}$$

Using Proposition (2.1), we have

$$
\begin{aligned}
-k_1\int_{t-k_1}^{t} x^T(s)Tx(s)\,ds &\le -\left(\int_{t-k_1}^{t} x(s)\,ds\right)^T T\left(\int_{t-k_1}^{t} x(s)\,ds\right)\\
-(k_2-k_1)\int_{t-k_2}^{t-k_1} x^T(s)Ux(s)\,ds &\le -(k(t)-k_1)\int_{t-k(t)}^{t-k_1} x^T(s)Ux(s)\,ds \qquad (3.6)\\
&\le -\left(\int_{t-k(t)}^{t-k_1} x(s)\,ds\right)^T U\left(\int_{t-k(t)}^{t-k_1} x(s)\,ds\right).
\end{aligned}
$$

By using the following identity relation

$$
-\dot{x}(t)+A_0x(t)+A_1x(t-h(t))+A_2\int_{t-k(t)}^{t-k_1} x(s)\,ds+A_2\int_{t-k_1}^{t} x(s)\,ds+f(t)=0,
$$

we obtain

$$
\begin{aligned}
\theta(t) =& 2\Bigg[x^T(t)N_1^T+x^T(t-h(t))N_2^T+x^T(t-h_1)N_3^T+\dot{x}^T(t)N_4^T+\left(\int_{t-k(t)}^{t-k_1} x(s)\,ds\right)^T N_5^T\\
&+\left(\int_{t-k_1}^{t} x(s)\,ds\right)^T N_6^T+f^T(t)N_7^T\Bigg]\times\Bigg[-\dot{x}(t)+A_0x(t)+A_1x(t-h(t))\\
&+A_2\int_{t-k(t)}^{t-k_1} x(s)\,ds+A_2\int_{t-k_1}^{t} x(s)\,ds+f(t)\Bigg]=0.
\end{aligned}
\qquad (3.7)
$$

Therefore, from $(3.3)-(3.7)$, we obtain

$$
\begin{aligned}
&\dot{V}(t,x_t)+2\alpha V(t,x_t)\\
&\le x^T(t)[P(A_0+\alpha I)+(A_0+\alpha I)^TP+\varepsilon a_0^2 I+Q+k_1^2T+(k_2-k_1)^2U-e^{-2\alpha h_2}R]x(t)\\
&\quad+\dot{x}^T(t)[h_2^2R+(h_2-h_1)^2S]\dot{x}(t)-\varepsilon f^T(t)f(t)\\
&\quad+x^T(t-h(t))[\varepsilon a_1^2 I-e^{-2\alpha h_2}R-e^{-2\alpha h_2}S]x(t-h(t))\\
&+x^T(t-h_1)[-e^{-2\alpha h_2}S-e^{-2\alpha h_1}Q]x(t-h_1)\\
&\quad+\left(\int_{t-k(t)}^{t-k_1} x(s)\,ds\right)^T[2\varepsilon a_2^2 I-e^{-2\alpha k_2}U]\left(\int_{t-k(t)}^{t-k_1} x(s)\,ds\right)\\
&\quad+\left(\int_{t-k_1}^{t} x(s)\,ds\right)^T[2\varepsilon a_2^2 I-e^{-2\alpha k_1}T]\left(\int_{t-k_1}^{t} x(s)\,ds\right)\\
&\quad+2x^T(t)Rx(t-h(t))+2x^T(t-h_1)Sx(t-h(t))+2x^T(t)PA_1x(t-h(t))\\
&\quad+2x^T(t)PA_2\int_{t-k(t)}^{t-k_1} x(s)\,ds+2x^T(t)PA_2\int_{t-k_1}^{t} x(s)\,ds+2x^T(t)Pf(t)+\theta(t)\\
&\quad=\zeta^T(t)\Xi\zeta(t),
\end{aligned}
$$

where $\zeta^T(t)=[x^T(t),x^T(t-h(t)),x^T(t-h_1),\dot{x}^T(t),\left(\int_{t-k(t)}^{t-k_1} x(s)\,ds\right)^T,\left(\int_{t-k_1}^{t} x(s)\,ds\right)^T,f^T(.)]$.
By condition (3.1), we obtain

$$
\dot{V}(t,x_t)\le -2\alpha V(t,x_t),\quad \forall t\ge 0. \qquad (3.8)
$$

Integrating both sides of (3.8) from 0 to t, we obtain

$$V(t,x_t) \le V(\phi)e^{-2\alpha t}, \quad \forall t \ge 0.$$

Furthermore, taking condition (3.2) into account, we have

$$\lambda_1\|x(t,\phi)\|^2 \le V(x_t) \le V(\phi)e^{-2\alpha t} \le \lambda_2 e^{-2\alpha t}\|\phi\|^2,$$

then

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|, \quad t \ge 0,$$

which concludes the proof of the theorem.

