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EFFECT OF A TOXICANT ON THE DYNAMICS OF A SPATIAL FISHERY

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Abstract

In this paper, we present a mathematical model of a spatial fishery with two patches. The first patch is unpolluted while fishes are infected by some toxicants released by industrial wastes in the other one. The fish stock can move between these zones, on which they are harvested by fishing fleets. Fish movements between the zones, as well as vessels displacements, are assumed to take place at a faster time scale than the variation of the fish stock and the change of the fleet size. We take advantage of these two time scales to derive a reduced model governing the dynamics of the total fish stock and fishing effort. This reduced model is analyzed and we look for existence of limit cycle. We also present numerical simulations to illustrate the results.

AMS Subject Classification: 37N25; 92D25; 93D05.

Keywords:Spatial fishery, Population dynamics, Aggregation of variables, Toxicant, Stability.

1 Introduction

The effects of toxicants on ecological communities have become problems of major environmental concerns in the recent decades. Mathematical modelling in dealing with such ecotoxicological problems apparently started with the studies of Hallam and Clark (1982), Hallam et al. (1983), Hallam and De Luna (1984), De Luna and Hallam (1987), Freedman and Shukla (1990) and others. Some other mathematical studies on ecotoxicology include the works of Chattopadhyay (1996), Shukla and Dubey (1996), Mukhopadhyay et al. (1998), Dubey and Hussain (2000), Shukla et al. (2001), etc.

As with the growing human needs the industries are also producing a huge amount of toxicants part of which are released in marine water, the species living there are highly affected by these toxicants. Maynard Smith (1974) incorporated the effects of toxic substances in a two species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other only when the other is present. The idea of Maynard Smith was extended further by Kar and Chaudhuri (2003) to a two species competing fish

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species which are commercially exploited. Tapasi et al (2009) considered a prey-predator fishery in which the growth of both species is affected differently by some toxicants.

In a previous work (Mchich et al., 2000), the authors built a dynamical model of a spatial fishery in the simplest case of stock independent vessels displacement rate. Two different cases are distinguished: extinction of the fishery or coexistence of the fish stock and the fishing fleet at a constant stable equilibrium.

In the present work, we generalize a previous work (Mchich et al., 2000) by considering a stock-effort dynamical model in which the growth of the fish is affected by some toxicants released by some source for example industrial wastes. We consider the simplest situation of a population located on two patches with a polluted patch while the other one is unpolluted. We also assume that the displacement rates of vessels and fish between the zones are constant. Furthermore, we assume that the fish take place at a faster time scale than local interactions within the fishing zones. We use time scale separation methods in order to derive a reduced model governing the total fish and fishing effort. Such a simple reduced model can be more easily analyzed and we obtain some general results that we would not be able to get otherwise.

In the next section we present the complete fishery model which is a system of four ordinary differential equations, governing the two local fish stocks and the two fishing efforts on two fishing zones. The model includes two time scales, a fast one associated to quick movements of fish and boats between patches in comparison to a slow one corresponding to growth of the fish population and to variation of the total number of vessels involved in the fishery. We take advantage of these two time scales to build a reduced model. To achieve the reduction, we apply aggregation methods of variables (Iwasa et al., 1987, 1989, Auger and Poggiale, 1996, 1998, Auger et al., 2008), which are based on perturbation technics and on the application of an adequate version of the Center Manifold Theorem. The reduced model, called aggregated model, describes the dynamics of the total fish stock and the total fishing effort. The existence of the possible steady states along with their local and stability properties is discussed. Existence of the limit cycle is also analyzed. We end by presenting some numerical simulations.

2 Formulation of the problem : Presentation of the mathematical model

We consider a model which describes the dynamics of fish population harvested by a fishing fleet. Fishing boats are allowed to exploit the resource on two fishing patches with a polluted patch whereas the other one is unpolluted. Let $N(t)$ be the fish population density at time t, $N(t) = (n_1(t), n_2(t))^T$. The upper subscript T denotes the transpose vector. $n_1(t)$ (respectively and $n_2(t)$) is the fish population density on fishing area 1 (respectively 2), at time t. Similarly, the fishing effort is subdivided into two components on each fishing zone $E(t) = (E_1(t), E_2(t))^T$. $E_1(t)$ (respectively and $E_2(t)$) is the fishing effort on fishing area 1 (respectively 2), at time t.

