African Diaspora Journal of Mathematics
Volume 10, Number 2, pp. 72–80 (2010)
ISSN 1539-854X
www.math-res-pub.org/adjm

The Orthogonal Complement Relative to the Functor Extension of the Class of all Gorenstein Projective Modules

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Abstract

In this paper, we study the pair $(\mathcal{G}P(R), \mathcal{G}P(R)^{\perp})$ where $\mathcal{G}P(R)$ is the class of all Gorenstein projective modules. We prove that it is a complete hereditary cotorsion theory, provided $l.\mathrm{Ggldim}(R) < \infty$. We discuss also, when every Gorenstein projective module is Gorenstein flat.

AMS Subject Classification: 13D05, 13D02.

Keywords: Cotorsion theory, Gorenstein homological dimensions of modules and rings.

1 Introduction

Throughout the paper, all rings are associative with identity, and an R-module will mean left R-module unless explicitly stated otherwise.
Let R be a ring, and let M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M. We denote by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ the *character module* of M.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finitely generated R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [5, 6] introduced the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [8] introduced the Gorenstein flat dimension. Some references are [2, 3, 4, 5, 6, 8, 12].

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Recall that an R-module M is called *Gorenstein projective*, if there exists an exact sequence of projective R-modules:

$$\mathbf{P}: \cdots \longrightarrow P_{-1} \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P_1)$ and such that the functor $\mathrm{Hom}_R(-,Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. The complex $\mathbf{P}$ is called a *complete projective resolution*.

The *Gorenstein injective* R-modules is defined dually.

An R-module M is called *Gorenstein flat*, if there exists an exact sequence of flat R-modules:

$$\mathbf{F}: \cdots \longrightarrow F_{-2} \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F_1)$ and such that the functor $I \otimes_R -$ leaves $\mathbf{F}$ exact whenever I is a right injective R-module. The complex $\mathbf{F}$ is called a complete flat resolution.

The Gorenstein projective, injective, and flat dimensions are defined in terms of resolutions and denoted by $\mathrm{Gpd}(-)$, $\mathrm{Gid}(-)$, and $\mathrm{Gfd}(-)$, respectively (see [3, 7, 12]).

Notation. By $\mathcal{P}(R)$ and $\mathcal{I}(R)$ we denote the classes of all projective and injective R-modules, respectively, and by $\overline{\mathcal{P}(R)}$ and $\overline{\mathcal{I}(R)}$ we denote the classes of all modules with finite projective dimensions and injective dimensions, respectively. Furthermore, we let $\mathcal{GP}(R)$ and $\mathcal{GI}(R)$ denote the classes of all Gorenstein projective and injective R-modules, respectively.

In [2], the authors proved the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid \text{M is a (left) R-module}\} = \sup\{\mathrm{Gid}_R(M) \mid \text{M is a (left) R-module}\}.$$

They called the common value of the above quantities the *left Gorenstein global dimension* of R and denoted it by $l.\mathrm{Ggldim}(R)$. Similarly, they set

$$l.\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid \text{M is a (left) R-module}\}$$

which they called the *left weak Gorenstein global dimension* of R.

Given a class $\mathfrak{X}$ of R-modules we set:

$$\begin{array}{rcccl} \mathfrak{X}^{\perp} & = & \ker \mathrm{Ext}^1_R(\mathfrak{X},-) & = & \{M \mid \mathrm{Ext}^1_R(X,M) = 0 \text{ for all } X \in \mathfrak{X}\}. \\ {}^{\perp}\mathfrak{X} & = & \ker \mathrm{Ext}^1_R(-,\mathfrak{X}) & = & \{M \mid \mathrm{Ext}^1_R(M,X) = 0 \text{ for all } X \in \mathfrak{X}\}. \end{array}$$

The class $\mathfrak{X}^{\perp}$ (resp.,${}^{\perp}\mathfrak{X}$) is usually called the *right (resp., left) orthogonal complement relative to the functor* $\mathrm{Ext}^1_R(-,-)$ *of the class* $\mathfrak{X}$.

Definition 1.1 (**Precovers and Preenvelopes).** Let $\mathfrak{X}$ be any class of R-modules and let M be an R-module.

- An $\mathfrak{X}$-*precover* of M is an R-homomorphism $\varphi : X \to M$ where $X \in \mathfrak{X}$ and such that the sequence

$$\mathrm{Hom}_R(X',X) \xrightarrow{\mathrm{Hom}_R(X',\varphi)} \mathrm{Hom}_R(X',M) \longrightarrow 0$$

  is exact for every $X' \in \mathfrak{X}$. An $\mathfrak{X}$-precover is called *special*, if φ is surjective and $\ker(\varphi) \in \mathfrak{X}^{\perp}$.