Remark 3.1. The conditions of Theorem 3.1 are derived in terms of linear matrix inequalities, which can be solved easily by various efficient numerical algorithms (see, for instance [1]) .

When $f(t) = 0$, Theorem 3.1 reduces to the following result for exponential stability of linear systems with interval time-varying non-differentiable delays, which extends the results of [2, 12].

Corollary 3.1. *Given $\alpha > 0$. System (2.1), where $f(t) = 0$, is $\alpha-$exponentially stable if there exist matrices $N_1, i = 1,\ldots,6$, symmetric positive definite matrices P,Q,R,S,T,U such that LMI holds*

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & A_0^TN_3 & -N_1^T + A_0^TN_4 & \Phi_{15} & \Phi_{16} \\ * & \Phi_{22} & e^{-2\alpha h_2}S + A_1^TN_3 & A_1^TN_4 - N_2^T & N_2^TA_2 + A_1^TN_5 & N_2^TA_2 + A_1^TN_6 \\ * & * & -e^{-2\alpha h_2}S - e^{-2\alpha h_1}Q & -N_3^T & N_3^TA_2 & N_3^TA_2 \\ * & * & * & \Phi_{44} & N_4^TA_2 - N_5 & N_4^TA_2 - N_6 \\ * & * & * & * & \Phi_{55} & N_5^TA_2 + A_2^TN_6 \\ * & * & * & * & * & \Phi_{66} \end{bmatrix} < 0, \tag{3.9}$$

where

$$\Phi_{11} = P(A_0 + \alpha I) + (A_0 + \alpha I)^TP + Q + k_1^2T + (k_2 - k_1)^2U - e^{-2\alpha h_2}R + N_1^TA_0 + A_0^TN_1,$$
$$\Phi_{12} = PA_1 + N_1^TA_1 + A_0^TN_2 + e^{-2\alpha h_2}R,$$
$$\Phi_{15} = (N_1^T + P)A_2 + A_0^TN_5,$$
$$\Phi_{16} = (N_1^T + P)A_2 + A_0^TN_6,$$
$$\Phi_{22} = -e^{-2\alpha h_2}R - e^{-2\alpha h_2}S + N_2^TA_1 + A_1^TN_2,$$
$$\Phi_{44} = h_2^2R + (h_2 - h_1)^2S - N_4 - N_4^T,$$
$$\Phi_{55} = -e^{-2\alpha k_2}U + N_5^TA_2 + A_2^TN_5,$$
$$\Phi_{66} = -e^{-2\alpha k_1}T + N_6^TA_2 + A_2^TN_6.$$

Moreover, the solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \le \sqrt{\frac{\Lambda_2}{\Lambda_1}}e^{-\alpha t}\|\phi\|, \quad \forall t \ge 0,$$

where

$$\begin{aligned}\Lambda_1 =&\lambda_{\min}(P),\\ \Lambda_2 =&\lambda_{\max}(P)+h_1\lambda_{\max}(Q)+\frac{1}{2}(h_2-h_1)^2(h_2+h_1)\lambda_{\max}(S)+\frac{1}{2}h_2^3\lambda_{\max}(R)\\ &+\frac{k_1^3}{2}\lambda_{\max}(T)+\frac{1}{2}(k_2-k_1)^2(k_2+k_1)\lambda_{\max}(U).\end{aligned}$$

