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Global Dynamics for a Relativistic Charged Fluid with Potential in Temporal Gauge in a Robertson-Walker Space-Time

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Abstract

Global existence of solutions to the coupled Einstein-Maxwell system which rules the dynamics of the considered relativistic charged fluid is proved and asymptotic behavior is investigated, in the case of positive cosmological constant and positive initial velocity of the cosmological expansion factor.

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1 Introduction

Global dynamics of various relativistic kinetic matter remain an open research area in General Relativity. The Robertson-Walker background space-time we consider in the present paper is known to be the basic space-time in cosmology, where homogeneous phenomena such as the one we consider here are relevant. Notice that the whole universe is modeled and that what we call "*particles*" in the kinetic description, may be galaxies or even clusters of galaxies, for which only the evolution in time is really significant. The coupled Einstein-Maxwell system we study in the present paper and which rules the dynamics of the considered charged fluid models physical situations which exist for instance in some media at very high temperature such as: burning reactors, nebular galaxies, solar winds, etc, where massive particles of ionized gas evolve with high velocities under the action of

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both their common gravitational field and self-created electromagnetic forces. The Einstein equations for the gravitational field inquire the gravitational effects, whereas the Maxwell equations for the electromagnetic field inform about the electromagnetic effects.

The Einstein theory stipulates that, the gravitational field, represented by the metric tensor g, depending in the case we consider on a single positive real-valued function a called the cosmological expansion factor, is determined, through the Einstein equations, by the material and energetic content of the space-time. In the considered case, the space-time content is represented by both the Maxwell tensor associated to the electromagnetic field F deriving itself from a potential vector A taken in temporal gauge, and the matter tensor of a class of relativistic fluids. We consider the case of matter tensors generated by a matter density ρ, a unit vector u tangent to the geodesics flow, and a symmetric 2-tensor Θ called "pressure pseudo-tensor", following the terminology introduced by A.LICHNEROWICZ in [3] for a notion which generalizes both the notion of pure matter and the notion of relativistic perfect fluid of pure radiation type. We consider that the electromagnetic field F is generated through the Maxwell equations, by the Maxwell current defined by both an unknown charge density e and the unknown unit vector u tangent to the geodesics flow. The Einstein-Maxwell system, coupled to the conservation laws, turns out to be a non linear second order differential system in a, ρ, u, F, A and Θ; which by a suitable change of variables gives an equivalent non linear first order differential system to which the standard theory applies.

Now our motivation for considering the Einstein equations with the cosmological constant Λ is due to astrophysical applications. The cosmological constant appears to be a wonderful tool to model and to explain mathematically the acceleration phenomenon of the expansion of the universe, as revealed by astrophysical observations. Such models are also studied in [1] [5] [6] [8] [9].

By global solutions in the present paper, we mean solutions defined over the whole interval $[0,+\infty[$. In the present paper we prove that the Einstein-Maxwell system with cosmological constant Λ admits global solutions only if $\Lambda \geq 0$ and if the initial velocity of expansion is positive. We prove in details that there cannot exist global solutions if $\Lambda < 0$. We also prove that, even in the case $\Lambda \geq 0$ which is the most studied case in the literature, there cannot exist global solutions if the initial velocity of expansion is negative. The proof shows that in these two cases, there exists singularities in the space-time after a finite time; from a physical point of view, this can be interpreted as explosions occurring after a finite time in the space-time. Finally we prove by studying the asymptotic behavior in the case of global existence, that the space-time tends to the vacuum at late times, regardless to the size of the initial data as a consequence of the growth of the cosmological expansion factor which is exponential if $\Lambda > 0$ and slow if $\Lambda = 0$, modeling as we indicated above, the acceleration phenomenon of the expansion of the universe. The present paper extends the results of [7] to the case of a non-zero cosmological constant and to the case of relativistic fluids of more general type than the relativistic perfect fluid of pure radiation type. The paper is organized as follows:

- In section 1, we state all the equations. This section is the most important part of the paper since we analyze the problem in details, we specify the hypotheses and we give the final form of each equation.

- In section 2, we study the global existence of solutions.
- Section 3 is devoted to the study of asymptotic behavior in the case of global existence.

2 The equations

Greek indices α, β, $\lambda, \ldots$ range from 0 to 3, and Latin indices $i, j, k, \ldots$ from 1 to 3. We adopt the Einstein summation convention $a_\alpha b^\alpha \equiv \sum_{\alpha=0}^{3} a_\alpha b^\alpha$.

2.1 The Einstein-Maxwell system

We consider a kind of fast moving charged massive particles in the flat Robertson-Walker space-time $(\mathbb{R}^4, g)$ and denote by $x^\alpha = (x^0, x^i) = (t, x^i)$, the usual coordinates in $\mathbb{R}^4$; g stands for the unknown metric tensor of Lorentzian type with signature $(-,+,+,+)$ which can be written:

$$g = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{2.1}$$

where $a > 0$ is an unknown function of the single variable t, called the cosmological expansion factor. The Einstein-Maxwell system with cosmological constant can be written, following HAWKING and ELLIS [2]:

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} &= 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta}), & (2.2)\\ \nabla_\alpha F^{\alpha\beta} &= e u^\beta, & (2.3)\\ \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} &= 0, & (2.4) \end{aligned}$$

where:

- (2.2) are the Einstein equations which are the basic equations of General Relativity, for the metric tensor $g = (g_{\alpha\beta})$ which represents the gravitational field; $R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor; $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, contracted of the Ricci tensor, $(g^{\alpha\beta})$ being the inverse matrix of $(g_{\alpha\beta})$; Λ is a constant called the cosmological constant; $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are respectively the matter tensor and the Maxwell tensor we specify below.

- (2.3)-(2.4) are the two groups of the Maxwell equations written in covariant form, for the electromagnetic field $F = (F^{0i}, F_{ij})$, which is a closed unknown antisymmetric 2-form depending on the single variable t; F^{0i} and F_{ij} are respectively, its electric and magnetic parts. In (2.3), $e \geq 0$ is an unknown scalar function of the single variable t representing the charge density of the charged particles, and $u = (u^\alpha)$ is an unknown time-like future pointing vector; eu^β is the Maxwell current. Recall that ∇ is the Levi-Civita connection associated to g, and indices are raised and lowered following the rules: $V^\alpha = g^{\alpha\beta} V_\beta$; $V_\alpha = g_{\alpha\beta} V^\beta$.

Now by Poincare Lemma, the closed 2-form F is locally exact; but since $\mathbb{R}^4$ is simply connected, F is globally exact, see for instance MALLIAVIN [4]; so F globally derives on $\mathbb{R}^4$ from a potential vector A. F and A are linked by:

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha. \tag{2.5}$$

We take the potential A in temporal gauge which means:

$$A_0 = 0. \tag{2.6}$$

Now we deduce from (2.5) that: $\nabla_\alpha F^{\alpha\beta} = \nabla_\alpha \nabla^\alpha A^\beta - \nabla_\alpha \nabla^\beta A^\alpha$ which gives, using the usual formula $\nabla_\alpha \nabla^\beta A^\alpha - \nabla^\beta \nabla_\alpha A^\alpha = R_\alpha^{\ \beta} A^\alpha$:

$$\nabla_\alpha F^{\alpha\beta} = \nabla_\alpha \nabla^\alpha A^\beta - \nabla^\beta \nabla_\alpha A^\alpha - R_\alpha^{\ \beta} A^\alpha. \tag{2.7}$$

We next introduce $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$.