We suppose that two processes occur at two different time scales. At the fast time scale, the model only describes the displacement of fish and vessels between the two patches. Thus, the fast part of the model is conservative, i.e. the total stock and the total fishing effort are constant. At the slow time scale, the total fish stock and the total fishing effort are not constant. Regarding fish stocks, their evolution, in each specific zone, is represented by the stock-effort Schaefer model, also called Graham–Schaefer model (Schaefer 1954): the fish population grows logisticaly and its decrease is due to the amount of harvested fish per unit of time : $-a_i n_i E_i$, We assume that fishes in patch 1 are infected by some external toxic substance.

Concerning the fishing effort, it is assumed to change with respect to the investment proportion of the fishing revenue. That means that the fleet owners will invest (or not), with respect to their revenues. Note that the revenue, in the model, is the difference between the income and the cost. We assume that unit prices and unit costs are constant.

According to previous assumptions, at the fast time scale τ with respect to the slow time scale t : two equations describe the evolution of the fish densities on the two zones,

$$\left\{\begin{array}{lcl} \frac{dn_1}{d\tau} & = & (m_{12}n_2 - m_{21}n_1) + \varepsilon\left(r_1 n_1 (1-\frac{n_1}{K_1}) - a_1 n_1 E_1 - \beta_1 n_1^3\right) \\ \frac{dn_2}{d\tau} & = & (m_{21}n_1 - m_{12}n_2) + \varepsilon\left(r_2 n_2 (1-\frac{n_2}{K_2}) - a_2 n_2 E_2\right) \end{array}\right. \tag{2.1}$$

and two equations describe the evolution of the fishing effort on the two zones,

$$\left\{\begin{array}{lcl} \frac{dE_1}{d\tau} & = & (k_{12}E_2 - k_{21}E_1) + \varepsilon(-c_1 E_1 + p a_1 n_1 E_1) \\ \frac{dE_2}{d\tau} & = & (k_{21}E_1 - k_{12}E_2) + \varepsilon(-c_2 E_2 + p a_2 n_2 E_2) \end{array}\right. \tag{2.2}$$

where r_i and K_i $(i = 1,2)$ represent, respectively, the intrinsic growth rate and carrying capacity of the stock in zone i. Patches have distinct characteristics, so we suppose that parameters r_1 and r_2 are different, and similarly for the carrying capacities.

The catchability coefficient of the fleet on patch i $(i = 1,2)$ is a_i . Parameters p is the unit price of the catch and c_i $(i = 1,2)$ are, respectively, the unit of cost per fishing effort unit on patch i, and are also assumed to be constant. The constant coefficients m_{ij} represent the fish per capita migration rates from patch j to patch i . The corresponding migration coefficients for the fishing efforts k_{ij}. These displacement terms are also assumed to be constant.

The term $\beta_1 n_1^3$ comes directly through the infection of the fish by some external toxic substance, for example, industrial wastes, (Tapasi Das et al, 2009). Since $\frac{d(\beta_1 n_1^3)}{dn_1} = 3\beta_1 n_1^2 > 0$ and $\frac{d^2(\beta_1 n_1^3)}{dn_1^2} = 6\beta_1 n_1$, there is an accelerated growth in the production of the toxic substance to the density n_1 as more and more of the fish consume infected foods. We call β_1 the coefficient of toxicity for fish. All parameters are assumed to be positive.

3 Results : Analysis of an aggregated model

From the complete system (2.1-2.2), we apply aggregation methods (Auger and Bravo de la Parra, 2000) to obtain a reduced system: a two dimensional system of ordinary differential equations governing the total fish stock $n(t) = n_1(t) + n_2(t)$ and the total fishing effort $E(t) = E_1(t) + E_2(t)$ at the slow time scale.