We conclude this section with an application to derive exponential stability conditions for uncertain linear systems with mixed interval time-varying delays of the form

$$\begin{aligned}&\dot{x}(t)=[A_0+\Delta A_0(t)]x(t)+[A_1+\Delta A_1(t)]x(t-h(t))+[A_2+\Delta A_2(t)]\int_{t-k(t)}^{t}x(s)\,ds\\ &x(t)=\phi(t),\quad t\in[-d,0],\quad d=\max\{h_2,k_2\},\end{aligned}\tag{3.10}$$

where $x(t)\in\mathbb{R}^n$ is the state; A_0,A_1,A_2 are given matrices of appropriate dimensions and $\phi(t)\in C^1([-d,0],\mathbb{R}^n)$ is the initial function with the norm $\|\phi\|=\sup_{-d\le t\le 0}\sqrt{\|\phi(t)\|^2+\|\dot{\phi}(t)\|^2}$; $\Delta A_0(t),\Delta A_1(t),\Delta A_2(t)$ are time-varying uncertainties which satisfy the following conditions:

$$\Delta A_0(t)=E_0F_0(t)H_0,\quad \Delta A_1(t)=E_1F_1(t)H_1,\quad \Delta A_2(t)=E_2F_2(t)H_2,$$

where $E_i,H_i,i=0,1,2$ are real matrices with appropriate dimensions, $F_i,i=0,1,2$ are unknown time-varying matrix, which is Lesbesgue measurable in t and satisfy $\| F_i(t)\|\le 1$, $h(t),k(t)$ are interval time-varying delay satisfy

$$0\le h_1\le h(t)\le h_2;\quad 0\le k_1\le k(t)\le k_2.$$

Rewrite the system (3.10) as:

$$\begin{aligned}&\dot{x}(t)=A_0x(t)+A_1x(t-h(t))+A_2\int_{t-k(t)}^{t}x(s)\,ds+\overline{f}(t,x(t),x(t-h(t)),\int_{t-k(t)}^{t}x(s)\,ds)\\ &x(t)=\phi(t),\quad t\in[-d,0],\quad d=\max\{h_2,k_2\},\end{aligned}\tag{3.11}$$

where

$$\overline{f}(t,x(t),x(t-h(t)),\int_{t-k(t)}^{t}x(s)\,ds)=\Delta A_0(t)x(t)+\Delta A_1(t)x(t-h(t))+\Delta A_2(t)\int_{t-k(t)}^{t}x(s)\,ds.$$

It is easy to check that

$$\begin{aligned}\|\overline{f}(.)\|^2\le&2\lambda_{\max}(E_0^TE_0)\lambda_{\max}(H_0^TH_0)\|x(t)\|^2+4\lambda_{\max}(E_1^TE_1)\lambda_{\max}(H_1^TH_1)\|x(t-h(t))\|^2\\ &+4\lambda_{\max}(E_2^TE_2)\lambda_{\max}(H_2^TH_2)\|\int_{t-k(t)}^{t}x(s)\,ds\|^2.\end{aligned}$$

Therefore, applying stability result of Theorem 3.1 to system (3.11) gives

Corollary 3.2. *Given $\alpha > 0$. System (3.10) is $\alpha-$robustly stable if there exist matrices $N_i, i = 1,\ldots,7$, symmetric positive definite matrices P,Q,R,S,T,U and a positive number ε such that the following LMI holds*

$$\begin{bmatrix} \overline{\Xi}_{11} & \Xi_{12} & A_0^T N_3 & -N_1^T + A_0^T N_4 & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \overline{\Xi}_{22} & e^{-2\alpha h_2} S + A_1^T N_3 & A_1^T N_4 - N_2^T & N_2^T A_2 + A_1^T N_5 & \Xi_{26} & \Xi_{27} \\ * & * & \Xi_{33} & -N_3^T & N_3^T A_2 & N_3^T A_2 & N_3^T \\ * & * & * & \Xi_{44} & N_4^T A_2 - N_5 & N_4^T A_2 - N_6 & N_4^T - N_7 \\ * & * & * & * & \overline{\Xi}_{55} & \Xi_{56} & \Xi_{57} \\ * & * & * & * & * & \overline{\Xi}_{66} & \Xi_{67} \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix} < 0, \tag{3.12}$$

