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EXISTENCE RESULTS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH FOUR-POINT NONLOCAL TYPE INTEGRAL BOUNDARY CONDITIONS

BASHIR AHMAD*
Department of Mathematics, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, SAUDI ARABIA

SOTIRIS K. NTOUYAS†
Department of Mathematics, University of Ioannina,
451 10 Ioannina, GREECE

Abstract

In this paper, we investigate some new existence results for nonlinear fractional differential equations of order $q \in (1,2]$ with four-point nonlocal integral boundary conditions by applying standard fixed point theorems and Leray-Schauder degree theory. Our results are new in the sense that the nonlocal parameters in four-point integral boundary conditions for the problem appear in the integral part of the conditions in contrast to the available literature on four-point fractional boundary value problems which deals with the four-point boundary conditions restrictions on the solution or gradient of the solution of the problem. Some illustrative examples are presented.

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1 Introduction

Boundary value problems for nonlinear fractional differential equations have recently been studied by several researchers. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics,

*E-mail address: bashir_qau@yahoo.com
†E-mail address: sntouyas@uoi.gr

chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [16, 18, 19, 20]. Some recent work on boundary value problems of fractional order can be found in [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 15, 21, 22] and the references therein.

In this paper, we consider a boundary value problem of nonlinear fractional differential equations of order $q \in (1,2]$ with four-point integral boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ \ 0 < t < 1, \ \ 1 < q \le 2, \\ x(0) = \alpha \displaystyle\int_0^{\xi} x(s)ds, \quad x(1) = \beta \displaystyle\int_0^{\eta} x(s)ds, \ \ 0 < \xi, \ \eta < 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times X \to X$ is continuous and $\alpha, \beta \in \mathbb{R}$. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathcal{C} = \mathrm{C}([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [5, 6, 14] and references therein. It has been observed that the limits of integration in the integral part of the boundary conditions are taken to be fixed, for instance, from 0 to 1 in case the independent variable belongs to the interval $[0,1]$. In the present study, we have introduced a nonlocal type of integral boundary conditions with limits of integration involving the parameters $0 < \xi$, $\eta < 1$. It is imperative to note that the available literature on nonlocal boundary conditions is confined to the nonlocal parameters involvement in the solution or gradient of the solution of the problem.

The aim of our paper is to present some existence results for the problem (1.1). The first result relies on the Banach contraction principle. In the second result, we apply a fixed point theorem due to Krasnoselskii, while the third result is based on nonlinear alternative of Leray-Schauder type. The methods used are standard, however their exposition in the framework of problem (1.1) is new.

2 Preliminaries

Let us recall some basic definitions of fractional calculus [16, 18, 20].

Definition 2.1. For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \ \ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \ \ q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a continuous function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \ \ n = [q]+1,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Lemma 2.4. *([16]) For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [q]+1$).

In view of Lemma 2.4, it follows that

$$I^q \, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [q]+1$).

Lemma 2.5. *Let $g : [0,1] \to \mathbb{R}$ be a given continuous function. Then a unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = g(t), \ \ 0 < t < 1, \ \ 1 < q \leq 2, \\ x(0) = \alpha \displaystyle\int_0^{\xi} x(s)ds, \quad x(1) = \beta \int_0^{\eta} x(s)ds, \ \ 0 < \xi, \ \eta < 1, \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} x(t) \ = \ & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds \\ & + \frac{\alpha}{\gamma\Gamma(q)} \left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)t \right) \int_0^{\xi} \left(\int_0^s (s-m)^{q-1} g(m) dm \right) ds \\ & + \frac{\beta}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t \right) \int_0^{\eta} \left(\int_0^s (s-m)^{q-1} g(m) dm \right) ds \\ & - \frac{1}{\gamma\Gamma(q)} \left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t \right) \int_0^1 (1-s)^{q-1} g(s) ds, \end{aligned} \tag{2.3}$$

where

$$\gamma = \frac{1}{2}[(\alpha\xi - 1)(\beta\eta^2 - 2) - \alpha\xi^2(\beta\eta - 1)] \neq 0.$$

Proof. In view of Lemma 2.4, for some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q g(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)ds - c_0 - c_1 t. \tag{2.4}$$

