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ON SANDWICH THEOREMS FOR SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING MODIFIED SAIGO FRACTIONAL INTEGRAL OPERATOR

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Abstract

The purpose of this paper is to derive some subordination and superordination results for analytic functions involving modified Saigo fractional integral operator in the open unit disk.

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1 Introduction

Let **H**(U) denote the class of analytic functions in the unit disk $U := \{z \in \mathbb{C}, |z| < 1\}$. For n positive integer and $a \in \mathbb{C}$, let

$$\mathbf{H}[a,n] := \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\},$$

and $\mathbf{A}_n = \{f \in H(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\}$ with $\mathbf{A}_1 = \mathbf{A}$. A function $f \in \mathbf{H}[a,n]$ is convex in U if it is univalent and $f(U)$ is convex. It is well known that f is convex if and only if $f(0) \neq 0$ and

$$\Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0,\ z \in U.$$

Definition 1.1. [1] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where

$$E(f) := \{\zeta \in \partial U : lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

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Let F and G be analytic in the unit disk U. The function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. Alternatively, given two functions F and G, which are analytic in U, the function F is said to be subordination to G in U if there exists a function h, analytic in U with

$$h(0) = 0 \;\; and \;\; |h(z)| < 1 \; for\, all\, z \in U$$

such that

$$F(z) = G(h(z)) \; for\, all\, z \in U.$$

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

We shall need the following results.

Lemma 1.2. *[2] Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*
1. $Q(z)$ is starlike univalent in U, and
2. $\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.
If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.3. *[3] Let $q(z)$ be convex univalent in the unit disk U and ψ and $\gamma \in \mathbb{C}$ with $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$. If $p(z)$ is analytic in U and $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 1.4. *[4] Let $q(z)$ be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that*
1. $zq'(z)\varphi(q(z))$ is starlike univalent in U, and
2. $\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.
If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi(z)$ is univalent in U and $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Lemma 1.5. *[1] Let $q(z)$ be convex univalent in the unit disk U and $\gamma \in \mathbb{C}$. Further, assume that $\Re\{\overline{\gamma}\} > 0$. If $p(z) \in \mathbf{H}[q(0), 1] \cap Q$, with $p(z) + \gamma zp'(z)$ is univalent in U then $q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$ implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.*

Let $F(a, b; c; z)$ be the Gauss hypergeometric function (see [5]) defined, for $z \in U$, by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,$$

where is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)...(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

We need the following definitions of fractional operators in the modified Saigo type fractional calculus (see [6],[7]).

Definition 1.6. Let $0 \leq \alpha < 1$ and $\beta, \eta \in \mathbb{R}$ then

$$D_{0,z,m}^{\alpha,\beta,\eta} f(z) = \frac{d}{dz}[\frac{z^{m(\alpha-\beta)}}{\Gamma(1-\alpha)} \int_0^z (z^m - \zeta^m)^{-\alpha} F(\beta-\alpha, 1-\eta; 1-\alpha; 1-\frac{\zeta^m}{z^m}) f(\zeta) d\zeta^m]$$

where the function $f(z)$ is analytic in a simply-connected region of the $z-$plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon})(z \to 0),\ \varepsilon > max\{0, m(\beta-\eta)\} - m$$

and the multiplicity of $(z^m - \zeta^m)^{-\alpha}$ is removed by requiring $log(z^m - \zeta^m)$ to be real when $z^m - \zeta^m > 0$.

The operator $D_{0,z,m}^{\alpha,\beta,\eta}$ include the well-known Riemann-Liouville and Erdély-Kober operators of fractional calculus. Indeed, we have

$$D_{0,z,1}^{\alpha,\alpha,\eta} f(z) = D_z^{\alpha} f(z),$$

where D_z^{α} is the Riemann-Liouville fractional derivative operator [8]. Also,

$$D_{0,z,1}^{\alpha,1,\eta} z f(z) = E_{0,z}^{-\alpha,-\eta} f(z) + (\alpha-\eta) E_{0,z}^{1-\alpha,\eta} f(z),$$

in terms of Erdély-Kober operator (see [9]).

