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# AFFINE OSSERMAN CONNECTIONS ON 2-DIMENSIONAL MANIFOLDS *

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**Abstract**

This paper deals with affine Osserman connections on 2-dimensional manifolds. We give in an explicit form, a sufficient condition for an affine connection to be Osserman. As applications, examples of affine Osserman connections are given.

**AMS Subject Classification:** 53B05, 53B20.

**Keywords**: affine connection; Osserman condition.

## 1 Introduction

Let $(M,g)$ be a Riemannian manifold. Let $\mathcal{R}$ be the curvature operator. The *Jacobi operator* $J_{\mathcal{R}}(X) : Y \to \mathcal{R}(Y,X)X$ is a self-adjoint operator and it plays an important role in the curvature theory. A Riemannian manifold is said to be an *Osserman space* if the eigenvalues of the Jacobi operators are constant on the unit sphere bundle $S(M,g)$. The investigation of Osserman manifolds has been an extremely active and fruitful one in recent years; we refer to [2, 4] for further details.

The purpose of this paper is to study the generalization of these notions to the affine geometry. Let $\nabla$ be a torsion free connection on $TM$. The pair $(M,\nabla)$ is said to be an *affine manifold*. Let $\mathcal{R}^{\nabla}$ be the curvature operator and $J_{\mathcal{R}^{\nabla}}(\cdot)$ be the *affine Jacobi operator*; we will write $\mathcal{R}^{\nabla}$ and $J_{\mathcal{R}^{\nabla}}$ when it is necessary to distinguish the role of the connection. One says that $(M,\nabla)$ is *affine Osserman* at $p \in M$ if $J_{\mathcal{R}^{\nabla}}$ has the same characteristic polynomial for every $X \in T_pM$. Also $(M,\nabla)$ is called *affine Osserman* if $(M,\nabla)$ is affine Osserman at each $p \in M$. It is well-known that for any affine Osserman manifolds $Spect\{J_{\mathcal{R}^{\nabla}}(X)\} = \{0\}$.

The concept of affine Osserman connection originated from the effort to build up examples of pseudo-Riemannian Osserman manifolds via the construction called the *Riemann extension*. This construction assigns to every $m$-dimensional manifold $M$ with a torsion-free affine connection $\nabla$ a pseudo-Riemannian metric $g_{\nabla}$ of signature $(m,m)$ on the cotangent bundle $T^*M$. (See [6], for more details.)

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In this note, we give in an explicit form, a sufficient condition for an affine connection on 2-dimensional manifolds to be Osserman. We shall prove the following result:

**Theorem 1.1.** *Let $\mathbb{R}^2$ and let $\nabla$ be the torsion free connection given by*

$$\begin{cases} \nabla_{\partial_1}\partial_1 &= f_{11}^1(u_1,u_2)\partial_1 + f_{11}^2(u_1,u_2)\partial_2; \\ \nabla_{\partial_1}\partial_2 &= f_{12}^1(u_1,u_2)\partial_1 + f_{12}^2(u_1,u_2)\partial_2; \\ \nabla_{\partial_2}\partial_2 &= f_{22}^1(u_1,u_2)\partial_1 + f_{22}^2(u_1,u_2)\partial_2. \end{cases} \tag{1.1}$$

*Then $\nabla$ is affine Osserman if and only if the functions $f_{11}^1, f_{11}^2, f_{12}^1, f_{12}^2, f_{22}^1, f_{22}^2$ satisfy the following PDE's*

$$\begin{cases} \partial_2 f_{11}^2 - \partial_1 f_{12}^2 + f_{12}^2(f_{11}^1 - f_{12}^2) + f_{11}^2(f_{22}^2 - f_{12}^1) &= 0; \\ \partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{22}^1(f_{11}^1 - f_{12}^2) + f_{12}^1(f_{22}^2 - f_{12}^1) &= 0; \\ \partial_1 f_{12}^1 - \partial_2 f_{11}^1 - \partial_1 f_{22}^2 + \partial_2 f_{12}^2 + 2(f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1) &= 0. \end{cases} \tag{1.2}$$