Using the boundary conditions for (2.2), we find that

$$(\alpha\xi - 1)c_0 + \alpha\frac{\xi^2}{2}c_1 = \alpha A, \tag{2.5}$$

$$(\beta\eta - 1)c_0 + \left(\frac{\beta\eta^2}{2} - 1\right)c_1 = \beta B - C, \tag{2.6}$$

where

$$\begin{aligned} A &= \frac{1}{\Gamma(q)}\int_0^\xi \left(\int_0^s (s-m)^{q-1} g(m)dm\right) ds, \\ B &= \frac{1}{\Gamma(q)}\int_0^\eta \left(\int_0^s (s-m)^{q-1} g(m)dm\right) ds, \\ C &= \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1} g(s)ds. \end{aligned}$$

Solving (2.5) and (2.6) for c_0 and c_1, we have that

$$c_0 = \frac{1}{\gamma}\left[\left(\frac{\alpha\beta\eta^2}{2} - \alpha\right)A - \frac{\alpha\beta\xi^2}{2}B + \frac{\alpha\xi^2}{2}C\right]$$

and

$$c_1 = \frac{1}{\gamma}\left[\beta(\alpha\xi - 1)B - (\alpha\xi - 1)C - \alpha(\beta\eta - 1)A\right].$$

Substituting the values of c_0 and c_1 in (2.4), we obtain (2.3). □

In view of Lemma 2.5, we define an operator $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} (\mathbf{F}x)(t) = {} & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds \\ & + \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2} + (\beta\eta - 1)t\right)\int_0^\xi \left(\int_0^s (s-m)^{q-1} f(m,x(m))dm\right) ds \\ & + \frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m,x(m))dm\right) ds \\ & - \frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2} + (1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1} f(s,x(s))ds, \quad t \in [0,1]. \end{aligned}$$

For the sequel, we need the following assumptions:

$(\mathbf{A_1})$ $\|f(t,x) - f(t,y)\| \le L\|x-y\|, \forall t \in [0,1],\ L > 0,\ x,y \in X$;

$(\mathbf{A_2})$ $\|f(t,x)\| \le \mu(t),\ \ \forall (t,x) \in [0,1] \times X$, and $\mu \in L^1([0,1], \mathbb{R}^+)$.

For convenience, let us set

$$\Lambda = \frac{1}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right), \tag{2.7}$$

where

$$\Lambda_1 = |\alpha|(|2-\beta\eta^2|+2|\beta\eta-1|)\xi^{q+1},$$

and

$$\Lambda_2 = (|\alpha|\xi^2+2|1-\xi\alpha|)(|\beta|\eta^{q+1}+q+1).$$

3 Existence results in Banach space

Theorem 3.1. *Assume that $f:[0,1]\times X\to X$ is a jointly continuous function and satisfies the assumption (A_1) with $L<1/\Lambda$, where Λ is given by (2.7). Then the boundary value problem (1.1) has a unique solution.*

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)|=M$ and choosing $r\geq\dfrac{\Lambda M}{1-L\Lambda}$, we show that $\mathbf{F}B_r\subset B_r$, where $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$. For $x\in B_r$, we have:

$$\begin{aligned}
&\|(\mathbf{F}x)(t)\|\\
\leq\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s))\|ds\\
&+\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^\xi\left(\int_0^s (s-m)^{q-1}\|f(m,x(m))\|dm\right)ds\\
&+\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^\eta\left(\int_0^s (s-m)^{q-1}\|f(m,x(m))\|dm\right)ds\\
&+\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^1 (1-s)^{q-1}\|f(s,x(s))\|ds\\
\leq\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|)ds\\
&+\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^\xi\Big(\int_0^s (s-m)^{q-1}(\|f(m,x(m))-f(m,0)\|\\
&+\|f(m,0)\|)dm\Big)ds\\
&+\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^\eta\Big(\int_0^s (s-m)^{q-1}(\|f(m,x(m))-f(m,0)\|\\
&+\|f(m,0)\|)dm\Big)ds\\
&+\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^1 (1-s)^{q-1}(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|)ds\\
\leq\ & (Lr+M)\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\\
&+\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^\xi\Big(\int_0^s (s-m)^{q-1}dm\Big)ds
\end{aligned}$$

$$
\begin{aligned}
& +\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^\eta\Big(\int_0^s(s-m)^{q-1}dm\Big)ds \\
& +\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^1(1-s)^{q-1}ds\Big] \\
\leq\ & \frac{(Lr+M)}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right) \\
=\ & (Lr+M)\Lambda\leq r.
\end{aligned}
$$

Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we obtain

$$
\begin{aligned}
& \|(\mathbf{F}x)(t)-(\mathbf{F}y)(t)\| \\
\leq\ & \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|f(s,x(s))-f(s,y(s))\|ds \\
& +\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^\xi\left(\int_0^s(s-m)^{q-1}\|f(m,x(m))-f(m,y(m))\|dm\right)ds \\
& +\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^\eta\left(\int_0^s(s-m)^{q-1}\|f(m,x(m))-f(m,y(m))\|dm\right)ds \\
& +\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^1(1-s)^{q-1}\|f(s,x(s))-f(s,y(s))\|ds \\
\leq\ & L\|x-y\|\Big[\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds \\
& +\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^\xi\Big(\int_0^s(s-m)^{q-1}dm\Big)ds \\
& +\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^\eta\Big(\int_0^s(s-m)^{q-1}dm\Big)ds \\
& +\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^1(1-s)^{q-1}ds\Big] \\
\leq\ & \frac{L}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right)\|x-y\| \\
=\ & L\Lambda\|x-y\|,
\end{aligned}
$$

where Λ is given by (2.7). Observe that Λ depends only on the parameters involved in the problem. As $L<1/\Lambda$, therefore $\mathbf{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Our next existence result is based on Krasnoselskii's fixed point theorem [17].

Theorem 3.2. *(Krasnoselskii's fixed point theorem). Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be the operators such that (i) $Ax+By\in M$ whenever $x,y\in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 3.3. *Let $f:[0,1]\times X\to X$ be a jointly continuous function mapping bounded subsets of $[0,1]\times X$ into relatively compact subsets of X, and the assumptions (A_1) and (A_2) hold with*

$$
\frac{L}{\Gamma(q+1)}\left(\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right)<1. \tag{3.1}
$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$\bar{r} \geq \frac{\|\mu\|}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right),$$

and consider $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \leq \bar{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}}$ as

$$\begin{aligned}
(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds, \\
(\mathcal{Q}x)(t) &= \frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^\xi \Big(\int_0^s (s-m)^{q-1} f(m,x(m))dm\Big)ds \\
&\quad +\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^\eta \Big(\int_0^s (s-m)^{q-1} f(m,x(m))dm\Big)ds \\
&\quad -\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1} f(s,x(s))ds.
\end{aligned}$$

For $x,y \in B_{\bar{r}}$, we find that

$$\begin{aligned}
\|\mathcal{P}x+\mathcal{Q}y\| &\leq \frac{\|\mu\|}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right) \\
&\leq \bar{r}.
\end{aligned}$$

Thus, $\mathcal{P}x+\mathcal{Q}y \in B_{\bar{r}}$. It follows from the assumption (A_1) together with (3.1) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{P}x\| \leq \frac{\|\mu\|}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\bar{r}}}|f(t,x)| = \overline{f}$, and consequently we have

$$\begin{aligned}
\|(\mathcal{P}x)(t_1)-(\mathcal{P}x)(t_2)\| &= \Bigg\|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s))ds \\
&\quad +\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\Bigg\| \\
&\leq \frac{\overline{f}}{\Gamma(q+1)}|2(t_2-t_1)^q+t_1^q-t_2^q|,
\end{aligned}$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we have that $\mathcal{P}(\mathcal{A})(t)$ is relatively compact in X for every t, where $\mathcal{A}$ is a bounded subset of $\mathcal{C}$. So $\mathcal{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 3.2 are satisfied. So the conclusion of Theorem 3.2 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

4 Existence of solution via Leray-Schauder degree theory

Theorem 4.1. *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$. Assume that there exist constants $0\le\kappa<\frac{1}{\Lambda}$, where Λ is given by (2.7) and $M>0$ such that $|f(t,x)|\le\kappa|x|+M$ for all $t\in[0,1], x\in C[0,1]$. Then the boundary value problem (1.1) has at least one solution.*