$T_{\alpha\beta}$ is taken in the general form:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + \Theta_{\alpha\beta}. \tag{2.8}$$

Where $\rho \geq 0$ is an unknown scalar function of t, representing the matter density of the charged particles; $\Theta_{\alpha\beta}$ is a symmetric 2-tensor called the "pressure pseudo-tensor", following the terminology of A.LICHNEROWICZ who introduced this notion in [3]; notice that the particular case $\Theta_{\alpha\beta}$=0 corresponds to the case of pure relativistic matter, and that $\Theta_{\alpha\beta} = p g_{\alpha\beta}$, where $p \geq 0$ is a scalar function representing the pressure, corresponds to the case of a perfect relativistic fluid of pure radiation type. We make the following assumptions on $\Theta_{\alpha\beta}$:

$$\begin{cases} \nabla_\alpha \Theta^{\alpha\beta} = \rho u^\beta; & (2.9) \\ g^{ij}\Theta_{ij} = 0. & (2.10) \end{cases}$$

Assumption (2.9) is also due to A.LICHNEROWICZ [3] and assumption (2.10) which means that the spatial part of Θ is traceless is not so restrictive since to every 2-tensor L_{ij} on $(\mathbb{R}^3, \overline{g})$ where $\overline{g}$=(g_{ij}), one can always associate the traceless tensor $\overline{L_{ij}}$= $L_{ij} - \frac{1}{3} g_{ij}(g^{km}L_{km})$.

$\tau_{\alpha\beta}$ is defined by:

$$\tau_{\alpha\beta} = -\frac{1}{4} g_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu} + F_{\beta\lambda} F_\alpha^{\ \lambda}. \tag{2.11}$$

$\tau_{\alpha\beta}$ has the useful property (see [3]):

$$\nabla_\alpha \tau^{\alpha\beta} = F^\beta_\lambda \nabla_\alpha F^{\alpha\lambda}. \tag{2.12}$$

Now recall that solving the Einstein equations is determining both the gravitational field and its sources. This means that we have to determine every unknown function introduced above, namely: a, F, e, A, ρ, u, and Θ. First of all, expression (2.1) of g gives:

$$\begin{cases} g_{00} = -1;\ g_{11} = g_{22} = g_{33} = a^2; \\ g^{00} = -1;\ g^{11} = g^{22} = g^{33} = \frac{1}{a^2}; \\ g_{\alpha\beta} = g^{\alpha\beta} = 0 \ for\ \alpha \neq \beta, \end{cases} \tag{2.13}$$

and the expression $\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}\left[\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right]$ of the Christoffel symbols gives directly:

$$\begin{cases} \Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \frac{\dot{a}}{a}; \\ \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \dot{a}a; \\ \Gamma^{\lambda}_{\alpha\beta} = 0 \text{ otherwise,} \end{cases} \tag{2.14}$$

where the dot stands for the derivative with respect to t. Also recall that $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$. Since u=(u^α) is a unit vector, we have $u^\alpha u_\alpha$=-1. Expression (2.1) of g then implies, since u is future pointing $i.e. u^0 \geq 0$:

$$u^0 = \sqrt{1 + a^2[(u^1)^2 + (u^2)^2 + (u^3)^2]}, \tag{2.15}$$

implying also that $u^0 \geq 1$, and in particular u^0 never vanishes.

1^0) **Determination of *e***

The Maxwell equation (2.3) for β=0 writes $\nabla_\alpha F^{\alpha 0} = eu^0$; but we have $\nabla_\alpha F^{\alpha 0} = \partial_\alpha F^{\alpha 0} + \Gamma^{\alpha}_{\alpha\sigma}F^{\sigma 0} + \Gamma^{0}_{\alpha\beta}F^{\alpha\beta}$=0 since F^{00}=0, F depends only on t, and using expression (2.14) of $\Gamma^{\lambda}_{\alpha\beta}$ and the fact that $\Gamma^{0}_{\alpha\beta}F^{\alpha\beta}$=0 since $\Gamma^{0}_{\alpha\beta} = \Gamma^{0}_{\beta\alpha}$ and $F^{\alpha\beta} = -F^{\beta\alpha}$. We then have eu^0=0. Then e=0, since u^0 never vanishes. As a consequence, the Maxwell equations (2.3) write:

$$\nabla_\alpha F^{\alpha\beta} = 0. \tag{2.16}$$

2^0) **Determination of *u* and ρ**

We use the fact that the well known identities $\nabla_\alpha(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta})$=0 impose, given the Einstein equations (2.2), that the source terms must always satisfy the conservation laws:

$$\nabla_\alpha T^{\alpha\beta} + \nabla_\alpha \tau^{\alpha\beta} = 0. \tag{2.17}$$

But using (2.12) and (2.16), we have $\nabla_\alpha \tau^{\alpha\beta}$=0. (2.17) then reduces to $\nabla_\alpha T^{\alpha\beta}$=0. Expression (2.8) of $T_{\alpha\beta}$ then gives, using assumption (2.9) on Θ: $\nabla_\alpha(\rho u^\alpha u^\beta) + \rho u^\beta$=0, which can be written:

$$u^\beta \nabla_\alpha(\rho u^\alpha) + \rho u^\alpha \nabla_\alpha u^\beta = -\rho u^\beta. \tag{2.18}$$

The contracted multiplication of (2.18) by u_β gives, using $u_\beta u^\beta$=-1, which implies $u_\beta \nabla_\alpha u^\beta = 0$:

$$\nabla_\alpha(\rho u^\alpha) = -\rho. \tag{2.19}$$

(2.18) then gives, using the expression of $\nabla_\alpha(\rho u^\alpha)$ provided by (2.19):

$$\rho u^\alpha \nabla_\alpha u^\beta = 0. \tag{2.20}$$

Computing the left hand side (l.h.s) of (2.19) gives, since ρ is a scalar function, the equation in ρ:

$$\dot{\rho} + (\frac{1 + \nabla_\alpha u^\alpha}{u^0})\rho = 0. \tag{2.21}$$

Solving (2.21) as an ordinary differential equation (o.d.e.) in ρ over $[0,t]$, $t > 0$, gives:

$$\rho = \rho(0)\exp\left(-\int_0^t (\frac{1+\nabla_\alpha u^\alpha}{u^0})(s)ds\right),$$

which shows that $(\rho(0) > 0) \Longrightarrow (\rho > 0)$. We will make the assumption that:

$$\rho(0) > 0. \tag{2.22}$$

So that we will always have:

$$\rho > 0. \tag{2.23}$$

(2.20) then gives, using (2.23), the differential system in u=(u^α):

$$u^\alpha \nabla_\alpha u^\beta = 0, \tag{2.24}$$

which shows that u=(u^α) is tangent to geodesics flow. So, if ρ and u satisfy (2.21) and (2.24), then the conservation laws (2.17) are satisfied.