## 2 Preliminaries

Let $M$ a two-dimensional manifold and $\nabla$ a smooth torsion-free connection. We choose a fixed coordinates domain $\mathcal{U}(u_1,u_2) \subset M$. In $\mathcal{U}$, the connection is given by

$$\begin{aligned} \nabla_{\partial_1}\partial_1 &= f_{11}^1(u_1,u_2)\partial_1 + f_{11}^2(u_1,u_2)\partial_2; \\ \nabla_{\partial_1}\partial_2 &= f_{12}^1(u_1,u_2)\partial_1 + f_{12}^2(u_1,u_2)\partial_2; \\ \nabla_{\partial_2}\partial_2 &= f_{22}^1(u_1,u_2)\partial_1 + f_{22}^2(u_1,u_2)\partial_2; \end{aligned}$$

where we denote $\partial_i = (\partial/\partial u_i)\,(i = 1,2)$. We will denote the functions $f_{11}^1(u_1,u_2)$, $f_{11}^2(u_1,u_2)$, $f_{12}^1(u_1,u_2)$, $f_{12}^2(u_1,u_2)$, $f_{22}^1(u_1,u_2)$, $f_{22}^2(u_1,u_2)$ by $f_{11}^1$, $f_{11}^2$, $f_{12}^1$, $f_{12}^2$, $f_{22}^1$, $f_{22}^2$ respectively, if there is no risk of confusion.

**Lemma 2.1.** *The components of the curvature operator are given by*

$$\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_1 = a\partial_1 + b\partial_2, \;\; \mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2 = c\partial_1 + d\partial_2.$$

*where*

$$\begin{aligned} a &= \partial_1 f_{12}^1 - \partial_2 f_{11}^1 + f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1, \\ b &= \partial_1 f_{12}^2 - \partial_2 f_{11}^2 + f_{11}^2 f_{12}^1 + f_{12}^2 f_{12}^2 - f_{11}^1 f_{12}^2 - f_{11}^2 f_{22}^2, \\ c &= \partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{11}^1 f_{22}^1 + f_{12}^1 f_{22}^2 - f_{12}^1 f_{12}^1 - f_{12}^2 f_{22}^1, \\ d &= \partial_1 f_{22}^2 - \partial_2 f_{12}^2 + f_{11}^2 f_{22}^1 - f_{12}^1 f_{12}^2. \end{aligned}$$

We say that the affine connection $\nabla$ is *flat* if and only if the curvature tensor $\mathcal{R}^\nabla$ vanishes on M. It is well-known that $\nabla$ is flat if and only if around each point it exists a local coordinate system such that all Christoffel symbols vanish.

**Lemma 2.2.** *If $X = \sum_1^2 \alpha_i \partial_i$ is a vector on $M$, then the affine Jacobi operator is given by*

$$J_{\mathcal{R}^\nabla}(X)\partial_1 = A\partial_1 + B\partial_2, \quad J_{\mathcal{R}^\nabla}(X)\partial_2 = C\partial_1 + D\partial_2,$$

*where*

$$A = \alpha_1\alpha_2 a + \alpha_2^2 c,\ \ B = \alpha_1\alpha_2 b + \alpha_2^2 d,\ \ C = -\alpha_1^2 a - \alpha_1\alpha_2 c,\ \ \text{and}\ \ D = -\alpha_1^2 b - \alpha_1\alpha_2 d.$$

**Lemma 2.3.** *The components of the Ricci tensor are given by*

$$\begin{aligned}
Ric^\nabla(\partial_1,\partial_1) &= \partial_2 f_{11}^2 - \partial_1 f_{12}^2 + f_{12}^2(f_{11}^1 - f_{12}^2) + f_{11}^2(f_{22}^2 - f_{12}^1);\\
Ric^\nabla(\partial_1,\partial_2) &= \partial_2 f_{12}^2 - \partial_1 f_{22}^2 + f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1;\\
Ric^\nabla(\partial_2,\partial_1) &= \partial_1 f_{12}^1 - \partial_2 f_{11}^1 + f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1;\\
Ric^\nabla(\partial_2,\partial_2) &= \partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{22}^1(f_{11}^1 - f_{12}^2) + f_{12}^1(f_{22}^2 - f_{12}^1).
\end{aligned}$$

The *skew-symmetric* of $Ric^\nabla$ means that, in local coordinates

$$Ric^\nabla(\partial_1,\partial_1) = Ric^\nabla(\partial_2,\partial_2),\ Ric^\nabla(\partial_1,\partial_2) + Ric^\nabla(\partial_2,\partial_1) = 0. \tag{2.1}$$

We easily see that the conditions (2.1) reduce to:

$$\begin{aligned}
\partial_2 f_{11}^2 - \partial_1 f_{12}^2 + f_{12}^2(f_{11}^1 - f_{12}^2) + f_{11}^2(f_{22}^2 - f_{12}^1) &= 0;\\
\partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{22}^1(f_{11}^1 - f_{12}^2) + f_{12}^1(f_{22}^2 - f_{12}^1) &= 0;\\
\partial_1 f_{12}^1 - \partial_2 f_{11}^1 - \partial_1 f_{22}^2 + \partial_2 f_{12}^2 + 2(f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1) &= 0.
\end{aligned}$$

The authors of [1], characterized affine connections on surfaces which are affine Osserman by skew-symmetric of their Ricci tensor.

