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AN EXTENSION OF HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract

New kinds of Hardy-Hilbert's integral inequalities are presented.

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1 Introduction

If $f,g \geq 0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,$

$$0<\int_0^\infty f^p(x)dx<\infty \quad and \quad 0<\int_0^\infty g^q(x)dx<\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(x)dx\right)^{1/q} \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Inequality (1.1) is called Hardy-Hilbert's integral inequality (see [1]) and is important in analysis and applications (cf.Mitrinovic et al. [2]). Hardy-et al.[1] gave an inequality similar to (1.1) as:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy \leq pq\left(\int_0^\infty f^p(t)dt\right)^{1/p}\left(\int_0^\infty g^q(t)dt\right)^{1/q}, \tag{1.2}$$

where the constant factor pq is best possible.
Other mathematicians present generalizations or new kinds of (1.2) as follows:

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Theorem 1.1. *[3]. If*

$$\lambda > 0,\ p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ f,g \geq 0$$

such that

$$0 < \int_0^\infty t^{p-1-\lambda} f^p(t)dt < \infty \quad , \quad 0 < \int_0^\infty t^{q-1-\lambda} g^q(t)dt < \infty$$

then one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy \quad \leq \quad \frac{pq}{\lambda} \left(\int_0^\infty t^{p-1-\lambda} f^p(t)dt \right)^{1/p} \left(\int_0^\infty t^{q-1-\lambda} g^q(t)dt \right)^{1/q}, \tag{1.3}$$

where the constant factor $\frac{pq}{\lambda}$ *is the best possible.*

Theorem 1.2. *[4]. Suppose*

$$f,g \geq 0, \quad 0 < \int_0^\infty f^2(x)dx < \infty, \quad 0 < \int_0^\infty g^2(x)dx < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dxdy \quad \leq \quad c \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{1/2}, \tag{1.4}$$

where $c = \sqrt{2}\left(\pi - 2\arctan\sqrt{2}\right) \approx 1.7408.$

The Beta function denoted by B(p,q), is defined by

$$B(p,q) \quad = \quad \int_0^\infty t^{p-1}(1-t)^{q-1}dt \quad = \quad \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}dt, \quad p>0,\ q>0. \tag{1.5}$$

2 Main Results

The following is our main result

Lemma 2.1. *Let* $p > 0,\ q > 0$. *Then*

$$B(p,q) \quad = \quad \int_1^\infty \frac{x^{-p-q}}{(x-1)^{1-q}} dx. \tag{2.1}$$

Proof. The proof follows by putting $x = 1/t$ in (1.5). □

Theorem 2.2. *Assume that $f,g,h \geq 0$, h is homogeneous of degree λ, $a,b,\gamma,\lambda > 0$, $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{h(a\max\{x,y\},b\min\{x,y\})(a\max\{x,y\}+b\min\{x,y\})^\gamma}dxdy \tag{2.2}$$

$$\leq \left(C\int_0^\infty t^{(\alpha-1)(1-p)+\beta-\gamma-\lambda}f^p(t)dt\right)^{1/p}\left(K\int_0^\infty t^{(\beta-1)(1-q)+\alpha-\gamma-\lambda}g^q(t)dt\right)^{1/q},$$

provided the integrals on the R.H.S do exist, where

$$C = C_1 b^{-\beta}a^{\beta-\gamma-\lambda}+C_2 a^{-\beta}b^{\beta-\gamma-\lambda},\ K = K_1 b^{-\alpha}a^{\alpha-\gamma-\lambda}+K_2 a^{-\alpha}b^{\alpha-\gamma-\lambda},$$

$$C_1 = \int_0^\infty \frac{t^{\beta-1}}{h(1,t)(1+t)^\gamma}dt, \qquad C_2 = \int_0^\infty \frac{t^{\beta-1}}{h(t,1)(t+1)^\gamma}dt,$$

$$K_1 = \int_0^\infty \frac{t^{\alpha-1}}{h(1,t)(1+t)^\gamma}dt, \quad \textit{and } K_2 = \int_0^\infty \frac{t^{\beta-1}}{h(t,1)(t+1)^\gamma}dt.$$

Proof.