Now using the formula $\nabla_\alpha u^\beta = \partial_\alpha u^\beta + \Gamma^\beta_{\alpha\lambda} u^\lambda$, (2.24) gives:

$$\dot{u}^\beta = -\Gamma^\beta_{\alpha\lambda} \frac{u^\lambda u^\alpha}{u^0}. \tag{2.25}$$

But we know by (2.15) that a and the u^i determine u^0. Then, (2.25) gives, using (2.14), the differential system in u^i:

$$\dot{u}^i + 2(\frac{\dot{a}}{a})u^i = 0; \;\; i = 1,2,3,$$

which solves at once over [0,t] to give:

$$u^i = u^i(0)\left(\frac{a(0)}{a}\right)^2; \;\; i = 1,2,3. \tag{2.26}$$

(2.26) shows that the cosmological expansion factor a determines the u^i, i=1,2,3. We are now able to state the final form of the equation in ρ. (2.25) gives for β=0, and using (2.14):

$$\dot{u}^0 = -\frac{\dot{a}a}{u^0}\sum_{i=1}^{3}\left(u^i\right)^2.$$

Now we have, using once more (2.14): $\nabla_\alpha u^\alpha$= $\dot{u}^0 + 3\frac{\dot{a}}{a}u^0$. We then deduce, using expression (2.15) of u^0 and expression (2.26) of u^i, that equation (2.21) in ρ can be written:

$$\begin{cases} \dot{\rho} = -\left[\dfrac{\dot{a}}{a}\left(2 + \dfrac{1}{1+\frac{k_0^2}{a^2}}\right) + \dfrac{1}{(1+\frac{k_0^2}{a^2})^{\frac{1}{2}}}\right]\rho; & (2.27) \\ \text{where:} k_0 = (a(0))^2\left[\displaystyle\sum_{i=1}^{3}\left(u^i(0)\right)^2\right]^{\frac{1}{2}}. & (2.28) \end{cases}$$

3^0) **Determination of F and A**

- Concerning F: the Maxwell equation (2.16) for β=i, i.e., $\nabla_\alpha F^{\alpha i}$=0, gives: $\nabla_\alpha F^{\alpha i}$= $\partial_\alpha F^{\alpha i} + \Gamma^\alpha_{\alpha\lambda}F^{\lambda i} + \Gamma^i_{\alpha\lambda}F^{\alpha\lambda} = \partial_0 F^{0i} + \Gamma^\alpha_{\alpha\lambda}F^{\lambda i}$=0 and, using (2.14) we obtain $\dot{F}^{0i} + 3\frac{\dot{a}}{a}F^{0i} = 0$, which solves directly over [0,t], $t > 0$ to give:

$$F^{0i} = F^{0i}(0)\left(\frac{a(0)}{a}\right)^3. \tag{2.29}$$

Observe that (2.5) gives $F^{\alpha\beta} = \nabla^\alpha A^\beta - \nabla^\beta A^\alpha$ and a direct calculation using the temporal gauge $A_0 = -A^0 = 0$ gives, using (2.14): $F^{0i} = -\dot{A}^i - 2\frac{\dot{a}}{a}A^i$; so that in (2.29) we have:

$$F^{0i}(0) = -\dot{A}^i(0) - 2\frac{\dot{a}(0)}{a(0)}A^i(0). \tag{2.30}$$

Next, (2.5) also gives $F_{\alpha\beta}$= $\partial_\alpha A_\beta$ - $\partial_\beta A_\alpha$; then, since A depends only on t, we have $\partial_i A_j - \partial_j A_i$=0, so that:

$$F_{ij} = 0, \tag{2.31}$$

which means that the electromagnetic field F reduces to its electric part F^{0i}. Also notice that the Maxwell equations (2.4) are identically satisfied since (2.4) is equivalent to:

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0,\, \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0,$$

relations always satisfied given (2.31) and the fact that F depends only on t.

- Concerning A: we deduce from the temporal gauge $A_0 = 0$, which implies A^0=0, and using (2.14), that: $\nabla_\alpha A^\alpha = \partial_\alpha A^\alpha + \Gamma^\alpha_{\alpha\lambda}A^\lambda = \partial_0 A^0 + \Gamma^\alpha_{\alpha i}A^i = 0$, which means that in the homogeneous case we consider, A also satisfies the LORENTZ gauge:

$$\nabla_\alpha A^\alpha = 0. \tag{2.32}$$

Then, since $A_0 = 0$ is known, (2.7), (2.16) and (2.32) give the second order differential system in A^i, which is a system of wave equations in a curved space-time:

$$\nabla_\alpha \nabla^\alpha A^i = R^i{}_j A^j.$$

A straight forward calculation yields, using the expression of the Ricci tensor

$$R_{\alpha\beta} = \left(\partial_\lambda \Gamma^\lambda_{\beta\alpha} - \partial_\beta \Gamma^\lambda_{\lambda\alpha}\right) + \left(\Gamma^\nu_{\beta\alpha}\Gamma^\lambda_{\lambda\nu} - \Gamma^\nu_{\lambda\alpha}\Gamma^\lambda_{\beta\nu}\right), \tag{2.33}$$

also using expression (2.14) of $\Gamma^\lambda_{\alpha\beta}$ and $\nabla_\alpha\nabla^\alpha = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, then gives the second order differential system in A^i, i= 1, 2, 3:

$$\ddot{A}^i + 5\left(\frac{\dot{a}}{a}\right)\dot{A}^i + 2\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right]A^i = 0. \tag{2.34}$$

4^0) **<u>Determination of Θ</u>**

Here we need the detailed expressions of each side of the Einstein equations (2.2). Let us introduce the Einstein tensor:

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}. \tag{2.35}$$

We obtain, using (2.13), (2.14), (2.33), also see [7] or [8]:

$$\begin{cases} S_{00} = 3\left(\frac{\dot{a}}{a}\right)^2; \\ S_{11} = S_{22} = S_{33} = -2a\ddot{a} - (\dot{a})^2; \\ S_{\alpha\beta} = 0 \ if \ \alpha \neq \beta. \end{cases} \tag{2.36}$$

Next, expression (2.11) of $\tau_{\alpha\beta}$ gives, using $F_{ij} = 0$ given by (2.31) and which implies $F^{ij} = g^{ik}g^{jl}F_{kl} = 0$:

$$\begin{cases} \tau_{00} = \frac{a^2}{2}\sum_{i=1}^{3}\left(F^{0i}\right)^2; \\ \tau_{0i} = 0; \\ \tau_{ii} = a^4\left(\frac{1}{2}\sum_{j=1}^{3}\left(F^{0j}\right)^2 - \left(F^{0i}\right)^2\right); \\ \tau_{ij} = -a^4F^{0i}F^{0j} \ \ if \ \ i \neq j. \end{cases} \tag{2.37}$$

Now, since $g_{\alpha\beta} = 0$ if $\alpha \neq \beta$, (2.36) gives, using (2.35): $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$ if $\alpha \neq \beta$. The Einstein equations (2.2) then impose that we must also have: $T_{0i} + \tau_{0i}$=0 and $T_{ij} + \tau_{ij}$=0 if $i \neq j$; expression (2.37) of $\tau_{\alpha\beta}$ then implies, using expression (2.8) of $T_{\alpha\beta}$, that we must have:

$$\begin{cases} \Theta_{0i} = -\rho u_0 u_i; \\ \Theta_{ij} = -\rho u_i u_j + a^4F^{0i}F^{0j} \ \ if \ \ i \neq j, \end{cases} \tag{2.38}$$

which shows, using $u_0 = -u^0$, $u_i = a^2u^i$, the expressions (2.26) and (2.29) of u^i and F^{0i}, that ρ and a determine Θ_{0i} and Θ_{ij} for $i \neq j$.