- An $\mathfrak{X}$-*preenvelope* of M is an R-homomorphism $\varphi : M \to X$ where $X \in \mathfrak{X}$ and such that the sequence,

$$\mathrm{Hom}_R(X,X') \xrightarrow{\mathrm{Hom}_R(\varphi,X')} \mathrm{Hom}_R(M,X') \longrightarrow 0$$

is exact for every $X' \in \mathfrak{X}$. An $\mathfrak{X}$-preenvelope is called *special*, if φ is injective and $\mathrm{coker}(\varphi) \in {}^{\perp}\mathfrak{X}$.

For more details about precovers (and preenvelopes), the reader may consult [7, Chapters 5 and 6].

Definition 1.2 ([12], **Resolving classes 1.1**). For any class $\mathfrak{X}$ of R-modules.

- We call $\mathfrak{X}$ *projectively resolving*, if $\mathcal{P}(R) \subseteq \mathfrak{X}$ and for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X" \longrightarrow 0$ with $X" \in \mathfrak{X}$ the conditions $X' \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent.

- We call $\mathfrak{X}$ *injectively resolving*, if $\mathcal{I}(R) \subseteq \mathfrak{X}$ and for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X" \longrightarrow 0$ with $X' \in \mathfrak{X}$ the conditions $X" \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent.

A pair $(\mathfrak{X},\mathfrak{Y})$ of classes of R-modules is called a *cotorsion theory* [7], if $\mathfrak{X}^{\perp} = \mathfrak{Y}$ and ${}^{\perp}\mathfrak{Y} = \mathfrak{X}$. In this case, we call $\mathfrak{X} \cap \mathfrak{Y}$ the kernel of $(\mathfrak{X},\mathfrak{Y})$. Note that each element K of the kernel is a splitter in the sense of [11], *i.e.,* $\mathrm{Ext}^1_R(K,K) = 0$. If $\mathfrak{C}$ is any class of modules, then $({}^{\perp}\mathfrak{C},({}^{\perp}\mathfrak{C})^{\perp})$ is easily seen be a cotorsion theory, called a *cotorsion theory generated* by $\mathfrak{C}$ (see [13, Definition 1.10]). A cotorsion theory $(\mathfrak{X},\mathfrak{Y})$ is called *complete* [13], if every R-module has a special $\mathfrak{Y}$-preenvelope (or equivalently every R-module has a special $\mathfrak{X}$-precover; see [13, Lemma 1.13]). A cotorsion theory $(\mathfrak{X},\mathfrak{Y})$ is said to be *hereditary* [10], if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathfrak{X}$, then L' is also in $\mathfrak{X}$, or equivalently, if $0 \to M' \to M \to M" \to 0$ is exact $M', M \in \mathfrak{Y}$, then M'' is also in $\mathfrak{Y}$.

The aim of this paper is the study of the pair $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$.
Note: Below, we have only proved the results concerning the Gorenstein projective modules. The proofs of the Gorenstein injective ones are dual, and we can find a dual of the results using in the proofs in [12].

2 Lemmas

In this section, we recall some fundamental results about Gorenstein projective modules and dimensions. These results are extracted from the work of Holm in [12].

The first lemma shows that the class of Gorenstein projective modules is projectively resolving:

Lemma 2.1 ([12], Theorems 2.5)**.** *Let R be a ring. Then, The class* $\mathcal{GP}(R)$ *is projectively resolving. Moreover, it is closed under direct sums and direct summands.*

The next lemma study the $\mathcal{GP}(R)$-precovers of R-modules with finite Gorenstein projective dimension.

Lemma 2.2 ([12], Theorems 2.10)**.** *Let M be an R-module with finite Gorenstein projective dimension n. Then M admits a surjective $\mathcal{GP}(R)$-precover, $\varphi : G \twoheadrightarrow M$, where $K = \ker\varphi$ satisfies $\mathrm{pd}_R K = n-1$ (if $n = 0$, this should be interpreted as $K = 0$).*

In the the following lemma, Holm gave a functorial description of the finite Gorenstein projective dimension of modules.