Proof. Lets us define an operator $\mathcal{F} : C[0,1]\to C[0,1]$ as

$$x=\mathcal{F}(x), \tag{4.1}$$

where

$$\begin{aligned}(\mathcal{F}x)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,x(s))ds\\ &+\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\int_0^{\xi}\left(\int_0^s (s-m)^{q-1}f(m,x(m))dm\right)ds\\ &+\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^{\eta}\left(\int_0^s (s-m)^{q-1}f(m,x(m))dm\right)ds\\ &-\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\int_0^1 (1-s)^{q-1}f(s,x(s))ds.\end{aligned}$$

In view of the fixed point problem (4.1), we just need to prove the existence of at least one solution $x\in C[0,1]$ satisfying (4.1). Define a suitable ball $B_R\subset C[0,1]$ with radius $R>0$ as

$$B_R=\left\{x\in C[0,1] : \max_{t\in[0,1]}|x(t)|<R\right\},$$

where R will be fixed later. Then, it is sufficient to show that $\mathcal{F} : \overline{B}_R\to C[0,1]$ satisfies

$$x\neq\lambda\mathcal{F}x,\ \ \forall\, x\in\partial B_R \ \text{ and } \ \forall\,\lambda\in[0,1]. \tag{4.2}$$

Let us set

$$H(\lambda,x)=\lambda\mathcal{F}x,\ \ x\in C(\mathbb{R})\ \ \lambda\in[0,1].$$

Then, by the Arzelá-Ascoli Theorem, $h_\lambda(x)=x-H(\lambda,x)=x-\lambda\mathcal{F}x$ is completely continuous. If (4.2) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda,B_R,0) &= \deg(I-\lambda\mathcal{F},B_R,0)=\deg(h_1,B_R,0)\\ &= \deg(h_0,B_R,0)=\deg(I,B_R,0)=1\neq0,\ \ 0\in B_r,\end{aligned}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(t)=x-\lambda\mathcal{F}x=0$ for at least one $x\in B_R$. In order to prove (4.2), we assume that $x=\lambda\mathcal{F}x$ for some $\lambda\in[0,1]$ and for all $t\in[0,1]$ so that

$$\begin{aligned}|x(t)| &= |\lambda(\mathcal{F}x)(t)|\\ &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))|ds\end{aligned}$$

$$
\begin{aligned}
& +\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^{\xi}\left(\int_0^s (s-m)^{q-1}|f(m,x(m))|dm\right)ds \\
& +\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^{\eta}\left(\int_0^s (s-m)^{q-1}|f(m,x(m))|dm\right)ds \\
& +\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^{1}(1-s)^{q-1}|f(s,x(s))|ds \\
\leq\ & (\kappa|x|+M)\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
& +\left|\frac{\alpha}{\gamma\Gamma(q)}\left(\frac{2-\beta\eta^2}{2}+(\beta\eta-1)t\right)\right|\int_0^{\xi}\left(\int_0^s (s-m)^{q-1}dm\right)ds \\
& +\left|\frac{\beta}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^{\eta}\left(\int_0^s (s-m)^{q-1}dm\right)ds \\
& +\left|\frac{1}{\gamma\Gamma(q)}\left(\frac{\alpha\xi^2}{2}+(1-\xi\alpha)t\right)\right|\int_0^{1}(1-s)^{q-1}ds\Big] \\
\leq\ & \frac{\kappa|x|+M}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right) \\
=\ & (\kappa|x|+M)\Lambda,
\end{aligned}
$$

which, on taking norm ($\sup_{t\in[0,1]}|x(t)|=\|x\|$) and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\Lambda}{1-\kappa\Lambda}.$$

Letting $R=\dfrac{M\Lambda}{1-\kappa\Lambda}+1$, (4.2) holds. This completes the proof. □

5 Examples

Example 5.1. Consider the following four-point integral fractional boundary value problem

$$
\begin{cases}
{}^cD^{3/2}x(t)=\dfrac{1}{(t+9)^2}\dfrac{\|x\|}{1+\|x\|},\ \ t\in[0,1], \\
x(0)=\dfrac{1}{2}\displaystyle\int_0^{1/4}x(s)ds, \quad x(1)=\int_0^{3/4}x(s)ds.
\end{cases}
\tag{5.1}
$$