Next, since $g_{11} = g_{22} = g_{33} = a^2$, (2.36) shows that the Einstein equations (2.2) for $\alpha = \beta = i$ have exactly the same left hand side which is:

$$L = -2\ddot{a}a - (\dot{a})^2 + \Lambda a^2. \tag{2.39}$$

The Einstein equations (2.2) for $\alpha = \beta = i$ then impose that these equations must also have the same right hand side, i.e., using (2.8):

$$\begin{cases} \rho(u_1)^2 + \Theta_{11} + \tau_{11} = \rho(u_2)^2 + \Theta_{22} + \tau_{22}; \\ \rho(u_2)^2 + \Theta_{22} + \tau_{22} = \rho(u_3)^2 + \Theta_{33} + \tau_{33}. \end{cases} \tag{2.40}$$

The expressions (2.26) and (2.29) of u^i and F^{0i} show that if we take:

$$\begin{cases} F^{01}(0) = F^{02}(0) = F^{03}(0) \neq 0, \\ u^1(0) = u^2(0) = u^3(0), \end{cases} \tag{2.41}$$

then we will have:

$$\begin{cases} F^{01} = F^{02} = F^{03}, \\ u^1 = u^2 = u^3, \end{cases} \tag{2.42}$$

and since $u_i = a^2u^i$ and $F_{0i} = -a^2F^{0i}$, we will also have:

$$\begin{cases} F_{01} = F_{02} = F_{03}, \\ u_1 = u_2 = u_3. \end{cases} \tag{2.43}$$

The expression of τ_{ii} in (2.37) then implies, using (2.42) and (2.29):

$$\tau_{11} = \tau_{22} = \tau_{33} = \frac{a^6(0)\left(F^{01}(0)\right)^2}{a^2}. \tag{2.44}$$

(2.43) and (2.44) show that (2.40) reduces to:

$$\Theta_{11} = \Theta_{22} = \Theta_{33}. \tag{2.45}$$

Assumption (2.10) on Θ which gives $\Theta_{11} + \Theta_{22} + \Theta_{33} = 0$ then implies, using (2.45) that:

$$\Theta_{11} = \Theta_{22} = \Theta_{33} = 0. \tag{2.46}$$

In what follows, we assume that (2.41) holds. Notice that $F^{01}(0) \neq 0$ implies, given (2.29) that F^{01} never vanishes so that the electromagnetic field is not trivial. By (2.38) and (2.46) it appears that, between the ten independent components Θ_{00}, Θ_{0i}, Θ_{ij} of Θ, only Θ_{00} is still to determine. For this purpose, we consider assumption (2.9) on Θ for β=0; i.e: $\nabla_\alpha \Theta^{\alpha 0} = \rho u^0$. We have, using (2.14): $\nabla_\alpha \Theta^{\alpha 0} = \partial_0 \Theta^{00} + \Gamma^\alpha_{\alpha\lambda}\Theta^{\lambda 0} + \Gamma^0_{\alpha\lambda}\Theta^{\alpha\lambda} = \partial_0 \Theta^{00} + 3\frac{\dot a}{a}\Theta^{00}$, since by (2.46), $\Theta^{ii} = \frac{1}{a^4}\Theta_{ii} = 0$. Hence Θ^{00} satisfies:

$$\dot\Theta^{00} + 3\frac{\dot a}{a}\Theta^{00} = \rho u^0. \tag{2.47}$$

But $\Theta_{00} = \Theta^{00}$; then equation (2.47) determines Θ_{00}.

5^0) **<u>Determination of *a*</u>**

According to what we said above, a will be determined by the Einstein equations (2.2), which reduce, using (2.8), (2.46), the notation (2.35) to the two equations (2.2) for $\alpha = \beta$=0 and $\alpha = \beta$=1:

$$\begin{cases} S_{00} + \Lambda g_{00} = 8\pi(\rho(u_0)^2 + \Theta_{00} + \tau_{00}); \\ S_{11} + \Lambda g_{11} = 8\pi(\rho(u_1)^2 + \tau_{11}). \end{cases} \tag{2.48}$$

Now, using (2.13), $u_0 = -u^0$, $u_1 = a^2 u^1$, (2.26), (2.29), (2.37), (2.42), (2.39), system (2.48) can be written in terms of a:

$$\begin{cases} 3\left(\dfrac{\dot a}{a}\right)^2 - \Lambda = 8\pi\left[\rho(1+\dfrac{c_0^2}{a^2}) + \dfrac{d_0^2}{a^4} + \Theta_{00}\right]; & (2.49) \\ -2\dfrac{\ddot a}{a} - \left(\dfrac{\dot a}{a}\right)^2 + \Lambda = 8\pi\left[\dfrac{c_0^2}{3}\dfrac{\rho}{a^2} + \dfrac{d_0^2}{3a^4},\right], & (2.50) \end{cases}$$

where, using the expression of $|F^{01}(0)|$ provided by (2.30):

$$\begin{cases} c_0 = \sqrt{3}a^2(0)|u^1(0)|; \\ d_0 = \frac{\sqrt{6}}{2}a^3(0)|\dot A^1(0) + 2\frac{\dot a(0)}{a(0)}A^1(0)|. \end{cases} \tag{2.51}$$

Now, in order to simplify the problem, and given (2.30) and (2.41) we assume to take:

$$\begin{cases} A^i(0) = 0, \ \ i = 1,2,3; \\ \dot A^1(0) = \dot A^2(0) = \dot A^3(0) \neq 0. \end{cases} \tag{2.52}$$

By this assumption we justify in what follows, (2.51) gives:

$$d_0 = \frac{\sqrt{6}}{2}a^3(0)|\dot A^1(0)| > 0. \tag{2.53}$$

Notice that, given (2.52), the system (2.34) in A^i reduces to the single equation for i=1.

2.2 Cauchy problem and constraints

We have to study the coupled system (2.49)-(2.50)-(2.27)-(2.34)-(2.47) in (a, ρ, A^1, Θ_{00}) given that by (2.15), (2.26), (2.29), (2.31), (2.38), and (2.46) the solution will provide the unknown u, F, Θ_{0i}, Θ_{ij}.