A affine connection $\nabla$ on $M$ is *locally symmetric* if and only if:

$$\nabla\mathcal{R}^\nabla = 0. \tag{2.2}$$

Writing this formula in local coordinates, we find that any locally symmetric affine connections must satisfy eight equations.

**Proposition 2.4.** *The connection $\nabla$ defined by (1.1) is locally symmetric if and only if the functions $f_{11}^1, f_{11}^2, f_{12}^1, f_{12}^2, f_{22}^1, f_{22}^2$ are solutions of the following:*

$$\begin{aligned}
\partial_1 a + f_{11}^1 a + f_{12}^1 b &= 0,\\
\partial_1 b + f_{11}^2 a + f_{12}^2 b &= 0,\\
\partial_1 c + f_{11}^1 c + f_{12}^1 d &= 0,\\
\partial_1 d + f_{11}^2 c + f_{12}^2 d &= 0,\\
\partial_2 a + f_{12}^1 a + f_{22}^1 b &= 0,\\
\partial_2 b + f_{12}^2 a + f_{22}^2 b &= 0,\\
\partial_2 c + f_{12}^1 c + f_{22}^1 d &= 0,\\
\partial_2 d + f_{12}^2 c + f_{22}^2 d &= 0.
\end{aligned}$$

*Proof.* Let $X_k = \alpha_i^k \partial_i$, $k = 1,2,3,4$, $i = 1,2$. The condition

$$\nabla_{X_1} \mathcal{R}^\nabla (X_2, X_3) X_4 = 0$$

leads to

$$\nabla_{\alpha_i^1 \partial_i} \mathcal{R}^\nabla (\alpha_i^2 \partial_i, \alpha_i^3 \partial_i) \alpha_i^4 \partial_i = 0, \quad i,j,k = 1,2.$$

Equivalently,

$$\nabla_{\alpha_1^1 \partial_1} \mathcal{R}^\nabla (\alpha_j^2 \partial_j, \alpha_k^3 \partial_k) \alpha_l^4 \partial_l + \nabla_{\alpha_2^1 \partial_2} \mathcal{R}^\nabla (\alpha_j^2 \partial_j, \alpha_k^3 \partial_k) \alpha_l^4 \partial_l = 0, \; j,k,l = 1,2.$$

Straightforward calculation give

$$\begin{cases} \nabla_{\partial_1} \mathcal{R}^\nabla (\partial_1, \partial_2) \partial_1 &= [\partial_1 a + f_{11}^1 a + f_{12}^1 b] \partial_1 + [\partial_1 b + f_{11}^2 a + f_{12}^2 b] \partial_2, \\ \nabla_{\partial_1} \mathcal{R}^\nabla (\partial_1, \partial_2) \partial_2 &= [\partial_1 c + f_{11}^1 c + f_{12}^1 d] \partial_1 + [\partial_1 d + f_{11}^2 c + f_{12}^2 d] \partial_2, \\ \nabla_{\partial_2} \mathcal{R}^\nabla (\partial_1, \partial_2) \partial_1 &= [\partial_2 a + f_{12}^1 a + f_{22}^1 b] \partial_1 + [\partial_2 b + f_{12}^2 a + f_{22}^2 b] \partial_2, \\ \nabla_{\partial_2} \mathcal{R}^\nabla (\partial_1, \partial_2) \partial_2 &= [\partial_2 c + f_{12}^1 c + f_{22}^1 d] \partial_1 + [\partial_2 d + f_{12}^2 c + f_{22}^2 d] \partial_2. \end{cases}$$

The proof is complete. □

A smooth connection $\nabla$ on $M$ is *locally homogeneous* if and only if it admits, in neighborhoods of each point $p \in M$; at least two linearly independent affine Killing vectors fields. An affine Killing vector field $X$ is characterized by the equation:

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0 \tag{2.3}$$

which has to be satisfied for arbitrary vectors fields $Y, Z$ (see [5]). It is sufficient to satisfy (2.3) for the choices $(Y,Z) \in \{(\partial_1, \partial_1), (\partial_1, \partial_2), (\partial_2, \partial_1), (\partial_2, \partial_2)\}$. Moreover, we easily check from the basic identities for the torsion and the Lie brackets, that the choice $(Y,Z) = (\partial_1, \partial_2)$ gives the same conditions as the choice $(Y,Z) = (\partial_2, \partial_1)$.