where

$$\begin{aligned} \overline{\Xi}_{11} &= P(A_0+\alpha I) + (A_0+\alpha I)^T P + Q + k_1^2 T + (k_2-k_1)^2 U - e^{-2\alpha h_2} R + N_1^T A_0 + A_0^T N_1 \\ &\quad + 2\varepsilon\lambda_{\max}(E_0^T E_0)\lambda_{\max}(H_0^T H_0) I, \\ \overline{\Xi}_{22} &= 4\varepsilon\lambda_{\max}(E_1^T E_1)\lambda_{\max}(H_1^T H_1) I - e^{-2\alpha h_2} R - e^{-2\alpha h_2} S + N_2^T A_1 + A_1^T N_2, \\ \overline{\Xi}_{55} &= 8\varepsilon\lambda_{\max}(E_2^T E_2)\lambda_{\max}(H_2^T H_2) I - e^{-2\alpha k_2} U + N_5^T A_2 + A_2^T N_5, \\ \overline{\Xi}_{66} &= 8\varepsilon\lambda_{\max}(E_2^T E_2)\lambda_{\max}(H_2^T H_2) I - e^{-2\alpha k_1} T + N_6^T A_2 + A_2^T N_6, \end{aligned}$$

and $\Xi_{12},\Xi_{15},\Xi_{16},\Xi_{17},\Xi_{26},\Xi_{27},\Xi_{33},\Xi_{44},\Xi_{56},\Xi_{57},\Xi_{67},\Xi_{77}$ are defined as in Theorem 3.1.

Moreover, the solution $x(t,\phi)$ of the system satisfies

$$\| x(t,\phi) \| \le \sqrt{\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}} \|\phi\| e^{-\alpha t}, \quad \forall t \ge 0.$$

Where $\tilde{\lambda}_1 = \lambda_{\min}(P)$, and

$$\begin{aligned} \tilde{\lambda}_2 =& \lambda_{\max}(P) + h_1\lambda_{\max}(Q) + \frac{1}{2}(h_2-h_1)^2(h_2+h_1)\lambda_{\max}(S) + \frac{1}{2}h_2^3\lambda_{\max}(R) \\ &+ \frac{k_1^3}{2}\lambda_{\max}(T) + \frac{1}{2}(k_2-k_1)^2(k_2+k_1)\lambda_{\max}(U). \end{aligned}$$

Remark 3.2. Note that Corollary 3.2 gives sufficient conditions for the robust exponential stability of uncertain linear systems with interval time-varying delays, which extend the results obtained in [3, 4, 6].

4 Numerical examples

Example 4.1. Consider the system (2.1), where

$$\begin{cases} h(t) = 0.1 + 0.1\sin^2 t & if \quad i \in I = \cup_{k\ge 0}[2k\pi,(2k+1)\pi] \\ h(t) = 0 & if \quad t \in R^+ \setminus I, \end{cases}$$

$$\begin{cases} k(t) = 0.1 + 0.3\cos^2 t & if \quad i \in I = \cup_{k\geq 0}[2k\frac{\pi}{2}, (2k+1)\frac{\pi}{2}] \\ k(t) = 0 & if \quad t \in R^+ \setminus I, \end{cases}$$

$$A_0 = \begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1 & 0 \\ 0.2 & -0.3 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & 0.3 \\ -0.2 & 0.1 \end{bmatrix},$$

and the nonlinear function

$$f(.) = \begin{bmatrix} 0.1\cos(t|x_1(t)|) \\ 0.1\sin(t|x_1(t)|) \end{bmatrix} + \begin{bmatrix} \sqrt{0.005}\cos(t|x_1(t-h(t))|) \\ \sqrt{0.005}\sin(t|x_1(t-h(t))|) \end{bmatrix} + \begin{bmatrix} \sqrt{0.005}\cos(t|\int_{t-k(t)}^t x_2(s)\,ds|) \\ \sqrt{0.005}\sin(t|\int_{t-k(t)}^t x_2(s)\,ds|) \end{bmatrix}.$$

It is easy to check that

$$\|f(.)\| \leq 0.1^2\|x(t)\|^2 + 0.1^2\|x(t-h(t))\|^2 + 0.1^2\left\| \int_{t-k(t)}^t x(s)\,ds \right\|^2.$$