Definition 1.7. For $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$, the fractional integral operator $I_{0,z,m}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z,m}^{\alpha,\beta,\eta} f(z) = \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m - \zeta^m)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta^m}{z^m}) f(\zeta) d\zeta^m$$

where the function $f(z)$ is analytic in a simply-connected region of the $z-$plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon})(z \to 0),\ \varepsilon > max\{0, m(\beta-\eta)\} - m$$

and the multiplicity of $(z^m - \zeta^m)^{\alpha-1}$ is removed by requiring $log(z^m - \zeta^m)$ to be real when $z^m - \zeta^m > 0$.

The main object of the present paper is to find the sufficient conditions for certain normalized analytic functions $f(z), g(z)$ to satisfy

$$[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g_1(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} g_2(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$$

and

$$q_1(z) \prec [\frac{I^{\alpha,\beta,\eta}_{0,z,m} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \prec q_2(z),$$

where $\mu \neq 0,$, $\rho_{\alpha,\beta,m}(z) \neq 0$ and q_1, q_2 are given univalent functions in U. Also, we obtain the results as special cases. Further, in this paper, we study the existence of univalent solution for the fractional differential equation

$$D^{\alpha,\beta,\eta}_{0,z,m} u(z) = F(z,u(z)), \tag{1.1}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbb{C}$ is an analytic function for all $z \in U$ and $F : U \times \mathbb{C} \to \mathbb{C}$ is a normalized analytic function on U. The existence is obtained by applying Schauder fixed point theorem. Moreover, we discuss some properties of this solution involving fractional differential subordination and superordination. The following results are used in the sequel.

Theorem 1.8. *Arzela-Ascoli (see [10]) Let E be a compact metric space and* $\mathbf{C}(E)$ *be the Banach space of real or complex valued continuous functions normed by*

$$\|f\| := sup_{t\in E}|f(t)|.$$

If $A = \{f_n\}$ *is a sequence in* $\mathbf{C}(E)$ *such that* f_n *is uniformly bounded and equi-continuous, then* $\overline{A}$ *is compact.*

Let M be a subset of Banach space X and $A : M \to M$ an operator. The operator A is called *compact* on the set M if it carries every bounded subset of M into a compact set. If A is continuous on M (that is, it maps bounded sets into bounded sets) then it is said to be *completely continuous* on M. A mapping $A : X \to X$ is said to a contraction if there exists a real number ρ, $0 \leq \rho < 1$ such that $\|Ax - Ay\| \leq \rho\|x - y\|$ for all $x, y \in X$.

Theorem 1.9. *(Schauder) (see [11]) Let X be a Banach space,* $M \subset X$ *a nonempty closed bounded convex subset and* $P : M \to M$ *is compact. Then P has a fixed point.*

2 Subordination and superordination results.

In this section, we study the subordination and superordination involving fractional integral. Assume that f, g are analytic functions in U.

Theorem 2.1. *Let the function* $q(z)$ *be univalent in the unit disk* U *such that* $q(z) \neq 0$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let*

$$\Re\{\frac{b}{a}q(z) + \frac{2c}{a}q^2(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\} > 0,\ a, b, c, \in \mathbb{C}, a \neq 0. \tag{2.1}$$

If the subordination

$$b[\frac{I^{\alpha,\beta,\eta}_{0,z,m} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I^{\alpha,\beta,\eta}_{0,z,m} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I^{\alpha,\beta,\eta}_{0,z,m} f(z))'}{I^{\alpha,\beta,\eta}_{0,z,m} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}] \prec bq(z) + cq^2(z) + \frac{azq'(z)}{q(z)}$$

holds. Then

$$[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu},\ z \in U,\ \rho_{\alpha,\beta,m}(z) \neq 0,\ \mu \neq 0.$$

By setting

$$\theta(\omega) := b\omega + c\omega^2 \ \textit{and}\ \phi(\omega) := \frac{a}{\omega},$$

it can easily be observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$, $\omega \in \mathbb{C}\backslash\{0\}$. Also we obtain

$$Q(z) = zq'(z)\phi(q(z)) = a\frac{zq'(z)}{q(z)} \ \textit{and}$$

$$h(z) = \theta(q(z)) + Q(z) = bq(z) + cq^2(z) + a\frac{zq'(z)}{q(z)}.$$

By the assumption of the theorem we find $Q(z)$ is starlike univalent in U,

$$\Re\{\frac{zh'(z)}{Q(z)} = \Re\{\frac{b}{a}q(z) + \frac{2c}{a}q^2(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\} > 0$$

and that

$$\begin{aligned} bp(z) + cp^2(z) + \frac{azp'(z)}{p(z)} &= b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + az\frac{([\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu})'}{[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}} \\ &= b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}] \\ &\prec bq(z) + cq^2(z) + \frac{azq'(z)}{q(z)} \end{aligned}$$