In the sequel, let us express the vector field $X$ in the form

$$X = F(u_1, u_2) \partial_1 + G(u_1, u_2) \partial_2.$$

Writing the formula (2.3) in local coordinates, we find that any affine Killing vector field $X$ must satisfy six basics equations. We shall write these equations in the simplified notation:

$$\begin{aligned} \partial_{11} F + f_{11}^1 \partial_1 F + \partial_1 f_{11}^1 F - f_{11}^2 \partial_2 F + \partial_2 f_{11}^1 G + 2 f_{12}^1 \partial_1 G &= 0, \\ \partial_{11} G + 2 f_{11}^2 \partial_1 F + (2 f_{12}^2 - f_{11}^1) \partial_1 G - f_{11}^2 \partial_2 G + \partial_1 f_{11}^2 F + \partial_2 f_{11}^2 G &= 0, \\ \partial_{12} F + (f_{11}^1 - f_{12}^2) \partial_2 F + f_{22}^1 \partial_1 G + f_{12}^1 \partial_2 G + \partial_1 f_{12}^1 F + \partial_2 f_{12}^1 G &= 0, \\ \partial_{12} G + f_{12}^2 \partial_1 F + f_{11}^2 \partial_2 F + (f_{22}^2 - f_{11}^2) \partial_1 G + \partial_1 f_{12}^2 F + \partial_2 f_{12}^2 G &= 0, \\ \partial_{22} F - f_{22}^1 \partial_1 F + (2 f_{12}^1 - f_{22}^2) \partial_2 F + 2 f_{22}^1 \partial_2 G + \partial_1 f_{22}^1 F + \partial_2 f_{22}^1 G &= 0, \\ \partial_{22} G + 2 f_{12}^2 \partial_2 F - f_{22}^1 \partial_1 G + f_{22}^2) \partial_2 G \partial_1 f_{22}^2 F + \partial_2 f_{22}^2 G &= 0. \end{aligned}$$

A complete description of locally homogeneous affine Osserman surfaces is given by Kowalski, Opozda and Vlášek in [5].

## 3 Proof of Theorem

The matrix associated to $J_{\mathcal{R}^\nabla}(X)$ with respect to the basis $\{\partial_1, \partial_2\}$ is given by

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

It follows from the matrix associated to $J_{\mathcal{R}^\nabla}(X)$, that its characteristic polynomial satisfies

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = \lambda^2 - \lambda(A+D) + (AD-BC).$$

Through the results of [1], $(M,\nabla)$ is affine Osserman if and only if

$$Spect\{J_{\mathcal{R}^\nabla}(X)\} = \{0\}.$$

Since $J_{\mathcal{R}^\nabla}(X)X = 0$, we conclude that

$$\det\{J_{\mathcal{R}^\nabla}(X)\} = (AD-BC) = 0.$$

Thus $Spect\{J_{\mathcal{R}^\nabla}(X)\} = \{0\}$ if and only if $A+D=0$. Straightforward computations of give

$$\begin{array}{rcl} \partial_1 f_{12}^1 - \partial_2 f_{11}^1 - \partial_1 f_{22}^2 + \partial_2 f_{12}^2 + 2 f_{12}^1 f_{12}^2 - 2 f_{11}^2 f_{22}^1 & = & 0; \\ \partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{11}^1 f_{22}^1 + f_{12}^1 f_{22}^2 - f_{12}^1 f_{12}^1 - f_{12}^2 f_{22}^1 & = & 0; \\ \partial_1 f_{12}^2 - \partial_2 f_{11}^2 + f_{11}^2 f_{12}^1 + f_{12}^2 f_{12}^2 - f_{11}^1 f_{12}^2 - f_{11}^2 f_{22}^2 & = & 0. \end{array}$$

The proof is complete. □

**Corollary 3.1.** *[1] Let $\nabla$ be the affine connection on $\mathbb{R}^2$ given by*

$$\nabla_{\partial_1}\partial_1 = f_{11}^1(u_1,u_2)\partial_1, \quad \nabla_{\partial_1}\partial_2 = 0, \quad \nabla_{\partial_2}\partial_2 = f_{22}^2(u_1,u_2)\partial_2. \tag{3.1}$$

*Then $\nabla$ is affine Osserman if and only if the functions $f_{11}^1, f_{22}^2$ satisfy the following equation:*

$$\partial_2 f_{11}^1 + \partial_1 f_{22}^2 = 0.$$

The authors of [2] used the connection defined by (3.1) to construct examples of pseudo-Riemannian nonsymmetric Osserman manifolds of signature $(2,2)$.