It is worth noting that, the delay functions $h(t), k(t)$ are non-differentiable. Morever, both A_0 and $A_0 + A_1$ are not stable. Therefore, the methods is used in [8, 9, 10, 13, 14] are not applicable to this system. We have $0.1 \leq h(t) \leq 0.2; \quad 0.1 \leq k(t) \leq 0.4, \forall t \geq 0$. Given $\alpha = 1$. By using LMI toolbox of Matlab, we can verify that, the LMI (3.1) is satisfied with $h_1 = 0.1, h_2 = 0.2, k_1 = 0.1, k_2 = 0.4, a_0 = a_1 = a_2 = 0.1, \varepsilon = 1$ and

$$P = \begin{bmatrix} 0.1300 & -0.6977 \\ -0.6977 & 5.7105 \end{bmatrix}, Q = \begin{bmatrix} 0.3531 & -0.9369 \\ -0.9369 & 8.2456 \end{bmatrix}, R = \begin{bmatrix} 0.6233 & -0.4653 \\ -0.4653 & 4.6066 \end{bmatrix},$$

$$S = \begin{bmatrix} 50.5169 & 0 \\ 0 & 50.5169 \end{bmatrix}, T = \begin{bmatrix} 17.3888 & -3.6206 \\ -3.6206 & 47.6912 \end{bmatrix}, U = \begin{bmatrix} 2.9413 & -3.7199 \\ -3.7199 & 34.3751 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -31.7536 & 142.1580 \\ 47.9102 & -0.0014 \end{bmatrix}, N_2 = \begin{bmatrix} -5.2569 & 7.0175 \\ 2.2701 & 0.7940 \end{bmatrix}, N_3 = \begin{bmatrix} -0.0394 & -0.0069 \\ -0.0116 & 0.0209 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 4.2415 & 23.5712 \\ -24.1186 & 1.2637 \end{bmatrix}, N_5 = \begin{bmatrix} 4.3547 & -3.5013 \\ 2.4576 & 7.1659 \end{bmatrix}, N_6 = \begin{bmatrix} 4.3799 & -3.4338 \\ 2.4491 & 7.1659 \end{bmatrix},$$

$$N_7 = \begin{bmatrix} -4.0573 & -23.6503 \\ 23.9990 & -0.3389 \end{bmatrix}.$$

By Theorem 3.1, the system is exponentially stable with decay rate $\alpha = 1$. We derive the value $\beta = 13.0850$ and the solution $x(t, \phi)$ satisfies

$$\| x(t,\phi) \| \leq 13.0850e^{-t} \| \phi \|, \quad t \in \mathbb{R}^+.$$

Example 4.2. Consider the system (2.1), where $f(t) = 0$, and

$$h(t) = 0.1 + 0.2\sin|t|,$$

$$\begin{cases} k(t) = 0.1 + 0.4\cos^2 t & if \quad i \in I = \cup_{k\geq 0}[2k\frac{\pi}{2}, (2k+1)\frac{\pi}{2}] \\ k(t) = 0 & if \quad t \in R^+ \setminus I, \end{cases}$$

$$A_0 = \begin{bmatrix} 0.5 & 0.1 \\ 0 & -4 \end{bmatrix}, A_1 = \begin{bmatrix} 0.09 & 0.2 \\ 0.1 & -0.1 \end{bmatrix}, A_2 = \begin{bmatrix} -0.1 & 0.5 \\ 0.3 & -0.5 \end{bmatrix}.$$

Note that the delay functions $h(t), k(t)$ are non-differentiable and the result obtained in [2, 12] are not be applicable to this system. We have $0.1 \leq h(t) \leq 0.3; \quad 0.1 \leq k(t) \leq 0.5, \quad \forall t \geq 0$. Given $\alpha = 1.5$. By using LMI toolbox of Matlab, we can verify that, the LMI (3.9) is satisfied with

$$P = \begin{bmatrix} 0.0230 & 0.0445 \\ 0.0445 & 7.7931 \end{bmatrix}, Q = \begin{bmatrix} 0.0648 & 0.0035 \\ 0.0035 & 4.8178 \end{bmatrix}, R = S = \begin{bmatrix} 344.1091 & 0.3510 \\ 0.3510 & 345.7965 \end{bmatrix},$$

$$T = \begin{bmatrix} 6.2113 & 0.2945 \\ 0.2945 & 195.7174 \end{bmatrix}, U = \begin{bmatrix} 0.3288 & 0.0255 \\ 0.0255 & 27.8962 \end{bmatrix}, N_1 = \begin{bmatrix} -45.4807 & -321.0531 \\ -43.9910 & 25.3128 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -0.5594 & -25.7371 \\ -8.7566 & -16.4693 \end{bmatrix}, N_3 = \begin{bmatrix} -0.0875 & -0.1782 \\ -0.0140 & -0.0006 \end{bmatrix}, N_4 = \begin{bmatrix} 91.6182 & -78.0273 \\ 87.9980 & 10.2593 \end{bmatrix}$$

$$, N_5 = \begin{bmatrix} 32.6407 & -84.6095 \\ 6.2832 & -39.7831 \end{bmatrix}, N_6 = \begin{bmatrix} 32.5293 & -83.9797 \\ 6.3172 & -39.8154 \end{bmatrix}.$$