Let the real numbers:

$$a_0 > 0,\ \ \rho_0 > 0,\ \ b_0, B_0^1 \neq 0,\ \ \omega_0, u_0^1 \tag{2.54}$$

be given. We look for solutions $(a, \rho, A^1, \Theta_{00})$ on [0,T[, $T \leq +\infty$, of (2.49)-(2.50)-(2.27)-(2.34)-(2.47) satisfying the relations:

$$\begin{cases} a(0) = a_0;\ \ \dot{a}(0) = b_0;\ \ \rho(0) = \rho_0;\ \ A^1(0) = 0; \\ \dot{A}^1(0) = B_0^1 \neq 0;\ \ \Theta^{00}(0) = \omega_0; \end{cases} \tag{2.55}$$

called the initial conditions, and such that, in (2.26) for i=1, $u^1(0) = u_0^1$ and in (2.29), given (2.30), (2.41) and (2.52), $F^{01}(0) = B_0^1$. This is the Cauchy problem for the considered system with the prescribed initial data (2.54). Notice that in (2.54) and (2.55) we take $B_0^1 \neq 0$ and $A^1(0) = 0$ according to (2.52).

Now, it is well known that equation (2.49) called the Hamiltonian constraint is satisfied in the whole domain of the solutions of equation (2.50) called the Einstein evolution equation, if and only if, (2.49) is satisfied for t=0. This means, using (2.55), if the initial data satisfy:

$$3\left(\frac{b_0}{a_0}\right)^2 - \Lambda = 8\pi\left[\rho_0(1 + \frac{c_0^2}{a_0^2}) + \frac{d_0^2}{a_0^4} + \omega_0\right], \tag{2.56}$$

called the initial constraint. It shows useful as we will see, to have in (2.49), $\Theta_{00} \geq 0$. But solving equation (2.47) in Θ^{00} over [0,t], $t > 0$, yields, using (2.55):

$$\Theta^{00}(t) = \left(\frac{a_0}{a(t)}\right)^3\left[\omega_0 + \int_0^t \left(\frac{a(s)}{a_0}\right)^3 (\rho u^0)(s)ds\right], \tag{2.57}$$

which shows, since $a > 0$, $\rho > 0$, $u^0 > 0$, that we will have $\Theta^{00} \geq 0$ if we take $\omega_0 \geq 0$. But notice that ω_0 cannot be chosen freely, since ω_0 is linked to the other initial data by the initial constraint (2.56). Also notice that by (2.51), (2.53) and $u^1(0) = u_0^1$, we have in (2.56): $c_0^2 = 3a_0^4(u_0^1)^2$ and $d_0^2 = \frac{3}{2}a_0^3(B_0^1)^2$. Now, if the constants Λ, ρ_0, a_0, u_0^1, B_0^1 are given, we can always choose $|b_0|$ sufficiently large to have $k_0 := 3\left(\frac{b_0}{a_0}\right)^2 - \Lambda - 8\pi\left[\rho_0(1 + \frac{c_0^2}{a_0^2}) + \frac{d_0^2}{a_0^4}\right] \geq 0$ and this allow to define the positive number $\omega_0 = \frac{k_0}{8\pi}$, so that (2.56) holds.

In what follows, we assume that the initial constraint (2.56) with $\omega_0 \geq 0$ holds, so that we will always have:

$$\Theta^{00} \geq 0. \tag{2.58}$$

Then, the Hamiltonian constraint (2.49) is always satisfied and we will use it as a property of the solutions of the evolution equation (2.50). Also notice that equality (2.56) makes sense only if: $\Lambda + 8\pi\left[\rho_0(1 + \frac{c_0^2}{a_0^2}) + \frac{d_0^2}{a_0^4}\right] + \omega_0 \geq$, which shows that the cosmological constant Λ has a lower bound which is strictly negative since $d_0 > 0$, and that, (2.56) provides two possible choices of $b_0 = \dot{a}(0)$, called the initial velocity of expansion; namely $b_0 \leq 0$ and $b_0 > 0$. As we will see, this choice will play a key role, as far as the global existence of solutions is concerned. Finally, notice that if we supposed $A^1(0) \neq 0$ in expression (2.51) of d_0, then the initial constraint (2.56) would turn out to be a quadratic equation in b_0, whose resolution would require more severe assumptions on the initial data than taking $A^1(0) = 0$ as assumed and adopted by (2.52).

2.3 Change of variables

Now, considering the whole system to study and in order to have an equivalent first order system to which the standard theory applies, we set:

$$\begin{cases} u = \frac{\dot{a}}{a}, \\ v = \frac{1}{a^2}, \\ \omega = \Theta^{00}, \\ \phi = A^1, \\ \psi = \dot{A}^1. \end{cases} \tag{2.59}$$

u is called the Hubble variable; (2.59) gives:

$$\begin{cases} \frac{\ddot{a}}{a} = \dot{u} + u^2, \\ \dot{v} = -2uv, \\ \dot{\omega} = \dot{\Theta}^{00}, \\ \dot{\phi} = \psi, \\ \dot{\psi} = \ddot{A}^1. \end{cases} \tag{2.60}$$

One then deduces from (2.49), (2.50), (2.27), (2.34) and (2.47) by a direct calculation, the equivalent first order autonomous differential system in u, v, ρ, ω, ϕ and ψ:

$$(S)\begin{cases} 3u^2 - \Lambda = 8\pi\left(\rho(1+c_0^2 v) + d_0^2 v^2 + \omega;\right) & (2.61) \\ \dot{u} = \dfrac{\Lambda}{2} - \dfrac{3}{2}u^2 - \dfrac{4\pi}{3}(c_0^2\rho v + d_0^2 v^2); & (2.62) \\ \dot{v} = -2uv; & (2.63) \\ \dot{\omega} = -3u\omega + \rho(1+c_0^2 v)^{\frac{1}{2}}; & (2.64) \\ \dot{\rho} = -\left[(1+c_0^2 v)^{-\frac{1}{2}} + u(2 + \dfrac{1}{1+c_0^2 v})\right]\rho; & (2.65) \\ \dot{\phi} = \psi; & (2.66) \\ \dot{\psi} = -5u\psi - \left[\Lambda + 3u^2 - \dfrac{8\pi}{3}(c_0^2\rho v + d_0^2 v^2)\right]\phi. & (2.67) \end{cases}$$

We study the Cauchy problem for (S) with initial conditions deduced from (2.55) and (2.59):

$$\begin{cases} u_0 := u(0) = \frac{b_0}{a_0}, v_0 := v(0) = \frac{1}{a_0^2}, \omega_0 := \omega(0) \geq 0, \\ \rho_0 := \rho(0), \phi_0 := \phi(0) = 0, \psi_0 := \psi(0) = B_0^1 \neq 0. \end{cases} \tag{2.68}$$

Remark 2.1. Solving equation (2.63) in v over [0,t], $t \geq 0$ gives, using (2.68): $v(t) = \frac{1}{a_0^2}\exp(-2\int_0^t u(s)ds) > 0$. Then also using (2.23), (2.58) and (2.59), we conclude that in the first order differential system (S) above, the solutions v, ρ and ω always satisfy

$$v > 0; \;\; \rho > 0; \;\; \omega \geq 0. \tag{2.69}$$

Remark 2.2. Equation (2.61) which involves no derivatives is the Hamiltonian constraint (2.49), written in terms of u, v, ρ and ω.