The assertion of the theorem follows by an application of Lemma 1.2. □

Corollary 2.2. *Assume that (2.1) holds and* q *is convex univalent in* U*. If* f *is analytic in* U *and*

$$b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]$$

$$\prec b[\frac{1+Az}{1+Bz}]^{\mu} + c[\frac{1+Az}{1+Bz}]^{2\mu} + \frac{a\mu(A-b)z}{(1+Az)(1+Bz)}$$

then

$$\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}\prec\Big[\frac{1+Az}{1+Bz}\Big]^{\mu},\ -1\le B<A\le 1$$

and $q(z)=[\frac{1+Az}{1+Bz}]^{\mu}$ *is the best dominant.*

Corollary 2.3. *Assume that (2.1) holds and q is convex univalent in U. If $f\in\mathcal{A}$ and*

$$b\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}+c\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{2\mu}+a\mu\Big[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta}f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}\Big]\prec b\Big[\frac{1+z}{1-z}\Big]^{\mu}+c\Big[\frac{1+z}{1-z}\Big]^{2\mu}+\frac{2a\mu z}{(1-z^2)}$$

for $z\in U,\ z\neq 0,\mu\neq 0,$ *and* $a\neq 0$, *then*

$$\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}\prec\Big[\frac{1+z}{1-z}\Big]^{\mu}$$

and $q(z)=[\frac{1+z}{1-z}]^{\mu}$ *is the best dominant.*

Corollary 2.4. *Assume that (2.1) holds and q is convex univalent in U. If f is analytic in U and*

$$b\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}+c\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{2\mu}+a\mu\Big[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta}f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}\Big]\prec be^{\mu Az}+ce^{2\mu Az}+aA\mu z$$

for $z\in U,\ \mu\neq 0,$ *and* $a\neq 0$, *then*

$$\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}\prec e^{\mu Az}$$

and $q(z)=e^{\mu Az}$ *is the best dominant.*

Next, by applying to Lemma 1.3, we prove the following theorem.

Theorem 2.5. *Let f,g be analytic in U, $[\frac{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ be convex univalent in U, $[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ be analytic in U and*

$$\Re\Big\{1+\frac{zG'(z)}{G(z)}+(\mu-1)\frac{z\rho_{\alpha,\beta,m}(z)}{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}G(z)+\frac{1}{\gamma}\Big\}>0,\ \gamma\in\mathbb{C},\ \mu\neq 0,\ G(z)\neq 0,\ z\in U,$$

where $G(z):=[\frac{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}{\rho_{\alpha,\beta,m}(z)}]'$. *If the subordination*

$$\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}\Big\{1+\gamma\mu\Big[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta}f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}\Big]\Big\}\prec\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}\Big\{1+\gamma\mu\Big[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta}g(z))'}{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}\Big]\Big\}$$

holds then

$$\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}f(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}\prec\Big[\frac{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}{\rho_{\alpha,\beta,m}(z)}\Big]^{\mu}$$

and $[\frac{I_{0,z,m}^{\alpha,\beta,\eta}g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ *is the best dominant.*

Proof. Denotes
$$p(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \text{ and } q(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}.$$

We can observe that

$$\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} = \Re\{1 + \frac{zG'(z)}{G(z)} + (\mu - 1)\frac{z\rho_{\alpha,\beta,m}(z)}{I_{0,z,m}^{\alpha,\beta,\eta} g(z)} G(z) + \frac{1}{\gamma}\} > 0$$

where $\psi = 1$. By the assumption of the theorem we have

$$\begin{aligned} p(z) + \gamma z p'(z) &= [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\} \\ &\prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} g(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} g(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\} \\ &= q(z) + \gamma z q'(z). \end{aligned}$$

Hence in view of Lemma 1.3, we obtain $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

Theorem 2.6. *Let the function $q(z)$ be analytic and convex univalent in the unit disk U such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Let*

$$\Re\{q(z)q'(z)[\frac{b}{a} + \frac{2c}{a}q(z)]\} > 0,\ a, b, c, \in \mathbb{C}, a \neq 0. \tag{2.2}$$

