African Diaspora Journal of Mathematics
Volume 11, Number 1, pp. 24–28 (2011)
ISSN 1539-854X
www.african-j-math.org

Left Multipliers and Jordan Ideals in Rings with Involution

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Abstract

The purpose of this paper is to study left multipliers satisfying certain identities on Jordan ideals of rings with involution. Some well known results characterizing commutativity of prime rings by left multipliers have also been extended to Jordan ideals.

AMS Subject Classification: 16W10, 16W25, 16U80.

Keywords: rings with involution, $*$-prime rings, Jordan ideals, left multipliers.

1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Recall that R is 2-torsion free if $2x = 0$ yields $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R admits an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ forces $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime, but the converse need not be true in general. For example, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x,y) = (y,x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.
In all that follows $Sa_*(R) = \{x \in R / x^* = \pm x\}$ will denote the set of symmetric and skew-symmetric elements of R. For $x,y \in R$, $[x,y] = xy - yx$ and $x \circ y = xy + yx$. An additive subgroup J of R is a Jordan ideal if $x \circ r \in J$ for all $x \in J$ and $r \in R$. Moreover, if $J^* = J$, then J is called a $*$-Jordan ideal. An additive mapping $H : R \to R$ is a left (resp. right) multiplier if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$) for all $x,y \in R$. Considerable work has been done on left (right) multipliers in prime and semiprime rings during the last couple of decades (see [4], [5] and [7] for a partial bibliography). An additive mapping $F : R \to R$ is a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. Clearly, generalized derivation with $d = 0$ covers the concept of left multipliers.

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There has been an ongoing interest concerning the relationship between the commutativity of a prime ring R and the behavior of a generalized derivation of R, with associated *nonzero* derivation. Moreover, many of obtained results extend other ones proven previously just for the action of the generalized derivation on the whole ring. So, it is natural to ask what can we say about the commutativity of R if the generalized derivation is replaced by a left multiplier. In the present paper, we have investigated this problem for certain situations involving left multipliers acting on Jordan ideals.

2 Left multipliers acting on Jordan ideals

We shall use without explicit mention the fact that if J is a Jordan ideal of a ring R, then $2[R,R]J \subseteq J$ and $2J[R,R] \subseteq J$ ([6], Lemma 1).
We recall the following lemmas which are essential for developing the proof of our results.

Lemma 2.1. ([2], Lemma 2) *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.*

Remark 2.2. If $aJ = 0$ or $Ja = 0$, then $a = 0$. Indeed, if $aJ = 0$ (resp. $Ja = 0$), then $aJa = 0 = aJa^*$ (resp. $aJa = 0 = a^*Ja$) and Lemma 2.1 yields $a = 0$.

Lemma 2.3. ([2], Lemma 3) *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $[J,J] = 0$, then $J \subseteq Z(R)$.*

Theorem 2.4. *Let J be a nonzero $*$-Jordan ideal of a 2-torsion free ring R and let F be a left multiplier such that $F([x,y]) = [x,y]$ for all $x,y \in J$. If R is $*$-prime, then F is trivial or $J \subseteq Z(R)$.*

Proof. From $F([x,y]) = [x,y]$ it follows that

$$F([2[x,y]u,v]) = [2[x,y]u,v]] \quad \text{for all } u,v,x,y \in J. \tag{2.1}$$

On the other hand, the fact that F is a left multiplier yields

$$F([2[x,y]u,v]) = 2[x,y]uv - 2F(v)[x,y]u \quad \text{for all } u,v,x,y \in J. \tag{2.2}$$

Comparing (2.1) and (2.2), as char$R \neq 2$, we get $(F(v)-v)[x,y]u = 0$ and thus

$$(F(v)-v)[x,y]J = 0 \text{ for all } v,x,y \in J. \tag{2.3}$$

Applying Remark 2.2, equation (2.3) assures that

$$(F(v)-v)[x,y] = 0 \text{ for all } v,x,y \in J. \tag{2.4}$$

Replacing y by $2y[r,s]$ in (2.4), where $r,s \in R$, and using (2.4) we find that $(F(v)-v)y[x,[r,s]] = 0$ and therefore

$$(F(v)-v)J[x,[r,s]] = 0 \text{ for all } v,x \in J,\ r,s \in R. \tag{2.5}$$

Since J is a $*$-ideal, then (2.5) yields

$$(F(v)-v)J[x,[r,s]]^{*}=0 \text{ for all } v,x\in J,\ r,s\in R. \tag{2.6}$$

In view of (2.5) and (2.6), Lemma 1 forces

$$F(v)-v=0 \text{ for all } v\in J \quad\text{or}\quad [x,[r,s]]=0 \text{ for all } x\in J,\, r,s\in R.$$