The sufficient conditions for a system to be perfectly as well as approximately aggregated have been investigated in the frame of general population models by Iwasa et al. (1987), Iwasa et al. (1989) and Levin and Pacala (1997). Some aggregation models permit to reduce a system with a large number of variables involving different time scales into an aggregated system with a few global variables. The method is based on perturbation technics and on the application of an adequate version of the center manifold Theorem (fenichel, 1971): see Poggiale (1994), Auger and Roussarie (1994) and Michalski et al. (1997). The aggregation of the complete model consists in supposing that the fast dynamics has attained a stable equilibrium and in substituting this fast equilibrium into the equation of the complete model. The first step to achieve aggregation is to neglect the small terms of the order of ε in Eq. (2.1-2.2) and look for the existence of stable equilibria for its fast part.

3.1 Fast equilibria

We notice that $n(t)$ and $E(t)$, the global variables, are constants of motion of the fast process: migration. Fast equilibria are solutions of the following system

$$\begin{cases} \frac{dn_1}{d\tau} = & (m_{12}n_2 - m_{21}n_1) = 0 \\ \frac{dn_2}{d\tau} = & (m_{21}n_1 - m_{12}n_2) = 0 \\ \frac{dE_1}{d\tau} = & (k_{12}E_2 - k_{21}E_1) = 0 \\ \frac{dE_2}{d\tau} = & (k_{21}E_1 - k_{12}E_2) = 0 \end{cases} \tag{3.1}$$

and a simple calculation leads to the following result:

$$\begin{pmatrix} n_1^* & n_2^* & E_1^* & E_2^* \end{pmatrix} = \begin{pmatrix} \upsilon_1^* n & \upsilon_2^* n & \eta_1^* E & \eta_2^* E \end{pmatrix} \tag{3.2}$$

Where:

$$\begin{cases} \upsilon_1^* = & \frac{m_{12}}{m_{12}+m_{21}} \\ \upsilon_2^* = & \frac{m_{21}}{m_{12}+m_{21}} \\ \eta_1^* = & \frac{k_{12}}{k_{12}+k_{21}} \\ \eta_2^* = & \frac{k_{21}}{k_{12}+k_{21}} \end{cases} \tag{3.3}$$

The constants υ_1^* and υ_2^* represent the fast equilibrium proportions of the stock on each patch, whereas η_1^* and η_2^* admit the same interpretation for the fishing effort. As we see there is a fast equilibrium for each pair of values of the global variables n and E. The proof of stability for these fast equilibria is straightforward.

3.2 The aggregated model

The aggregated model is obtained by substituting the fast and stable equilibrium for fish and fishing effort (equation 3.2) into the complete system (equations 2.1 and 2.2) and by adding the two fish and the two fishing effort equations. This leads to the following structurally stable model:

$$\begin{cases} \frac{dn}{dt} &= rn(1-\frac{n}{K}) - anE - \beta n^3 \\ \frac{dE}{dt} &= (-c+pan)E \end{cases} \tag{3.4}$$

where,

$$\begin{cases} r &= r_1\nu_1^* + r_2\nu_2^* \\ \frac{r}{K} &= r_1\frac{\nu_1^{*2}}{K_1} + r_2\frac{\nu_2^{*2}}{K_2} \\ a &= a_1\nu_1^*\eta_1^* + a_2\nu_2^*\eta_2^* \\ \beta &= \beta_1\nu_1^{*3} \\ c &= c_1\eta_1^* + c_2\eta_2^* \end{cases} .$$

The dynamics of Eq. (3.4) is a good approximation of the real dynamics of the global variables in the complete model Eq. (2.1-2.2) if the aggregated model that is derived, Eq. (3.4), is structurally stable, which is the case, and if ε is small enough, which is also assumed.