Lemma 2.3 ([12], Theorem 2.20)**.** *Let M be an R-module with finite Gorenstein projective dimension, and let n be an integer. Then the following conditions are equivalent:*

1. $\mathrm{Gpd}_R(M) \leq n$.
2. $\mathrm{Ext}^i_R(M,L) = 0$ *for all $i > n$, and all R-modules L with finite* $\mathrm{pd}_R(L)$.
3. $\mathrm{Ext}^i_R(M,Q) = 0$ *for all $i > n$, and all projective R-modules Q.*
4. *For every exact sequence of R-module $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ where $G_0,...,G_{n-1}$ are Gorenstein projectives, K_n is also Gorenstein projective.*

Recall that the finitistic projective dimension of R is defined as:

$$\mathrm{FPD}(R) = \sup\{\mathrm{pd}_R(M) \mid \text{M is an } R\text{-module with } \mathrm{pd}_R(M) < \infty\}$$

Lemma 2.4 ([12], Theorems 2.28)**.** *For any ring R there is an equality*

$$\mathrm{FPD}(R) = \sup\{\,\mathrm{Gpd}_R(M) \mid M \textit{ is an R-module with finite Gorenstein projective dimension}\}.$$

3 Main results

We begin with the following theorem:

Theorem 3.1. *For any ring R, the following holds:*

1. $\mathrm{Ext}^i_R(G,M) = 0$ *for all $i > 0$, all $G \in \mathcal{GP}(R)$, and all $M \in \mathcal{GP}(R)^\perp$.*
2. $\mathrm{Ext}^i_R(M,G) = 0$ *for all $i > 0$, all $G \in \mathcal{GI}(R)$, and all $M \in {}^\perp\mathcal{GI}(R)$.*
3. *$\mathcal{GP}(R)^\perp$ and ${}^\perp\mathcal{GI}(R)$ are projectively resolving.*
4. *$\mathcal{GP}(R)^\perp$ and ${}^\perp\mathcal{GI}(R)$ are injectively resolving.*

Proof. (1) Let M and G be an arbitrary elements of $\mathcal{GP}(R)^\perp$ and $\mathcal{GP}(R)$, respectively, and let $n > 1$ be an integer. Pick an exact sequence $0 \to G' \to P_1 \to ... \to P_n \to G \to 0$ where all P_i are projectives. By the projectively resolving of $\mathcal{GP}(R)$ (Lemma 2.1), G' is clearly Gorenstein projective. Consequently, we have $\mathrm{Ext}^n_R(G,M) = \mathrm{Ext}^1_R(G',M) = 0$, as desired.

(2) By a dual argument to (1).

(3) We claim that $\mathcal{GP}(R)^\perp$ is projectively resolving. Using the long exact sequence in homology, we conclude that $\mathcal{GP}(R)^\perp$ is closed by extension, *i.e.,* if $0 \to M \to M' \to M'' \to 0$ is an exact sequence where M and M'' are in $\mathcal{GP}(R)^\perp$, then so is M'. In addition, from the

definition of Gorenstein projective modules, it is clear that $\mathcal{P}(R) \subseteq \mathcal{GP}(R)^{\perp}$. Now, consider a short exact sequence $0 \to M \to M' \to M'' \to 0$ where M' and M'' are in $\mathcal{GP}(R)^{\perp}$. For an arbitrary Gorenstein projective R-module G, consider a short exact sequence $0 \to G \to P \to G' \to 0$ where P is projective and G' is Gorenstein projective (such a sequence exists by the definition of Gorenstein projective modules). From the long exact sequence of homology, we have:

$$\cdots \to \mathrm{Ext}_R^1(G', M'') \to \mathrm{Ext}_R^2(G', M) \to \mathrm{Ext}_R^2(G', M') \to \dots$$

Then, $\mathrm{Ext}_R^2(G', M) = 0$ since $\mathrm{Ext}_R^1(G', M'') = \mathrm{Ext}_R^2(G', M') = 0$ (from (1) above). Accordingly, $\mathrm{Ext}_R^1(G, M) = \mathrm{Ext}_R^2(G', M) = 0$, as desired.

(4) We claim that $\mathcal{GP}(R)^{\perp}$ is injectively resolving. Clearly, $\mathcal{I}(R) \subseteq \mathcal{GP}(R)^{\perp}$, and $\mathcal{GP}(R)^{\perp}$ is closed by extension. Now, consider a short exact sequence $0 \to M \to M' \to M'' \to 0$ where M and M' belongs to $\mathcal{GP}(R)^{\perp}$. Using the long exact sequence of homology, for all Gorenstein projective module G, we have

$$\cdots \to \mathrm{Ext}_R^1(G, M') \to \mathrm{Ext}_R^1(G, M'') \to \mathrm{Ext}_R^2(G, M) \to \dots$$

Thus, from (1), $\mathrm{Ext}_R^1(G, M'') = 0$. Hence, $M'' \in \mathcal{GP}(R)^{\perp}$. Consequently, $\mathcal{GP}(R)^{\perp}$ is injectively resolving.