**Corollary 3.2.** *Let $\nabla$ be the affine connection on $\mathbb{R}^2$ given by*

$$\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = f_{12}^1(u_1,u_2)\partial_1, \quad \nabla_{\partial_2}\partial_2 = f_{22}^1(u_1,u_2)\partial_1.$$

*Then $\nabla$ is affine Osserman if and only if the functions $f_{12}^1$ and $f_{22}^1$ have the form*

$$f_{12}^1(u_1,u_2) = f(u_2), \quad \textit{and} \quad f_{22}^1(u_1,u_2) = u_1 G(u_2),$$

*where G depending only $u_2$ satisfying $G(u_2) = \partial_2 f_{12}^1 + (f_{12}^1)^2$.*

**Corollary 3.3.** *Let $\nabla$ be the affine connection on $\mathbb{R}^2$ given by*

$$\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = f_{12}^2(u_1,u_2)\partial_2, \quad \nabla_{\partial_2}\partial_2 = f_{22}^2(u_1,u_2)\partial_2.$$

*Then $\nabla$ is affine Osserman if and only if the functions $f_{12}^1$ and $f_{22}^1$ have the form:*

$$f_{12}^2(u_1,u_2) = \frac{1}{u_1}, \quad \textit{and} \quad f_{22}^2(u_1,u_2) = f(u_2).$$

One has the following observation:

**Theorem 3.4.** *Let $(M,\nabla)$ be a 2-dimensional affine Osserman manifold. If $\nabla$ is locally symmetric, then the Ricci tensor of $\nabla$ is zero.*

**Example 3.5.** Let $\nabla$ the connection on the plane $\mathbb{R}^2$ defined by

$$\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = u_2\partial_1, \quad \nabla_{\partial_2}\partial_2 = u_1(1+u_2^2)\partial_1.$$

A straightforward calculation shows that $\nabla$ is a locally symmetric affine Osserman connection.

**Example 3.6.** Let $\nabla$ the connection on the plane $\mathbb{R}^2$ defined by

$$\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = \frac{1}{u_1}\partial_2, \quad \nabla_{\partial_2}\partial_2 = e^{u_2}\partial_2.$$

A straightforward calculation shows that $\nabla$ is a symmetric affine Osserman connection.

**Theorem 3.7.** *Let $(M,\nabla)$ be a 2-dimensional affine Osserman manifold. If $\nabla$ is nonsymmetric, then the Ricci tensor of $\nabla$ is skew-symmetric.*

**Example 3.8.** ( [2]) Consider the connection $\nabla$ on $\mathbb{R}^2$ defined by

$$\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = e^{u_2}u_1\partial_1, \quad \nabla_{\partial_1}\partial_2 = \frac{1}{2}e^{u_2}u_1^2\partial_1 + e^{u_2}u_1\partial_2.$$

We have

$$\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_1 = e^{u_2}\partial_1, \quad \mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2 = e^{u_2}\partial_2.$$

Now the nonvanishing components of the Ricci tensor are given by

$$Ric^\nabla(\partial_1,\partial_2) = -e^{u_2}, \quad Ric^\nabla(\partial_2,\partial_1) = e^{u_2}.$$

It follows that the Ricci tensor of $\nabla$ is skew symmetric, and thus, an affine Osserman connection. We use (2.2) in order to show that $(\mathbb{R}^2,\nabla)$ is nonsymmetric.

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## References

[1] E. García-Rio, D. N. Kupeli, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Affine Osserman connections and their Riemannian extensions. *Differential Geom. Appl.* **11** (1999), 145-153.

[2] E. García-Rio, D. N. Kupeli and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*. Lectures Notes in Mathematics 1777, Springer-Verlag, Berlin 2002.

[3] E. García-Rio, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds. *Proc. Amer. Math. Soc.* **126** (1998), 2771-2778.

[4] P. Gilkey, *Geometric Properties of Natural Operators defined by the Riemann Curvature Tensor*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[5] O. Kowalski, B. Opozda, Z. Vlásek, A classification of locally homogeneous affine connections with skew-symmetric Ricci tensor on 2-dimensional manifolds. *Monatsh. Math.* **130** (2000), 109-125.

[6] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*. Marcel Dekker, New York, 1973.