4 Steady states and local stability analysis

The possible steady states of the dynamical aggregated system of equations (3.4) are:

• $P_0 = (0,0)$,

• $P_1 = (\bar{n},0)$ where

$$\bar{n} = \frac{-\frac{r}{K} + \sqrt{\left(\frac{r}{K}\right)^2 + 4r\beta}}{2\beta} < K.$$

• $P_2 = (n^*, E^*)$ where n^*, E^* are the positive solutions of the following algebraic equations:

$$\begin{cases} r(1-\frac{n}{K}) - aE - \beta n^2 = & 0 \\ -c+pan = & 0 \end{cases} \tag{4.1}$$

We have

$$n^* = \frac{c}{pa}$$

and

$$E^* = \frac{1}{a}\left(r(1-\frac{n^*}{K}) - \beta n^{*2}\right)$$

which implies that

$$E^* = \frac{1}{a}\left[r - \frac{c}{pa}\left(\frac{r}{K} + \frac{\beta c}{pa}\right)\right]$$

which is positive when

$$r > \frac{c}{pa}\left(\frac{r}{K} + \frac{\beta c}{pa}\right).$$

Let us now proceed to local stability analysis of the previous steady states. To do this, we have to look for eigenvalues of the Jacobian matrices calculated at these equilibrium points.

The Jacobian matrix is given by:

$$J(n,E) = \begin{bmatrix} r - \frac{2r}{K}n - aE - 3\beta n^2 & -an \\ paE & -c + pan \end{bmatrix}$$

• For the equilibrium point $P_0 = (0,0)$, we have,

$$J(0,0) = \begin{bmatrix} r & 0 \\ 0 & -c \end{bmatrix}$$

This matrix has two real eigenvalues with opposite signs: r and $-c$, so $(0,0)$ is always a saddle point.

• For the equilibrium point $P_1 = (\bar{n},0)$, we have,

$$J(\bar{n},0) = \begin{bmatrix} -\left(\frac{r}{K}\bar{n} + 2\beta\bar{n}^2\right) & -a\bar{n} \\ 0 & -c + pa\bar{n} \end{bmatrix}$$

therefore, two possible situations can occur:

(i) if $c > pa\bar{n}$, i.e. the determinant of this matrix is positive and the trace is negative, then the point $(\bar{n},0)$ is asymptotically stable (it is a sink).

(ii) if $c < pa\bar{n}$, then $(\bar{n},0)$ is a saddle point.

• For the equilibrium the point $S_2 = (n^*,E^*)$, we have

$$J(n^*,E^*) = \begin{bmatrix} -\left(\frac{r}{K}n^* + 2\beta n^{*2}\right) & -an^* \\ paE^* & 0 \end{bmatrix}$$

This matrix has positive determinant D when the equilibrium is positive:

$$D = pa^2n^*E^*$$

On the other hand, we have:

$$trJ(n^*,E^*) = -\left(\frac{r}{K}n^* + 2\beta n^{*2}\right) < 0$$

We can consider two cases of interest, the case of absence and presence of toxicant.

In absence of toxicity, we have $\beta = 0$ and then $trJ(n^*,E^*) < 0$, therefore, we have either two negative real or complex conjugates eigenvalues with negative real parts. Hence the steady state P_2 is either a locally stable node or a locally stable focus.

In presence of toxicity $\beta > 0$. we have $trJ(n^*, E^*) < 0$, therefore, we have either two negative real or complex conjugates eigenvalues with negative real parts. Hence the steady state P_2 is either a locally stable node or a locally stable focus in the presence or absence of toxicity.

Therefore, the local stability of the system is not directly dependent on the intensities of the toxicants.

5 Global stability

In this section, we look for global stability of the aggregated system of equations (3.4) by presenting a suitable Lyapunov function:

$$V(n,E) = \left[n - n^* - n^* \log\left(\frac{n}{n^*}\right)\right] + h\left[E - E^* - E^* \log\left(\frac{E}{E^*}\right)\right]$$

where h is a suitable constant to be determined in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium point (n^*, E^*) and is positive for all other values n, E. The time derivative of V along the trajectories of (3.4) is

$$\begin{aligned} \frac{dV}{dt} &= \frac{n-n^*}{n}\frac{dn}{dt} + h\frac{E-E^*}{E}\frac{dE}{dt} \\ &= (n-n^*)\left[r(1-\tfrac{n}{K}) - aE - \beta n^2\right] + h(E-E^*)(-c+pan) \end{aligned} \tag{5.1}$$

Also we have the set of equilibrium equations

$$\begin{cases} r(1-\frac{n^*}{K}) - aE^* - \beta n^{*2} &= 0 \\ -c + pan^* &= 0 \end{cases}$$

corresponding to the steady state $P_2 = (n^*, E^*)$.