Assume that $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$ and

$$b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} g(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} g(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]$$

is univalent in U. If the subordination

$$bq(z) + cq^2(z) + \frac{azq'(z)}{q(z)} \prec b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]$$

holds. Then
$$q(z) \prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$$

and $q(z)$ is the best subordinant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu},\ z \in U,\ \rho_{\alpha,\beta,m}(z) \neq 0, \mu \neq 0.$$

By setting

$$\vartheta(\omega) := b\omega + c\omega^2 \textit{ and } \varphi(\omega) := \frac{a}{\omega},$$

it can easily be observed that $\vartheta(\omega)$ is analytic in $\mathbb{C}$, $\varphi(\omega)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\varphi(\omega) \neq 0$, $\omega \in \mathbb{C}\backslash\{0\}$. By the assumption of the theorem we find

$$\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))} = \Re\{q(z)q'(z)[\frac{b}{a} + \frac{2c}{a}q(z)]\} > 0$$

and that

$$bq(z) + cq^2(z) + \frac{azq'(z)}{q(z)} \prec b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]$$

$$= bp(z) + cp^2(z) + \frac{azp'(z)}{p(z)}.$$

The assertion of the theorem follows by an application of Lemma 1.4. □

Theorem 2.7. *Let* $f, g \in \mathbf{A}$, $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ *be convex univalent in* U, $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \in \mathbf{H}[q(0) \cap Q]$ *and*

$$[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\},\ \Re\{\gamma\} > 0,\ \mu \neq 0,$$

is univalent in U*. If the subordination*

$$[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} g(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} g(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\} \prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\}$$

holds then

$$[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$$

and $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ *is the best subordinant.*

Proof. Denotes

$$p(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \textit{ and } q(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}.$$

By the assumption of the theorem we have

$$q(z) + \gamma z q'(z) = [\frac{I_{0,z,m}^{\alpha,\beta,\eta} g(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} g(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} g(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\}$$

$$\prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\}$$

$$= p(z) + \gamma z p'(z).$$

Hence in view of Lemma 1.5, we obtain $q(z) \prec p(z)$ and $q(z)$ is the best subordinant. □

Combining Theorem 2.1 and Theorem 2.6 and Theorem 2.5 and Theorem 2.7, we get the following sandwich theorems.

Theorem 2.8. *Let the function $q_1(z)$ be analytic and convex univalent in the unit disk U such that $q_1(z) \neq 0$ and $\frac{zq_1'(z)}{q_1(z)}$ be starlike univalent in U with*

$$\Re\{q_1(z)q_1'(z)[\frac{b}{a} + \frac{2c}{a}q_1(z)]\} > 0,\ a, b, c, \in \mathbb{C},\ a \neq 0. \tag{2.3}$$

Let the function $q_2(z)$ be univalent in the unit disk U such that $q_2(z) \neq 0$. Suppose that $\frac{zq_2'(z)}{q_2(z)}$ is starlike univalent in U such that

$$\Re\{\frac{b}{a}q_2(z) + \frac{2c}{a}q_2^2(z) + \frac{zq_2''(z)}{q_2'(z)} - \frac{zq_2'(z)}{q_2(z)}\} > 0,\ a, b, c, \in \mathbb{C},\ a \neq 0 \tag{2.4}$$

Assume that $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \in \mathbf{H}[q_1(0), 1] \cap Q$ and

$$b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]$$

is univalent in U. If the subordination

$$bq_1(z) + cq_1^2(z) + \frac{azq_1'(z)}{q_1(z)} \prec b[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} + c[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{2\mu} + a\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]$$
$$\prec bq_2(z) + cq_2^2(z) + \frac{azq_2'(z)}{q_2(z)}$$

holds. Then

$$q_1(z) \prec [\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \prec q_2(z)$$

and $q_1(z), q_2(z)$ are respectively the best subordinant and best dominant.