3 Global existence of solutions

Observe first of all that in the case $\Lambda \geq 0$, since $\rho_0 > 0$ and $\omega_0 \geq 0$, the initial constraint (2.56) gives, using (2.53): $3\left(\frac{b_0}{a_0}\right)^2 > \frac{8\pi d_0^2}{a_0^4} > 0$, and this implies $b_0 \neq 0$. So, we can only take $b_0 < 0$ or $b_0 > 0$. Consider the first order differential system (S) to study; its right hand side is a C^∞ function of the variable $(u, v, \rho, \omega, \phi, \psi) \in \mathbb{R}^6$ and is then locally lipschitzian with respect the $\mathbb{R}^6$-norm. The standard theory on the first order differential systems then guarantees the <u>local</u> existence of a unique solution to the Cauchy problem. The problem here is to prove, whether or not, this solution is <u>global</u>. We prove:

Proposition 3.1. 1^0) *If* $\Lambda < 0$, *then the coupled Einstein-Maxwell system has no global solution.*

2^0) *If* $\Lambda \geq 0$ *and* $b_0 < 0$, *then the coupled Einstein-Maxwell system has no global solution.*

Proof 1^0) Let $\Lambda < 0$ be given. Suppose the Einstein-Maxwell system has a global solution over $[0, +\infty[$. Then, by (2.50), a is of class C^2 and by (2.59), u is of class C^1 on $[0, +\infty[$. Now, equation (2.62) in u implies, since $\Lambda < 0$, and using (2.69):

$$\begin{cases} \dot{u} < -\frac{3}{2}u^2; \\ \dot{u} \leq \frac{\Lambda}{2}. \end{cases} \tag{3.1}$$

Integrating the second inequality (3.1) over [0,t], $t > 0$, yields $u(t) \leq u_0 + \frac{\Lambda}{2}t$, which implies that $u(t) \longrightarrow -\infty$ as $t \longrightarrow +\infty$, since $\Lambda < 0$. So, there exists $t_0 > 0$ such that $u(t_0) < 0$. But it is well known, given the first inequality (3.1), that if a function z satisfies:

$$\dot{z} = -\frac{3}{2}z^2; \;\; z(t_0) = u(t_0) < 0, \tag{3.2}$$

then u(t)$\leq z(t)$ for $t \geq t_0$; (3.2) shows that z is a decreasing function for $t \geq t_0$ and hence $z(t) \leq z(t_0) < 0$ for $t \geq t_0$; then, $z(t) \neq 0$ for $t \geq t_0$ and the equation in z separates and integrates over $[t_0, t]$ to give, using $z(t_0) = u(t_0)$: $z(t) = 2u(t_0)\left[2 + 3u(t_0)(t - t_0)\right]^{-1}$, which shows that $z(t) \longrightarrow -\infty$ as $t \longrightarrow_< t^* := t_0 - \frac{2}{3u(t_0)} > t_0$. Since $u(t) \leq z(t)$ for $t \geq t_0$, we conclude that $u(t) \longrightarrow -\infty$ as $t \longrightarrow_< t^*$. But this is impossible, since u being continuous over $[0, +\infty[$ should remain bounded on the line segment $[t, t^*]$. So there can exist no global solution if $\Lambda < 0$.

2^0) Let $\Lambda \geq 0$ and $b_0 < 0$ be given. Suppose the Einstein-Maxwell system has a global solution over $[0,+\infty[$; then for the same reasons as in 1^0), u is class of C^1 over $[0,+\infty[$. Now the Hamiltonian constraint (2.61) gives, since $\Lambda \geq 0$ and using (2.69) which gives $v > 0$:

$$3u^2 \geq 8\pi d_0^2 v^2 > 0. \tag{3.3}$$

(3.3) shows in particular that u never vanishes. So, since $b_0 < 0$, (2.68) gives $u(0) = u_0 = \frac{b_0}{a_0} < 0$, then $u < 0$, since u is continuous. Equation (2.63) in v then gives: $\dot{v} = (-2u)v > 0$, since $-u > 0$ and $v > 0$. So v is an increasing function and this implies $v \geq v_0 > 0$. We then deduce from (3.3) that:

$$3u^2 \geq 8\pi d_0^2 v_0^2 := \gamma_0 > 0. \tag{3.4}$$

Now (3.4) implies, since u is continuous: $u \geq \sqrt{\frac{\gamma_0}{3}}$ or $u \leq -\sqrt{\frac{\gamma_0}{3}}$. But we know that $u < 0$. We then have:

$$u \leq -\sqrt{\frac{\gamma_0}{3}}. \tag{3.5}$$

Equation (2.63) in v implies, using (3.5) and $v \geq v_0$:

$$\dot{v} = (-2u)v \geq 2\sqrt{\frac{\gamma_0}{3}} v_0 := h_0 > 0. \tag{3.6}$$

Integrating (3.6) over $[0,t]$ yields: $v(t) \geq h_0 t + v_0 > 0$ which implies that: $v^2(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$; (3.3) then implies that:

$$u^2(t) \longrightarrow +\infty \text{ as } t \longrightarrow +\infty. \tag{3.7}$$

Now equation (2.62) in u gives, since $\rho > 0$, $v > 0$:

$$\dot{u} < \frac{\Lambda}{2} - \frac{3}{2}u^2 = -\frac{1}{2}u^2 + (\frac{\Lambda}{2} - u^2). \tag{3.8}$$

But by (3.7) $\frac{\Lambda}{2} - u^2(t) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. So there exists $t_0 > 0$ such that $\frac{\Lambda}{2} - u^2(t) < 0$ for $t \geq t_0$. Then (3.8) implies: $\dot{u} < -\frac{1}{2}u^2$ in $[t_0,+\infty[$. We then also use (3.5) and proceed as in 1^0) to conclude to the non global existence. This completes the proof of proposition 2.1. Next we prove:

Theorem 3.2. *If $\Lambda \geq 0$ and $b_0 > 0$, then the coupled Einstein-Maxwell system has a global solution.*

Proof Also following the standard theory on the first order differential system, to prove that the coupled Einstein-Maxwell system has a global solution, it will be sufficient if we could prove that every solution of the Cauchy problem for the differential system (S) remains uniformly bounded on every bounded interval $[0,T]$, $0 < T < +\infty$.

So let $\Lambda \geq 0$ and $b_0 > 0$ be given. Since $\Lambda \geq 0$ (3.3) still holds and implies that u never vanishes. But since this time we have $b_0 > 0$, (2.68) gives $u(0) = u_0 = \frac{b_0}{a_0} > 0$, then $u > 0$ since u is continuous.