By Corollary 3.1, the system is exponentially stable with decay rate $\alpha = 1.5$ and the solution $x(t, \phi)$ satisfies

$$\| x(t, \phi) \| \leq 27.0475 e^{-1.5t} \| \phi \|, \quad \forall t \geq 0.$$

Example 4.3. Consider the uncertain system with mixed interval time-varying delays (3.10), where

$$\begin{cases} h(t) = 0.1 + 0.3 \sin^2 t & if \quad i \in I = \cup_{k \geq 0} [2k\pi, (2k+1)\pi] \\ h(t) = 0 & if \quad t \in R^+ \setminus I, \end{cases}$$

$$\begin{cases} k(t) = \beta(t), & if \quad t \in [0, 1] \\ k(t) = \beta(t - k), & if \quad t \in [k, k+1], k = 1, 2, ..., \end{cases}$$

where $\beta(t) = t + 0.1, t \in [0, 0.5]; = -t + 1.1, t \in (0.5, 1]$.

$$A_0 = \begin{bmatrix} 0.1 & 1 \\ 0 & -6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.05 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 0.1 \\ 0 & -0.2 \end{bmatrix},$$

$$E_0 = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 & 0.2 \\ -0.3 & 0.2 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.3 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}.$$

Note that, the delay functions $h(t), k(t)$ are non-differentiable and. Morever, both A_0 and $A_0 + A_1$ are not stable. Therefore, the methods is used in [3, 4, 6] are not applicable to this system. We have $0.1 \leq h(t) \leq 0.4; \quad 0.1 \leq k(t) \leq 0.6, \forall t \geq 0$. Given $\alpha = 0.5$. By using LMI toolbox of Matlab, we can verify that, the LMI (3.12) is satisfied with $\varepsilon = 1$,

$$P = \begin{bmatrix} 0.0366 & -0.0575 \\ -0.0575 & 5.7096 \end{bmatrix}, Q = \begin{bmatrix} 0.1237 & -0.0939 \\ -0.0939 & 8.9722 \end{bmatrix}, R = S = \begin{bmatrix} 291.1839 & 0 \\ 0 & 291.1839 \end{bmatrix},$$

$$T=\begin{bmatrix}10.4855 & -2.2391\\ -2.2391 & 221.4136\end{bmatrix}, U=\begin{bmatrix}0.4501 & -0.3436\\ -0.3436 & 32.8175\end{bmatrix}, N_1=\begin{bmatrix}-7.0916 & -3.3135\\ -0.1638 & 19.0932\end{bmatrix},$$

$$N_2=\begin{bmatrix}-3.5934 & -0.0579\\ -0.4279 & -0.0243\end{bmatrix}, N_3=\begin{bmatrix}-0.0675 & -0.0022\\ -0.0082 & -0.0071\end{bmatrix}, N_4=\begin{bmatrix}71.3997 & 0.6261\\ 1.7412 & 4.4528\end{bmatrix},$$

$$N_5=\begin{bmatrix}6.9464 & -6.8282\\ 0.1718 & 0.5179\end{bmatrix}, N_6=\begin{bmatrix}6.8864 & -6.7702\\ 0.1708 & 0.5125\end{bmatrix}, N_7=\begin{bmatrix}-70.5047 & -0.6083\\ -1.7353 & -3.5051\end{bmatrix}.$$

By Corollary 3.2, the system is robustly stable with decay rate $\alpha = 0.5$ and the solution $x(t,\phi)$ satisfies

$$\| x(t,\phi) \| \le 26.5934e^{-0.5t} \| \phi \|, \quad \forall t \ge 0.$$

5 Conclusions

This paper has proposed new delay-depedent criteria for exponential stability of nonlinear systems with continuously distributed time-varying delays. By constructing a suitable augmented Lyapunovs functional, new criteria for the exponential stability of the system have benn established in terms of LMIs. The result has been applied to robust stability problem of uncertain systems with interval time-varying delays.

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