Theorem 2.9. *Let f, g_1, g_2 be analytic in U. $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g_1(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$, $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} g_2(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ be convex univalent in U,*

$$\Re\{1 + \frac{zG_2'(z)}{G_2(z)} + (\mu - 1)\frac{z\rho_{\alpha,\beta,m}(z)}{I_{0,z,m}^{\alpha,\beta,\eta} g_2(z)}G_2(z) + \frac{1}{\gamma}\} > 0,\ \gamma \in \mathbb{C},\ \mu \neq 0,\ G_2(z) \neq 0,\ z \in U,$$

where $G_i(z) := [\frac{I_{0,z,m}^{\alpha,\beta,\eta} g_i(z)}{\rho_{\alpha,\beta,m}(z)}]'$, $i = 1, 2$. Also, assume that $[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu} \in \mathbf{H}[q_1(0) \cap Q]$ and analytic in U with

$$[\frac{I_{0,z,m}^{\alpha,\beta,\eta} f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1 + \gamma\mu[\frac{z(I_{0,z,m}^{\alpha,\beta,\eta} f(z))'}{I_{0,z,m}^{\alpha,\beta,\eta} f(z)} - \frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\},\ \Re\{\gamma\} > 0,\ \mu \neq 0,$$

is univalent in U. If the subordination

$$[\frac{I^{\alpha,\beta,\eta}_{0,z,m}g_1(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1+\gamma\mu[\frac{z(I^{\alpha,\beta,\eta}_{0,z,m}g_1(z))'}{I^{\alpha,\beta,\eta}_{0,z,m}g_1(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\}\prec[\frac{I^{\alpha,\beta,\eta}_{0,z,m}f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1+\gamma\mu[\frac{z(I^{\alpha,\beta,\eta}_{0,z,m}f(z))'}{I^{\alpha,\beta,\eta}_{0,z,m}f(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\}$$

$$\prec[\frac{I^{\alpha,\beta,\eta}_{0,z,m}g_2(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\{1+\gamma\mu[\frac{z(I^{\alpha,\beta,\eta}_{0,z,m}g_2(z))'}{I^{\alpha,\beta,\eta}_{0,z,m}g_2(z)}-\frac{z\rho'_{\alpha,\beta,m}(z)}{\rho_{\alpha,\beta,m}(z)}]\}$$

holds then

$$[\frac{I^{\alpha,\beta,\eta}_{0,z,m}g_1(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}[\frac{I^{\alpha,\beta,\eta}_{0,z,m}f(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}\prec[\frac{I^{\alpha,\beta,\eta}_{0,z,m}g_2(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$$

and $[\frac{I^{\alpha,\beta,\eta}_{0,z,m}g_1(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$, $[\frac{I^{\alpha,\beta,\eta}_{0,z,m}g_2(z)}{\rho_{\alpha,\beta,m}(z)}]^{\mu}$ *are respectively the best subordinant and best dominant.*

3 Existence of univalent solution.

In this section, we establish the existence of univalent solution for equation (1.1). Let $\mathcal{B} := \mathcal{C}[U,\mathbb{C}]$ be a Banach space of all continuous functions on U endowed with the sup. norm

$$\|u\| := sup_{z\in U}|u(z)|.$$

(H1) Assume the function $F : U\times\mathcal{B}\to\mathbb{C}-\{0\}$ is analytic for $z\in U$ and continuous in u satisfies that there exist a positive function $\Omega : U\to\mathbb{R}_+$ such that

$$|F(z,u)|\le\Omega(z)\ z\in U,\ u\in\mathbb{C}.$$

(H2) Denotes $B_n := \frac{(\alpha+\beta)_n(-\eta)_n}{(\alpha)_n(1)_n}$.

Theorem 3.1. *(Existence) Let the assumptions (H1) and (H2) hold. Then (1.1) has at least one locally univalent solution.*

Proof. Operating by $I^{\alpha,\beta,\eta}_{0,z,m}$, the initial value problem (1.1) becomes

$$u(z) = I^{\alpha,\beta,\eta}_{0,z,m}F(z,u(z)),\ \alpha>0,\beta<0,\eta\in\mathbb{R}. \tag{3.1}$$

Define an operator $P : \mathcal{B}\to\mathcal{B}$ by

$$(Pu)(z) := \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)}\int_0^z(z^m-\zeta^m)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta^m}{z^m})F(\zeta,u(\zeta))d\zeta^m.$$

We need only to show that P has a fixed point by applying Theorem 1.9.