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{h(a\max\{x,y\},b\min\{x,y\})(a\max\{x,y\}+b\min\{x,y\})^\gamma}dxdy$$

$$= \int_0^\infty\int_0^\infty \frac{f(x)y^{\frac{\beta-1}{p}}}{x^{\frac{\alpha-1}{q}}h^{\frac{1}{p}}(a\max\{x,y\},b\min\{x,y\})(a\max\{x,y\}+b\min\{x,y\})^{\frac{\gamma}{p}}}$$

$$\times \frac{g(y)x^{\frac{\alpha-1}{q}}}{y^{\frac{\beta-1}{p}}h^{\frac{1}{q}}(a\max\{x,y\},b\min\{x,y\})(a\max\{x,y\}+b\min\{x,y\})^{\frac{\gamma}{q}}}dxdy$$

$$\leq \left(\int_0^\infty\int_0^\infty \frac{f^p(x)y^{\beta-1}}{x^{(\alpha-1)\frac{p}{q}}h(a\max\{x,y\},b\min\{x,y\})(a\max\{x,y\}+b\min\{x,y\})^\gamma}dxdy\right)^{1/p}$$

$$\times \left(\int_0^\infty\int_0^\infty \frac{g^q(y)x^{\alpha-1}}{y^{(\beta-1)\frac{q}{p}}h(a\max\{x,y\},b\min\{x,y\})(a\max\{x,y\}+b\min\{x,y\})^\gamma}dxdy\right)^{1/q}$$

$$= M^{1/p}N^{1/q}$$

We first consider

$$\begin{aligned} M &= \int_0^\infty\int_0^\infty \frac{f^p(x)y^{\beta-1}}{x^{(\alpha-1)\frac{p}{q}}h\left(a\max\{x,y\},b\min\{x,y\}\right)\left(a\max\{x,y\}+b\min\{x,y\}\right)^\gamma}dxdy \\ &= \int_0^\infty x^{(\alpha-1)(1-p)}f^p(x)(M_1+M_2)dx, \end{aligned}$$

where, for $x>0$,

$$M_1 = \int_0^x \frac{y^{\beta-1}}{h\left(a\max\{x,y\},b\min\{x,y\}\right)\left(a\max\{x,y\}+b\min\{x,y\}\right)^\gamma}dy$$

and

$$M_2 = \int_x^\infty \frac{y^{\beta-1}}{h\left(a\max\{x,y\},b\min\{x,y\}\right)\left(a\max\{x,y\}+b\min\{x,y\}\right)^\gamma}dy$$

Now,

$$\begin{aligned} M_1 &= \int_0^x \frac{y^{\beta-1}}{h\left(a\max\{x,y\},b\min\{x,y\}\right)\left(a\max\{x,y\}+b\min\{x,y\}\right)^\gamma}dy \\ &= \int_0^x \frac{y^{\beta-1}}{h(ax,by)(ax+by)^\gamma}dy \\ &= \int_0^x \frac{y^{\beta-1}}{(ax)^{\lambda+\gamma}h(1,by/ax)(1+by/ax)^\gamma}dy \\ &= b^{-\beta}(ax)^{\beta-\gamma-\lambda}\int_0^{b/a}\frac{u^{\beta-1}}{h(1,u)(1+u)^\gamma}du \qquad .(u=by/ax) \\ &\leq b^{-\beta}(ax)^{\beta-\gamma-\lambda}\int_0^\infty\frac{u^{\beta-1}}{h(1,u)(1+u)^\gamma}du \\ &= C_1a^{\beta-\gamma-\lambda}b^{-\beta}x^{\beta-\gamma-\lambda}. \end{aligned}$$