We can rewrite equation (5.1) taking into account the two previous equations leading to the next form:

$$\begin{aligned} \frac{dV}{dt} &= (n-n^*)\left[r(1-\tfrac{n}{K}) - aE - \beta n^2 - \left(r(1-\tfrac{n^*}{K}) - aE^* - \beta n^{*2}\right)\right] \\ &\quad + h(E-E^*)\left[-c + pan - (-c + pan^*)\right] \\ &= -(n-n^*)^2\left[\tfrac{r}{K} + \beta(n+n^*)\right] + (E-E^*)(n-n^*)(pah - a) \\ &= -(n-n^*)^2\left[\tfrac{r}{K} + \beta(n+n^*)\right] \quad \text{(where we have chosen } h = \tfrac{1}{p}\text{)} \end{aligned}$$

Now, since dV/dt is negative semidefinite in some neighborhood of (n^*, E^*), the interior equilibrium point (n^*, E^*) is globally asymptotically stable.

6 Existence of Limit Cycle

We use Bendixon-Dulac test to look for existence of limit cycle.

Let us consider the aggregated dynamical system that we recall below:

$$\begin{cases} \frac{dn}{dt} &= rn(1-\frac{n}{K})-anE-\beta n^3 = F(n,E) \\ \frac{dE}{dt} &= (-c+pan)E = G(n,E) \end{cases}$$

in which functions $F(n,E)$ and $G(n,E)$ are smooth in a given simply connected region D of the 1st quadrant of the (n,E) phase plane. Now, let us consider function:

$$H(n,E) = \frac{1}{nE}$$

which is also smooth in the region D.

Then we have:

$$\begin{aligned} B(n,E) &= \frac{\partial(HF)}{\partial n}+\frac{\partial(HG)}{\partial E} \\ &= \frac{\partial}{\partial n}\left[\frac{1}{nE}\left\{rn(1-\frac{n}{K})-anE-\beta n^3\right\}\right]+\frac{\partial}{\partial E}\left[\frac{1}{nE}\{(-c+pan)E\}\right] \\ &= \frac{\partial}{\partial n}\left(\frac{r}{E}-\frac{rn}{KE}-a-\frac{\beta n^2}{E}\right)+\frac{\partial}{\partial E}\left(\frac{-c}{n}+pa\right) \\ &= -\left(\frac{r}{KE}+\frac{2\beta n}{E}\right). \end{aligned}$$

We thus see that whatever may be the values of the parameters (as they are assumed to be positive) the above expression is always negative i.e. it keeps the same sign throughout the region D and hence there are no closed orbits lying entirely in the region D.

7 Numerical simulations

In this section, we consider some numerical examples:

We take the following parameter values:$r_1 = 1.5, r_2 = 1.3, m_{12} = 0.7, m_{21} = 0.4, k_{12} = 0.5, k_{21} = 0.5, K_1 = 100, K_2 = 60, a_1 = 0.8, a_2 = 0.7,$

$p = 1.5, c_1 = 1, c_2 = 0.5, \beta_1 = 1.5$; in appropriate units. for the above values, we get the following solutions:

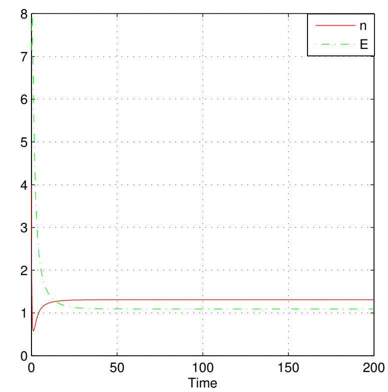

Figure 1. Solutions curves beginning with $n = 4$, $E = 7$ in presence of toxicity.