Assume that

$$[x,[r,s]]=0 \text{ for all } x\in J,\, r,s\in R. \tag{2.7}$$

Substituting sx for s in (2.7) and employing (2.7) we find that

$$[x,s][r,x]=0 \text{ for all } x\in J,\ r,s\in R. \tag{2.8}$$

Replacing r by tr in (2.8), where $t\in R$, we get $[x,s]t[r,x]=0$ and thus

$$[x,s]R[r,x]=0 \text{ for all } x\in J,\ r,s\in R. \tag{2.9}$$

From (2.9) it follows that

$$[x,s]R[r,x]R[r,x]^{*}=0 \text{ for all } r,s\in R. \tag{2.10}$$

Since $[r,x]R[r,x]^{*}$ is invariant under $*$, the $*$-primeness of R together with (2.10) force $[x,s]=0$ for all $s\in R$, in which case $x\in Z(R)$, or $[r,x]R[r,x]^{*}=0$. Suppose

$$[r,x]R[r,x]^{*}=0 \text{ for all } r\in R. \tag{2.11}$$

As $[r,x]R[r,x]=0$ by (2.9), then (2.11) assures that $[r,x]=0$ for all $r\in R$ and thus $x\in Z(R)$. Accordingly, $J\subseteq Z(R)$.
Now assume that

$$F(v)=v \quad \text{for all } v\in J. \tag{2.12}$$

Let $r\in R$ and $x,y\in J$, as $2r[x,y]\in J$, (2.12) yields $F(2r[x,y])=2r[x,y]$ and thus

$$(F(r)-r)[x,y]=0 \quad \text{for all } x,y\in J, r\in R. \tag{2.13}$$

Replacing r by rs in (2.13), where $s\in R$, we get $(F(r)-r)s[x,y]=0$ and therefore

$$(F(r)-r)R[x,y]=0 \quad \text{for all } x,y\in J,\ r\in R. \tag{2.14}$$

Since $J^{*}=J$, then equation (2.14) leads to

$$(F(r)-r)R[x,y]^{*}=0 \quad \text{for all } x,y\in J, r\in R. \tag{2.15}$$

Applying the $*$-primeness of R, from (2.14) and (2.15) it follows that either $F(r)=r$ for all $r\in R$ or $[J,J]=0$ in which case, because of Lemma 2.3, $J\subseteq Z(R)$. In conclusion, either F is trivial or $J\subseteq Z(R)$.

The following example proves necessity of $*$-primeness condition in Theorem 2.4.

Example 2.5. Let $R=\left\{\begin{pmatrix} x & 0\\ y & x\end{pmatrix} \mid x,y\in S\right\}$ where S is a ring such that the square of each element in S is zero. Let us consider

$$F\begin{pmatrix} x & 0\\ y & x\end{pmatrix}=\begin{pmatrix} x & 0\\ y-x & x\end{pmatrix} \text{ and } \begin{pmatrix} x & 0\\ y & x\end{pmatrix}^{*}=\begin{pmatrix} -x & 0\\ -y & -x\end{pmatrix}.$$

From ([3], Example 2.1) it follows that R is a non $*$-prime ring and F is a nontrivial left multiplier. Moreover, if we set $J=\left\{\begin{pmatrix} 0 & 0\\ y & 0\end{pmatrix} \mid y\in S\right\}$, then J is a $*$-Jordan ideal of R such that $F([x,y])=[x,y]$ for all $x,y\in J$ but, in view of ([3], Example 2.1), $J\nsubseteq Z(R)$. Accordingly, in Theorem 2.4 the $*$-primeness hypothesis is crucial.

Remark 2.6. In Theorem 2.4, if we replace the Jordan ideal by an ideal of R, then 2-torsion freeness hypothesis can be omitted.