Also, via similar steps,

$$\begin{aligned} M_2 &\leq a^{-\beta}(bx)^{\beta-\gamma-\lambda}\int_0^\infty\frac{v^{\beta-1}}{h(v,1)(v+1)^\gamma}dv \\ &= C_2a^{-\beta}b^{\beta-\gamma-\lambda}x^{\beta-\gamma-\lambda}, \end{aligned}$$

and hence

$$M \leq C\int_0^\infty x^{(\alpha-1)(1-p)+\beta-\gamma-\lambda} f^p(x)dx.$$

Similarly

$$N \leq K\int_0^\infty y^{(\beta-1)(1-q)+\alpha-\gamma-\lambda} g^q(x)dy.$$

Collecting the above estimates, we obtain

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{h\left(a\max\{x,y\},b\min\{x,y\}\right)\left(a\max\{x,y\}+b\min\{x,y\}\right)^\gamma}dxdy$$

$$\leq \left(C\int_0^\infty t^{(\alpha-1)(1-p)+\beta-\gamma-\lambda} f^p(t)dt\right)^{1/p}\left(K\int_0^\infty t^{(\beta-1)(1-q)+\alpha-\gamma-\lambda} g^q(t)dt\right)^{1/q}.$$

The proof is complete. □

3 Applications

Corollary 3.1. *Assume that*

$$f,g \geq 0,\ a,b>0,\ 1/2<\lambda<\ p>1,\ \frac{1}{p}+\frac{1}{q}=1$$

Then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\left|a\max\{x,y\}-b\min\{x,y\}\right|^\lambda\left(a\max\{x,y\}+b\min\{x,y\}\right)^\lambda}dxdy \tag{3.1}$$

$$\leq C\left(\int_0^\infty t^{(\alpha-1)(1-p)+\beta-\gamma-\lambda} f^p(t)dt\right)^{1/p}\left(\int_0^\infty t^{(\beta-1)(1-q)+\alpha-\gamma-\lambda} g^q(t)dt\right)^{1/q},$$

provided the integrals on the R.H.S do exist, where

$$C=\frac{1}{2}\left(a^{-2\lambda}\left(\frac{a}{b}\right)^{2-2\lambda}+b^{-2\lambda}\left(\frac{b}{a}\right)^{2-2\lambda}\right)\left(B\left(1-\lambda,1-\lambda\right)+B\left(1-\lambda,2\lambda-1\right)\right).$$

Proof. The proof follows from theorem 2.2 via lemma 2.1 putting

$$\gamma=\lambda,\ \ \alpha=\beta=2-2\lambda,$$

as follows

$$K_2 = K_1 = C_2 = C_1 = \int_0^\infty \frac{t^{1-2\lambda}}{|1-t|^\lambda (1+t)^\lambda} dt = \left(\int_0^1 + \int_1^\infty \right) \frac{t^{1-2\lambda}}{|1-t|^\lambda (1+t)^\lambda} dt$$
$$= C_{11} + C_{12},$$

where

$$C_{11} = \int_0^1 \frac{t^{1-2\lambda}}{|1-t|^\lambda (1+t)^\lambda} dt = \int_0^1 \frac{t^{1-2\lambda}}{(1-t^2)^\lambda} dt = \frac{1}{2} \int_0^1 \frac{t^{-\lambda}}{(1-t)^\lambda} dt = \frac{1}{2} B(1-\lambda, 1-\lambda),$$

$$C_{12} = \int_1^\infty \frac{t^{1-2\lambda}}{|1-t|^\lambda (1+t)^\lambda} dt = \int_1^\infty \frac{t^{1-2\lambda}}{(t^2-1)^\lambda} dt = \frac{1}{2} \int_1^\infty \frac{t^{-\lambda}}{(t-1)^\lambda} dt = \frac{1}{2} B(1-\lambda, 2\lambda-1).$$

□

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