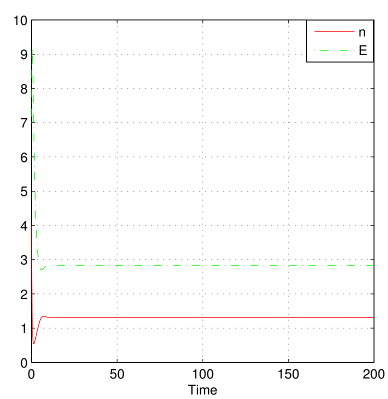

Figure 2. Solutions curves beginning with $n = 4$, $E = 7$ in absence of toxicity $(\beta = 0)$.

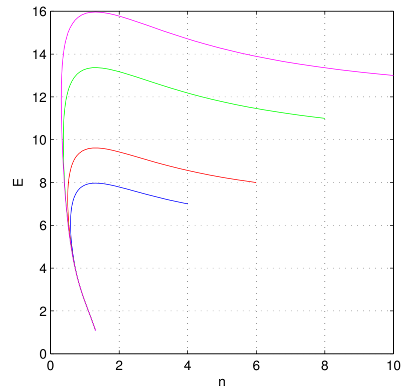

Figure 3. Phase plane trajectories with different initial values in presence of toxicity.

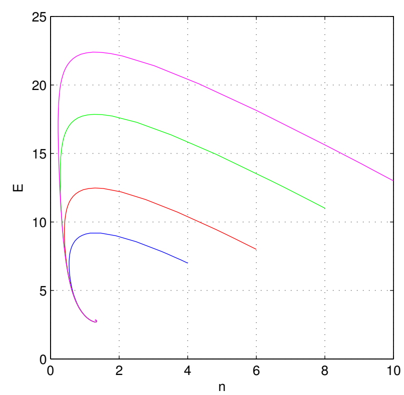

Figure 4. Phase plane trajectories with different initial values in absence of toxicity.

Trajectories clearly indicate that the equilibrium $P_2(1.3, 1.1)$ is globally asymptotically stable. In absence of toxicity. We get the equilibrium $P_2(1.3, 2.8)$ is globally asymptotically stable.

From numerical simulations, we note the following points:

In presence of toxicity, the fish population density at equilibrium exists at 1.3 and the corresponding fishing effort is 1.1.

In the absence of toxicity $\beta = 0$, the fish population density equilibrium also exists at 1.3 but the corresponding fishing effort is 2.8. So as expected, we get a higher fishing effort without pollution compared to the case of toxicity.

8 Concluding remarks

In this paper, a mathematical model of a spatial fishery has been proposed and analyzed. We have explored the effect of toxicants on fish population and fishing effort dynamics. This model can be used to assess the consequences of increased concentrations of toxicants in the natural environment on fish population dynamics and fishing effort. Using stability theory of ordinary differential equations it has been shown that the interior equilibrium of the aggregated model exists under certain conditions and is globally asymptotically stable. It has also been shown that the system under consideration does not have any limit cycle.

Our results are based on a very simple model but in both cases, absence or presence of toxicants, there exist a unique strictly positive equilibrium for the total system which is globally asymptotically stable. Our results also show that pollution has a major effect on the equilibrium total fishing effort which is severely reduced in the case of pollution.

In this work, we assumed that there are two time scales, a fast one for movements of fish and boats between patches, and a slow one corresponding to fish population growth and fishery dynamics. Of course, the use of aggregation methods allowed us to simplify the mathematical analysis of the complete model, a set of four equations, into a simple two dimensional aggregated model. However, is it realistic to consider that two different time scales exist? In practice, it has been shown on numerical examples that as soon as $\varepsilon = 10^{-1}$ or 10^{-2} the approximation made for "aggregating" the complete model into a reduced one is relevant, and the trajectories obtained with the aggregated model remain close to those obtained with the complete model (Poggiale and Auger 2004). Therefore, if we think about a fish stock that grows annually and about boats and fish that change patch every week, then the method could be applied and the aggregated model can be used to make predictions about the complete system, as we did here. This model developed here could be easily extended to any number of patches. However, in most cases, we can consider a two-patch system only, with a zone 1 where pollutant is present, and a zone 2 with no pollution.

In future work, we wish to generalize this model by using vessels movements between the two fishing areas which would be stock dependent, i.e. boats would remain on patches where the fish density is large.

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