From the above theorem, we conclude the following two corollary.

Corollary 3.2. *For any ring R,*

1. $\mathcal{P}(R) = \mathcal{GP}(R) \cap \mathcal{GP}(R)^{\perp}$.
2. $\mathcal{I}(R) = \mathcal{GI}(R) \cap {}^{\perp}\mathcal{GI}(R)$.

Proof. (1) Let $M \in \mathcal{GP}(R) \cap \mathcal{GP}(R)^{\perp}$ and consider a short exact sequence $0 \to M' \to P \to M \to 0$ where P is projective. Since $\mathcal{GP}(R)^{\perp}$ is projectively resolving (by Theorem 3.1), $M' \in \mathcal{GP}(R)^{\perp}$. Then, $\mathrm{Ext}_R^1(M, M') = 0$. Therefore, this short exact sequence splits. Consequently, M is a direct summand of P, and then projective.

(2) By a dual proof to (1).

Corollary 3.3.

1. *[12, Proposition 2.27] Every Gorenstein projective (resp., injective) module with finite projective (resp., injective) dimension is projective (resp., injective).*
2. *Every Gorenstein projective (resp., injective) module with finite injective (resp., projective) dimension is projective (resp., injective).*

Proof. (1) If M is a Gorenstein projective module with finite projective dimension, then $M \in \mathcal{GP}(R) \cap \mathcal{GP}(R)^{\perp}$ (by Lemma 2.3). Consequently, M is projective (by Corollary 3.2).

The injective case is dual.

(2) Note that every module I with $\mathrm{id}_R(I) := n < \infty$ belongs to $\mathcal{GP}(R)^{\perp}$. Indeed, by the definition of the Gorenstein projective modules, for each Gorenstein projective module G we can find an exact sequence $0 \to G \to P_{n-1} \to \dots \to P_0 \to G' \to 0$ where all P_i are projective and G' is Gorenstein projective. Thus, we have $\mathrm{Ext}_R^1(G, I) = \mathrm{Ext}_R^{n+1}(G', I) = 0$.

Now, if M is a Gorenstein projective module with finite injective dimension, then $M \in \mathcal{GP}(R) \cap \mathcal{GP}(R)^{\perp}$. Accordingly, by Corollary 3.2, M is projective.

Dually, we can prove that every module with finite projective dimension is an element of ${}^{\perp}\mathcal{GI}(R)$. Consequently, by Corollary 3.2, every Gorenstein injective module with finite projective dimension is injective.

The main result of this paper is the following theorem:

Theorem 3.4. *If* $l.\mathrm{Ggldim}(R) < \infty$, *then* $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ *and* $({}^{\perp}\mathcal{GI}(R), \mathcal{GI}(R))$ *are complete, hereditary cotorsion theories.*

Proof. (1) To show that $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a cotorsion theory, we have to prove that ${}^{\perp}(\mathcal{GP}(R)^{\perp}) = \mathcal{GP}(R)$. Let M be an element of ${}^{\perp}(\mathcal{GP}(R)^{\perp})$. Since $\mathrm{Gpd}_R(M) < \infty$ and from Lemmas 2.2 and 2.3, M admits a surjective $\mathcal{GP}(R)$-precover $\varphi : G \to M$ where $K = \ker(\varphi) \in \mathcal{GP}(R)^{\perp}$. Then, G is a special $\mathcal{GP}(R)$-precover of M, and the short exact sequence $0 \to K \to G \to M \to 0$ splits since $\mathrm{Ext}^1_R(M,K) = 0$. Thus, M is a direct summand of G. Hence, M is Gorenstein projective (by Lemma 2.1). Consequently, ${}^{\perp}(\mathcal{GP}(R)^{\perp}) \subseteq \mathcal{GP}(R)$, while the other inclusion is clear. Therefore, $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a cotorsion theory, and every R-module has a special $\mathcal{GP}(R)$-precover. This implies that $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is complete. Moreover, since $\mathcal{GP}(R)$ is projectively resolving and $\mathcal{GP}(R)^{\perp}$ is injectively resolving, this cotorsion theory is hereditary.