Here, $q=3/2,\ \alpha=1/2,\ \beta=1,\ \xi=1/4,\ \eta=3/4$, and $f(t,x)=\dfrac{1}{(t+2)^2}\dfrac{\|x\|}{1+\|x\|}$. As $\|f(t,x)-f(t,y)\|\leq\dfrac{1}{4}\|x-y\|$, therefore, (A_1) is satisfied with $L=\dfrac{1}{4}$. Further, $\gamma=81/128$, and

$$
\begin{aligned}
L\Lambda &= \frac{L}{\Gamma(q+1)}\left(1+\frac{\Lambda_1+\Lambda_2}{2|\gamma|(q+1)}\right) \\
&= \frac{1}{180\sqrt{\pi}}\left(19\sqrt{3}+\frac{7831}{27}\right)=0.50611188<1.
\end{aligned}
$$

Thus, by the conclusion of Theorem 3.1, the boundary value problem (5.1) has a unique solution on $[0,1]$.

Example 5.2. Consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{1}{(4\pi)}\sin(2\pi x) + \dfrac{|x|}{1+|x|}, \quad t \in [0,1], \\ x(0) = \displaystyle\int_0^{1/3} x(s)ds, \quad x(1) = \int_0^{2/3} x(s)ds. \end{cases} \tag{5.2}$$

Here, $q = 3/2,\ \alpha = \beta = 1, \xi = 1/3,\ \eta = 2/3,\ \gamma = 29/54$, and

$$\Big|f(t,x)\Big| = \left|\frac{1}{(4\pi)}\sin(2\pi x) + \frac{|x|}{1+|x|}\right| \le \frac{1}{2}\|x\| + 1.$$

Clearly $M = 1$ and

$$\kappa = \frac{1}{2} < \frac{1}{\Lambda} = \left[\frac{4}{3\sqrt{\pi}}\left(1 + \frac{6}{145}\left(\frac{65}{2} + \frac{20+52\sqrt{2}}{9\sqrt{3}}\right)\right)\right]^{-1} = 0.5126401766.$$

Thus, all the conditions of Theorem 4.1 are satisfied and consequently the problem (5.2) has at least one solution.

References

[1] R. P. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for fractional differential equations, *Georgian Math. J.* **16** (2009), 401-411.

[2] R. P. Agarwal, B. Ahmad, Existence of solutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations, *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.,* to appear.

[3] B. Ahmad, S. Sivasundaram, Existence and uniqueness results for nonlinear boundary value problems of fractional differential equations with separated boundary conditions, *Commun. Appl. Anal.* **13** (2009), 121-228.

[4] B. Ahmad, J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher order nonlinear fractional differential equations, *Abstr. Appl. Anal. 2009, Art. ID 494720, 9 pp.*

[5] B. Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, *Bound. Value Probl. 2009, Art. ID 708576, 11 pp.*

[6] B. Ahmad, A. Alsaedi, B. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal. Real World Appl.* **9** (2008), 1727-1740.

[7] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009) 1838-1843.

[8] B. Ahmad, S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, *Appl. Math. Comput.* **217** (2010), 480-487.

[9] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.* **23** (2010), 390-394.

[10] B. Ahmad, Existence of solutions for fractional differential equations of order $q \in (2,3]$ with anti-periodic boundary conditions, *J. Appl. Math. Comput.* **34** (2010), 385-391.

[11] Z. Bai, H. Liu, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311** (2005), 495-505.

[12] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal.* **72** (2010), 916-924.

[13] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.* **71** (2009) 2391-2396.

[14] A. Boucherif, Second-order boundary value problems with integral boundary conditions, *Nonlinear Anal.* **70** (2009), 364-371.

[15] S. Hamani, M. Benchohra, John R. Graef, Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions, *Electronic J. Differential Equations, Vol. 2010(2010), No. 20, pp. 1-16.*

[16] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[17] M.A. Krasnoselskii, Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk* **10** (1955), 123-127.

[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[19] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[21] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, *Comput. Math. Appl.* **59** (2010), 1300-1309.

[22] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differential Equations 2006, No. 36, 12 pp*.