1^0) Using $u > 0$ and $v > 0$, equation (2.63) in v gives: $\dot{v} = -2uv < 0$. So v is a decreasing function and we have $0 < v \leq v_0$. Then v is bounded.

2^0) Using $u > 0$, $\rho > 0$ and $v > 0$, equation (2.65) in ρ implies $\dot\rho < 0$; so ρ is a decreasing function and (2.69) implies $0 < \rho \leq \rho_0$. Then ρ is bounded.

3^0) The Hamiltonian constraint (2.61) implies, since $\rho > 0$, $v > 0$ and $\omega \geq 0$: $3u^2 > \Lambda$; then $\frac{3u^2}{2} > \frac{\Lambda}{2}$ and $\frac{\Lambda}{2} - \frac{3u^2}{2} < 0$. Equation (2.62) in u then implies, since $\rho v > 0$, that $\dot u < 0$. So u is a decreasing function and since $u > 0$, we have: $0 < u \leq u_0$. Then u is bounded.

4^0) Equation (2.64) in ω implies, using $0 < u \leq u_0$, $0 < v \leq v_0$, $0 < \rho \leq \rho_0$ and $\omega \geq 0$: $|\dot\omega| \leq 3u_0\omega + p_0$, where $p_0 = \rho_0(1 + c_0^2 v_0)^{\frac{1}{2}}$. So we have by integration over $[0,t]$:

$$\omega(t) \leq (\omega_0 + p_0 T) + 3u_0 \int_0^t \omega(s)ds, \ \ t \in [0,T]. \tag{3.9}$$

But by Gronwall lemma, we have $\omega(t) \leq z(t)$ where z satisfies the integral equation:

$$z(t) = (\omega_0 + p_0 T) + 3u_0 \int_0^t z(s)ds, \ \ t \in [0,T]. \tag{3.10}$$

(3.10) is equivalent to: $\dot z(t) = 3u_0 z(t)$; $z(0) = \omega_0 + p_0 T$, $(0 \leq t \leq T)$, which gives:

$$z(t) = (\omega_0 + p_0 T)\exp 3u_0 t \leq (\omega_0 + p_0 T)\exp 3u_0 T.$$

Then we have: $0 \leq \omega(t) \leq (\omega_0 + p_0 T)\exp 3u_0 T$. Then ω is uniformly bounded on [0,T].

5^0) Equation (2.67) in ψ gives, using the above results:

$$|\dot\psi| \leq 5u_0|\psi| + Q_0|\phi|, \tag{3.11}$$

where $Q_0 = \Lambda + 3u_0^2 + \frac{8\pi}{3}(c_0^2\rho_0 v_0 + d_0^2 v_0^2) > 0$. And then, add (3.11) and $|\dot\phi| = |\psi|$ provided by equation (2.66) in ϕ and integrate over [0,t] to obtain, using $\phi(0) = 0$:

$$|\phi(t)| + |\psi(t)| \leq |\psi_0| + (5u_0 + Q_0 + 1)\int_0^t (|\phi(s)| + |\psi(s)|)ds, \ \ t \in [0,T].$$

Then, proceeding as for (3.9) we obtain: $|\phi(t)| + |\psi(t)| \leq |\psi_0|\exp[(5u_0 + Q_0 + 1)T]$ $(t \in [0,T])$, which shows that ϕ and ψ are uniformly bounded on [0,T]. We then conclude to the global existence of solutions over $[0,+\infty[$. This completes the proof of theorem 2.2.

4 Asymptotic behavior

We prove:

Theorem 4.1. 1) *If $\Lambda \geq 0$ and $b_0 > 0$, then the space-time which exists globally tends to the vacuum at late times. In the case $\Lambda = 0$, suppose in addition that $\omega_0 > 0$.*

2) *If $\Lambda > 0$ and $b_0 > 0$, then the mean curvature of the space-time admits a strictly positive limit at late times.*

1) We have to show, using expression (2.8) of $T_{\alpha\beta}$, that:

$$(\rho u_\alpha u_\beta)(t) + \Theta_{\alpha\beta}(t) + \tau_{\alpha\beta}(t) \longrightarrow 0 \ \ as \ \ t \longrightarrow +\infty. \tag{4.1}$$

a) Suppose $\Lambda > 0$ and $b_0 > 0$.

Recall that $b_0 > 0$ implies $u = \frac{\dot{a}}{a} > 0$; so the Hamiltonian constraint (2.49) implies, using (2.69): $3\left(\frac{\dot{a}}{a}\right)^2 > \Lambda > 0$, from where we deduce: $\frac{\dot{a}}{a} > \sqrt{\frac{\Lambda}{3}} > 0$. Integrating over [0,t] yields:

$$a(t) \geq a_0 \exp(t\sqrt{\frac{\Lambda}{3}}), \tag{4.2}$$

which shows that $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$ with an exponential growth. Let us show that each term in (4.1) tends to zero at late times.

i) Expression (2.37) of $\tau_{\alpha\beta}$ gives, using expression (2.29) of F^{0i}:

$$|\tau_{00}| \leq \frac{K}{a^4};\ \ \tau_{0i} = 0;\ \ |\tau_{ij}| \leq \frac{K}{a^2};\ \ i,j = 1,2,3, \tag{4.3}$$

where K is an absolute constant depending only on the initial data. (4.2) then implies $\tau_{\alpha\beta}(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

ii) We deduce from equation (2.65) in ρ, using $v > 0$ and $u = \frac{\dot{a}}{a}$, that: $\dot{\rho} \leq -2\frac{\dot{a}}{a}\rho$, which integrates at once over [0,t] $(t > 0)$ to give, using $\rho > 0$:

$$0 < \rho \leq \rho_0\left(\frac{a_0}{a}\right)^2. \tag{4.4}$$

Now expression (2.15) of u^0, expression (2.26) of u^i give, using $u_0 = -u^0$, and $u_i = a^2 u^i$:

$$|u_0| \leq \left(1 + \frac{c_0^2}{a^2}\right)^{\frac{1}{2}};\ \ |u_i| \leq K_i, \tag{4.5}$$

where $K_i \geq 0$ is a constant depending only on the initial data; (4.4) and (4.5) give:

$$\begin{cases} \rho u_0^2 \leq \rho_0\left(\frac{a_0}{a}\right)^2\left(1 + \frac{c_0^2}{a^2}\right); \\ |\rho u_0 u_i| \leq \rho_0 K_i\left(\frac{a_0}{a}\right)^2\left(1 + \frac{c_0^2}{a^2}\right)^{\frac{1}{2}}; \\ |\rho u_i u_j| \leq K_i K_j \rho_0\left(\frac{a_0}{a}\right)^2. \end{cases} \tag{4.6}$$