(2) To prove the dual Gorenstein injective result, we use the dual result of Lemmas 2.1 and 2.2.

Proposition 3.5. *If* $l.\mathrm{Ggldim}(R) < \infty$, *then* $\mathcal{GP}(R)^{\perp} = \overline{\mathcal{P}(R)} = \overline{I(R)} = {}^{\perp}\mathcal{GI}(R)$.

Proof. Clearly, by Lemma 2.3, $\overline{\mathcal{P}(R)} \subseteq \mathcal{GP}(R)^{\perp}$. Now, let $M \in \mathcal{GP}(R)^{\perp}$ and N be an arbitrary R-module, and set $n := l.\mathrm{Ggldim}(R)$. We have, $\mathrm{Gpd}_R(N) \leq n$. Then, by Lemma 2.3, we can find an exact sequence

$$0 \longrightarrow G \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow N \longrightarrow 0$$

where all P_i are projective and G is Gorenstein projective. Thus, by Theorem 3.1, for all $j > 0$, $\mathrm{Ext}^{j+n}_R(N,M) = \mathrm{Ext}^j_R(G,M) = 0$. Consequently, $\mathrm{id}_R(M) \leq n$. Using [2, Corollary 2.7], $\overline{\mathcal{P}(R)} = \overline{I(R)}$ since $l.\mathrm{Ggldim}(R) < \infty$. Then, $M \in \overline{\mathcal{P}(R)}$. Accordingly, $\mathcal{GP}(R)^{\perp} = \overline{\mathcal{P}(R)}$. Similarly, we prove that ${}^{\perp}\mathcal{GI}(R) = \overline{I(R)}$. This finishes the proof.

Proposition 3.6. *If* $\mathcal{GP}(R) = {}^{\perp}(\overline{\mathcal{P}(R)})$ *and* $\mathcal{GP}(R)^{\perp} = \overline{\mathcal{P}(R)}$, *then* $\mathrm{FPD}(R) = l.\mathrm{Ggldim}(R)$.

Proof. From [13, Theorem 2.2], every R-module admits a special $\mathcal{GP}(R)^{\perp}$-preenvelope. On the other hand, by hypothesis, $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is the cotorsion theory generated by $\overline{\mathcal{P}(R)}$. Then, $(\mathcal{GP}(R), \overline{\mathcal{P}(R)})$ is a complete cotorsion theory. Therefore, every R-module M has a special $\mathcal{GP}(R)$-precover.

The inequality $\mathrm{FPD}(R) \leq l.\mathrm{Ggldim}(R)$ follows from Lemma 2.4. Now, suppose that $\mathrm{FPD}(R) \leq n$ and let M be an arbitrary R-module. We claim that $l.\mathrm{Ggldim}(R) < \infty$. From the first part of this proof, M admits a special $\mathcal{GP}(R)$-precover. Then, there is an exact sequence $0 \to K \to G \to M \to 0$ where G is Gorenstein projective and $K \in \mathcal{GP}(R)^{\perp} = \overline{\mathcal{P}(R)}$. Thus, $\mathrm{pd}_R(K) \leq n$, and so $\mathrm{Gpd}_R(M) \leq n+1$. Hence, $l.\mathrm{Ggldim}(R) \leq n+1 < \infty$. Consequently, by Lemma 2.4, $l.\mathrm{Ggldim}(R) = \sup\{\mathrm{Gpd}_R(M \mid \mathrm{Gpd}_R(M) < \infty\} = \mathrm{FPD}(R)$.

From the above propositions, we conclude the following characterization of the left Gorenstein global dimension of a ring R, provided $\mathrm{FPD}(R) < \infty$.

Corollary 3.7. *If* $\mathrm{FPD}(R) < \infty$*, then the following are equivalent:*

1. $l.\mathrm{Ggldim}(R) < \infty$.

2. $\mathcal{GP}(R) = {}^{\perp}\overline{\mathcal{P}(R)}$ *and* $\mathcal{GP}(R)^{\perp} = \overline{\mathcal{P}(R)}$.

Proof. $(1) \Rightarrow (2)$ The first equality follows from Lemma 2.3, whereas the second follows from Proposition 3.5.

$(2) \Rightarrow (1)$ Follows from Proposition 3.6.

Now, we discuss the rings over which "*every Gorenstein projective module is Gorenstein flat*".