$$
\begin{aligned}
|(Pu)(z)| &= |\frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m-\zeta^m)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta^m}{z^m}) F(\zeta,u(\zeta))d\zeta^m| \\
&\le |\Omega(z)||\frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m-\zeta^m)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta^m}{z^m}) d\zeta^m| \\
&\le |\Omega(z)|| \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n(-\eta)_n}{(\alpha)_n(1)_n} \frac{z^{-m(\alpha+\beta+n)}}{\Gamma(\alpha)} \int_0^z (z^m-\zeta^m)^{n+\alpha-1} d\zeta^m| \\
&\le \|\Omega\| \sum_{n=0}^{\infty} |B_n \frac{z^{-m\beta}}{\Gamma(\alpha+1)+n\Gamma(\alpha)}| \le \|\Omega\| \sum_{n=0}^{\infty} \frac{|B_n|}{\Gamma(\alpha+1)+n\Gamma(\alpha)} |z|^{-m\beta} \\
&< \sum_{n=0}^{\infty} \frac{\|\Omega\||B_n|}{\Gamma(\alpha+1)+n\Gamma(\alpha)} := r.
\end{aligned}
$$

Thus $P: B_r \to B_r$ and P maps B_r into itself. Now we proceed to prove that P is equicontinuous. For $z_1, z_2 \in U$ such that $z_1 \neq z_2$, we have $|z_1^m - z_2^m| < \delta,\ \delta > 0$. Then $\forall u \in S := \{u \in \mathbb{C},: |u| \le r, r > 0\}$

$$
\begin{aligned}
|(Pu)(z_1)-(Pu)(z_2)| &\le |\Omega(z)||\frac{z_1^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^{z_1} (z_1^m-\zeta^m)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta^m}{z_1^m}) d\zeta^m \\
&\quad - \frac{z_2^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^{z_2} (z_2^m-\zeta^m)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta^m}{z_2^m}) d\zeta^m| \\
&\le |\Omega(z)|| \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n(-\eta)_n}{(\alpha)_n(1)_n} [\frac{z_1^{-m(\alpha+\beta+n)}}{\Gamma(\alpha)} \int_0^{z_1} (z_1^m-\zeta^m)^{n+\alpha-1} d\zeta^m \\
&\quad - \frac{z_2^{-m(\alpha+\beta+n)}}{\Gamma(\alpha)} \int_0^{z_2} (z_2^m-\zeta^m)^{n+\alpha-1} d\zeta^m]| \\
&\le \|\Omega\| \sum_{n=0}^{\infty} \frac{|B_n|}{\Gamma(\alpha+1)+n\Gamma(\alpha)} |z_1^{-m\beta} - z_2^{-m\beta}| \\
&\le \|\Omega\| \sum_{n=0}^{\infty} \frac{|B_n|}{\Gamma(\alpha+1)+n\Gamma(\alpha)} |(z_1^m)^{-\beta} - (z_2^m)^{-\beta} + 2(z_1^m - z_2^m)^{-\beta}| \\
&< \sum_{n=0}^{\infty} \frac{2\|\Omega\||B_n|}{\Gamma(\alpha+1)+n\Gamma(\alpha)} |z_1^m - z_2^m|^{-\beta} \\
&< \sum_{n=0}^{\infty} \frac{2\|\Omega\||B_n|}{\Gamma(\alpha+1)+n\Gamma(\alpha)} \delta^{-\beta}
\end{aligned}
$$

which is independent on u. Hence P is an equicontinuous mapping on S. In view of the assumption of the theorem we have P is a univalent function (see [12]). The Arzela-Ascoli theorem (Theorem 1.8) yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a univalent solution of the initial value problem (1.1). □

In the following theorem we show the relation of univalent solution for the initial value problem (1.1).

Theorem 3.2. *Let the assumptions of Theorem 2.8 be satisfied. Then every univalent solution* $u(z)$ *of the problem (1.1) satisfies the subordination* $q_1(z) \prec u(z) \prec q_2(z)$, *where* $q_1(z)$ *and* $q_2(z)$ *are univalent function in* U.

Proof. Setting $\mu = 1$, $F(z,u(z)) := f(z)$ and $\rho_{\alpha,\beta,m}(z) \equiv 1$. □

Theorem 3.3. *Let the assumptions of Theorem 2.6 be satisfied. Then univalent solutions* u_1, u, u_2, *of the problem (1.1) are satisfying the subordination* $u_1 \prec u \prec u_2$.

Proof. Setting $\mu = 1$ and let $F(z,u_1(z)) := g_1(z), F(z,u_2(z)) := g_2(z), F(z,u(z)) := f(z)$ and $\rho_{\alpha,\beta,m}(z) \equiv 1$. □

Note also, the authors have recently established the existence solutions for different types of differential equations of fractional order in the complex plane (see [13, 14, 15, 16]).

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