(4.6) shows, using (4.2), that $\rho u_\alpha u_\beta)(t) \longrightarrow 0$ as $t \longrightarrow +\infty$

iii) Expression (2.38) of Θ_{0i} and Θ_{ij} give, using (4.6) and expression (2.29) of F^{0i}:

$$\begin{cases} |\Theta_{0i}| \leq K_i \rho_0\left(\frac{a_0}{a}\right)^2\left(1 + \frac{c_0^2}{a^2}\right)^{\frac{1}{2}}; \\ |\Theta_{ij}| \leq K_i K_j \rho_0\left(\frac{a_0}{a}\right)^2 + \frac{c_{ij}}{a^2},\ \ i \neq j, \end{cases} \tag{4.7}$$

where $c_{ij} \geq 0$ is a constant depending only on the initial data. Now concerning Θ_{00}, we first deduce from (4.5) using $\dot{a} > 0$ which implies $a \geq a_0$, that:

$$u^0 = |-u_0| \leq \left(1 + \frac{c_0^2}{a_0^2}\right)^{\frac{1}{2}}.$$

The expression (2.57) of $\Theta^{00} = \Theta_{00}$ then implies, using (4.4):

$$\Theta^{00}(t) \leq \left(\frac{a_0}{a(t)}\right)^3\left[\omega_0 + M_0\int_0^t a(s)ds\right], \tag{4.8}$$

where $M_0 = \frac{\rho_0}{a_0}\left(1+\frac{c_0^2}{a_0^2}\right)^{\frac{1}{2}}$. But since a is an increasing function, we have $a(s) \leq a(t)$ for $s \in [0,t]$, so that (4.8) gives:

$$\Theta^{00}(t) \leq \frac{\omega_0 a_0^3}{a^3(t)} + a_0^3 M_0 \frac{t}{a^2(t)}. \tag{4.9}$$

But (4.2) implies that $\frac{t}{a^2(t)} \longrightarrow 0$ as $t \longrightarrow +\infty$. We then deduce from (4.7), (4.9) and (2.46) which gives $\Theta_{11} = \Theta_{22} = \Theta_{33} = 0$, that $\Theta_{\alpha\beta} \longrightarrow 0$ as $t \longrightarrow +\infty$, and (4.1) follows.

$b)$ Suppose $\Lambda = 0$ and $b_0 > 0$. We then add in this case $\omega_0 > 0$. Expression (2.57) of Θ^{00} then implies, since $\rho u^0 > 0$ that:

$$\Theta^{00}(t) > \frac{a_0^3}{a^3}\omega_0 > 0. \tag{4.10}$$

Now the Hamiltonian constraint (2.49) with $\Lambda = 0$ gives, using $\rho > 0$:

$$3\left(\frac{\dot{a}}{a}\right)^2 > 8\pi\Theta^{00} > \Theta^{00}.$$

(4.10) then implies: $3\left(\frac{\dot{a}}{a}\right)^2 > \frac{a_0^3}{a^3}\omega_0 > 0$. We then deduce, since

$$(b_0 > 0) \Longrightarrow (\dot{a} > 0)$$

that:

$$\dot{a}a^{\frac{1}{2}} > \delta_0 > 0, \tag{4.11}$$

where $\delta_0 = \left(\frac{a_0^3\omega_0}{3}\right)^{\frac{1}{2}} > 0$. Observe that $\dot{a}a^{\frac{1}{2}} = \frac{2}{3}\dot{\widehat{a^{\frac{3}{2}}}}$, then integrating (4.11) over [0,t], $t > 0$, yields:

$$a(t) > \left(\frac{3}{2}\delta_0\right)^{\frac{2}{3}} t^{\frac{2}{3}}. \tag{4.12}$$

(4.12) shows that $a(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ with a slow growth. The proof of (4.1) in the case $\Lambda = 0$, $b_0 > 0$, $\omega_0 > 0$ then follows the same lines as the case $\Lambda > 0$, $b_0 > 0$, except for Θ^{00} for which (4.9), which remains valid, implies this time, using (4.12):

$$\Theta^{00}(t) \leq \frac{\omega_0 a_0^3}{a^3(t)} + \frac{\tilde{M}_0}{t^{\frac{1}{3}}}, \tag{4.13}$$

where $\tilde{M}_0 = a_0^3 M_0\left(\frac{3}{2}\delta_0\right)^{-\frac{4}{3}}$; (4.13) shows that $\Theta^{00}(t) = \Theta_{00}(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ and this completes the proof of point $1)$ in theorem (4.1).

$2)$ Suppose $\Lambda > 0$ and $b_0 > 0$.

The mean curvature of the space-time is a scalar function H defined by $H = -g^{ij}K_{ij}$, where K_{ij} is the second fundamental form defined in the present case by $K_{ij} = -\frac{1}{2}\dot{g}_{ij}$. Expression (2.13) of $g_{\alpha\beta}$ and $g^{\alpha\beta}$ gives: $K_{11} = K_{22} = K_{33} = -\dot{a}a$; $K_{ij} = 0$ for $i \neq j$. We then have $g^{ii}K_{ii} = -3\frac{\dot{a}}{a}$ and $H = 3\frac{\dot{a}}{a}$. We deduce from the Hamiltonian constraint (2.49), since by (4.2), (4.4) and (4.9) we have: $\frac{1}{a^2} \longrightarrow 0$, $\rho \longrightarrow 0$, $\Theta_{00} \longrightarrow 0$ as $t \longrightarrow +\infty$:

$$\left(\frac{\dot{a}}{a}\right)^2(t) - \frac{\Lambda}{3} = \left(\frac{\dot{a}}{a} - \sqrt{\frac{\Lambda}{3}}\right)\left(\frac{\dot{a}}{a} + \sqrt{\frac{\Lambda}{3}}\right)(t) \longrightarrow 0 \ \ as \ \ t \longrightarrow +\infty. \tag{4.14}$$

(4.14) implies:

$$\begin{cases} \left(9\left(\frac{\dot{a}}{a}\right)^2 - 3\Lambda\right)(t) = \left(3\frac{\dot{a}}{a} - \sqrt{3\Lambda}\right)\left(3\frac{\dot{a}}{a} + \sqrt{3\Lambda}\right)(t) = \\ (H - \sqrt{3\Lambda})\left(3\frac{\dot{a}}{a} + \sqrt{3\Lambda}\right)(t) \longrightarrow 0 \ \ as \ \ t \longrightarrow +\infty. \end{cases} \tag{4.15}$$

But since $\frac{\dot{a}}{a} > 0$ we have $3\frac{\dot{a}}{a} + \sqrt{3\Lambda} > \sqrt{3\Lambda} > 0$. (4.15) then implies that $H(t) \longrightarrow \sqrt{3\Lambda}$ as $t \longrightarrow +\infty$, which means that the mean curvature H admits the strictly positive limit $\sqrt{3\Lambda}$ at late times. This completes the proof of theorem 3.1.

Remark 4.2. If $\Lambda > 0$ and $b_0 > 0$, this property of H together with (4.2) which shows an exponential growth of the cosmological expansion factor a, confirm from a mathematical point of view, the acceleration phenomenon of the expansion of the universe. This recalls somewhat the Hubble low according to which stars constantly go away from each other.

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