Proposition 3.8. *For any ring R, the following are equivalent:*

1. *Every Gorenstein projective module is Gorenstein flat.*

2. $I^+ \in \mathcal{GP}(R)^{\perp}$ *for every right injective R-module I.*

3. $(F^+)^+ \in \mathcal{GP}(R)^{\perp}$ *for every flat R-module F.*

Proof. $(1) \Rightarrow (2)$ Let I be a right injective R-module. Since every Gorenstein projective R-module is Gorenstein flat, and by the definition of the Gorenstein flat modules, we have $\mathrm{Tor}^1_R(I,G) = 0$ for all $G \in \mathcal{GP}(R)$. By adjointness, we have $\mathrm{Ext}^1_R(G,I^+) = (\mathrm{Tor}^1_R(I,G))^+ = 0$. Consequently, $I^+ \in \mathcal{GP}(R)^{\perp}$.

$(2) \Rightarrow (1)$ Consider a complete projective resolution

$$\mathbf{P}: \quad \cdots \to P_{-2} \xrightarrow{f_2} P_{-1} \xrightarrow{f_{-1}} P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \to \ldots$$

We decompose it into a short exact sequences $0 \to G_i \to P_i \to G'_i \to 0$ where $G_i = \ker(f_i)$ and $G'_i = \mathrm{Im}(f_i)$. From [12, Observation 2.2], G_i and G'_i are Gorenstein projectives. Now, let I be a right injective R-module. By hypothesis, we have $(\mathrm{Tor}^1_R(I,G'_i))^+ = \mathrm{Ext}^1_R(G'_i,I^+) = 0$. Then, $\mathrm{Tor}^1_R(I,G'_i) = 0$. Therefore,

$$0 \to I \otimes_R G_i \to I \otimes_R P_i \to I \otimes_R G'_i \to 0$$

is exact. Thus, $I \otimes_R -$ keeps the exactness of $\mathbf{P}$. Then, $\mathbf{P}$ is a complete flat resolution. Consequently, every Gorenstein projective module is Gorenstein flat.

$(2) \Rightarrow (3)$ Let F be a flat R-module. Then, F^+ is a right injective R-module. Consequently, $(F^+)^+ \in \mathcal{GP}(R)^{\perp}$.

$(3) \Rightarrow (2)$ Let I be a right injective R-module. There exists a flat R-module F such that $F \to I^+ \to 0$ is exact. Then, $0 \to (I^+)^+ \to F^+$ is exact. However, $0 \to I \to (I^+)^+$ is exact (by [9, Proposition 3.52]). Thus, $0 \to I \to F^+$ is exact, and then I is a direct summand of F^+. Hence, I^+ is a direct summand of $(F^+)^+$. On the other hand, it is easy to see that $\mathcal{GP}(R)^{\perp}$ is closed under direct summands. Consequently, $I^+ \in \mathcal{GP}(R)^{\perp}$, as desired.

Proposition 3.9. *For any ring R,* $\sup\{\mathrm{Gfd}_R(M) \mid M \text{ is Gorenstein projective}\} = 0$ *or* ∞.

Proof. Note that if $\mathrm{Gfd}_R(M) \leq n$, then we have $\mathrm{Tor}^i_R(I,M) = 0$ for all $i > n$. Indeed, the case $n = 0$ follows directly from the definition of the Gorenstein flats modules, whereas the case $n > 0$ is deduced from the first case by an n-step projective resolution of M. Suppose that $\sup\{\mathrm{Gfd}_R(M) \mid M \text{ is Gorenstein projective}\} = n < \infty$. Then, $\mathrm{Ext}^{n+1}_R(G,I^+) = (\mathrm{Tor}^{n+1}_R(I,G))^+ = 0$ for every right injective module I and every Gorenstein projective module G. However, for every Gorenstein projective module G we can find an exact sequence $0 \to G \to P_{n-1} \to \ldots \to P_0 \to G' \to 0$ where all P_i are projective and G' is Gorenstein projective. Thus, $\mathrm{Ext}^1_R(G,I^+) = \mathrm{Ext}^{n+1}_R(G',I^+) = 0$. So, $I^+ \in \mathcal{GP}(R)^\perp$ for every right injective module I. Then, by Proposition 3.8, every Gorenstein projective module is Gorenstein flat. Consequently, $\sup\{\mathrm{Gfd}_R(M) \mid M \text{ is Gorenstein projective}\} = 0$, as desired.

A direct consequence of the above proposition is the following corollary:

Corollary 3.10. *If* $l.\mathrm{wGgldim}(R) < \infty$*, then every Gorenstein projective R-module is Gorenstein flat.*

Acknowledgement. The author would like to express his sincere thanks to the referee for his helpful suggestions.

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