Using the fact that a $*$-prime ring having a nonzero central $*$-ideal must be commutative, Theorem 2.4 together with Remark 2.6 yield a commutativity criterium as follows:

Corollary 2.7. *Let $(R,*)$ be a ring with involution and let F be a nontrivial left multiplier such that $F([x,y])=[x,y]$ for all x,y in a nonzero $*$-ideal I of R. If R is $*$-prime, then R is commutative.*

In ([1], Theorem 2.1) it is proved that if a prime ring R admits a nonzero left multiplier H, with $H(x)\neq x$ for all x in a nonzero ideal I of R, such that $H([x,y])=[x,y]$ for all $x,y\in I$, then R is commutative. However, one can easily see that the condition $H(x)\neq x$ for all $x\in I$ can be replaced by *there exists* $x\in I$ such that $H(x)\neq x$, that is H is nontrivial. Using this fact, the following Theorem extends ([1], Theorem 2.1) to Jordan ideals.

Theorem 2.8. *Let R be a 2-torsion free prime ring and let J be a nonzero Jordan ideal of R. If R admits a left multiplier F such that $F([x,y])=[x,y]$ for all $x,y\in J$, then F is trivial or $J\subseteq Z(R)$.*

Proof. Let $\mathcal{F}$ be the additive mapping defined on $\mathcal{R}=R\times R^0$ by $\mathcal{F}(x,y)=(F(x),y)$. Clearly, $\mathcal{F}$ is a left multiplier of $\mathcal{R}$. Moreover, if we set $\mathcal{J}=J\times J$, then $\mathcal{J}$ is a $*_{\text{ex}}$-Jordan ideal of $\mathcal{R}$. As $F([x,y])=[x,y]$ for all $x,y\in J$, it's easy to check that $\mathcal{F}([u,v])=[u,v]$ for all $u,v\in\mathcal{J}$. Since $\mathcal{R}$ is $*_{\text{ex}}$-prime, in view of Theorem 2.4 we deduce that $\mathcal{J}\subseteq Z(\mathcal{R})$ or $\mathcal{F}$ is trivial. Accordingly, either $J\subseteq Z(R)$ or F is trivial.

Now if F is a left multiplier such that $F\neq -Id$, then $-F$ is a nontrivial left multiplier. This fact together with Theorem 2.8 yield the following result:

Theorem 2.9. *Let R be a 2-torsion free prime ring and let J be a nonzero Jordan ideal of R. If R admits a left multiplier F such that $F([x,y])=-[x,y]$ for all $x,y\in J$, then $J\subseteq Z(R)$ or $F(r)=-r$ for all $r\in R$.*

The following theorem extends Theortem 2.5 of [1] to Jordan ideals.

Theorem 2.10. *Let R be a 2-torsion free prime ring and let J be a nonzero Jordan ideal of R. If R admits a left multiplier F such that neither F nor $-F$ is trivial, then the following*

conditions are equivalent:
(1) $F([x,y]) = [x,y]$ *for all* $x,y \in J$;
(2) $F([x,y]) = -[x,y]$ *for all* $x,y \in J$;
(3) *for all* $x,y \in J$, *either* $F([x,y]) = [x,y]$ *or* $F([x,y]) = -[x,y]$*;*
(4) $J \subseteq Z(R)$.

Proof. Using Theorem 2.8 together with Theorem 2.9, it remains only to show that (3) $\Longrightarrow$ (4). Let us consider $J_1 = \{x \in J \mid F([x,y]) = [x,y] \text{ for all } y \in J\}$ and $J_2 = \{x \in J \mid F([x,y]) = -[x,y] \text{ for all } y \in J\}$. It is clear that J_1 and J_2 are additive subgroups of J such that $J = J_1 \cup J_2$. But a group cannot be a union of two of its proper subgroups and hence $J = J_1$ or $J = J_2$. Thus, we find that either $F([x,y]) = [x,y]$ for all $x,y \in J$ and $J \subseteq Z(R)$ by Theorem 2.8 or $F([x,y]) = -[x,y]$ for all $x,y \in J$ in which case $J \subseteq Z(R)$ by Theorem 2.9.

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