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EXCELLENT EXTENSION AND COMPARABILITY OF REGULAR RINGS

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Abstract

Let S be an excellent extension of a (von Neumann) regular ring R. In this note, we study comparability of S related to comparability of R. We show that if R has the n-unperforation property, then R satisfies s-comparability, almost comparability or weak comparability if and only if so does S.

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1 Introduction

Recall that S is a ring extension of R if there is a (unital) ring homomorphism $f : R \to S$. Let S be a ring and let R be a subring of S (with the same 1). S is called a finite normalizing extension of R if there exist elements $a_1, \ldots, a_n \in S$ such that $a_1 = 1,\ S = Ra_1 + \cdots + Ra_n$, $a_iR = Ra_i$ for all $i = 1, \ldots, n$. Finite normalizing extensions have been studied in many papers such as [10, 11, 12, 13], S is called a free normalizing extension of R if $a_1 = 1,\ S = Ra_1 + \cdots + Ra_n$ is finite normalizing extension and S is free with basis $\{a_1, \ldots, a_n\}$ as both a right R-module and a left R-module. S is said to be an excellent extension of R in case S is a free normalizing extension of R and S is right R-projective (that is, if M_S is a right S-module and N_S is a submodule of M_S , then $N_R \mid M_R$ implies $N_S \mid M_S$, where $N|M$ means N is a direct summand of M). Let S be an excellent extension of R. The following results are well-known:

(1) R is semisimple Artinian if and only if S is semisimple Artinian.

(2) R is regular if and only if S is regular.

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(3) R is right hereditary if and only if S is right hereditary.
More generally, $gldimR = gldimS$ and $wgldimR = wgldimS$, where $gldimR$ stands for the global dimension of R, and $wgldimR$ for the weak global dimension of R [7, Theorem 3].

Comparability concepts have proven to be particularly fruitful in the development of the theory of (von Neumann) regular rings. Goodearl and Handelman showed that directly finite regular rings satisfying 1-comparability have stable range one [3, Theorem 8.12]. Pardo showed that exchange ring satisfying s-comparability is separative, so has stable range 1, 2, or ∞.

The notion of almost comparability for regular rings was first introduced by Ara and Goodearl, for giving an alternative proof of the outstanding O'Meara's Theorem: directly finite simple regular rings with weak comparability are unit-regular. It was proved that for a regular ring R, R satisfies almost comparability if and only if every finitely generated projective R-module satisfies almost comparability [6, Theorem 1.9], and the almost comparability is Morita invariant [6, Theorem 1.11]. For the simple regular rings, s-comparability for some $s > 0$ is equivalent to the ring satisfying almost comparability [1, Theorem 1.4].

O'Meara first introduced the notion of weak comparability, and proved that simple directly finite regular rings with weak comparability must be unit-regular [3, Open Problem 3]. Many authors studied regular rings with weak comparability [2, 4, 5]. For the regular ring R with weak comparability, it was proved [5, Theorem 1.6] that $A \oplus C \prec B \oplus C$ implies $A \prec B$ for any finitely generated projective R-modules A, B and C with $B \neq 0$, and was proved [5, Theorem 1.8] that $nA \prec nB$ implies $A \prec B$ for any positive integer n and any finitely generated projective R-modules A and B. It was also proved that for a regular ring R, R satisfies weak comparability if and only if every finitely generated projective R-module satisfies weak comparability [5, Theorem 1.9],

For two R-modules M and N, we use $M \lesssim_{\oplus} N$ (respectively $M \lesssim N$) to denote that M is isomorphic to a direct summand of N (respectively M is isomorphic to a submodule of N), and $M \prec_{\oplus} N$ (respectively $M \prec N$) to denote that M is isomorphic to a proper direct summand of N (respectively M is isomorphic to a proper submodule of N). Let M and N be finitely generated projective R-modules. We write $M \lesssim_a N$ to mean that for any nonzero principal right ideal C of R, $M \lesssim_{\oplus} N \oplus C$, and $M \prec_a N$ to mean that for any nonzero principal right ideal C of R, $M \prec_{\oplus} N \oplus C$. Other basic notations can be found in [3]. Throughout this note, R is an associative ring with identity and R-modules are unitary right R-modules.

2 Main results

Lemma 2.1. *Let S be an excellent extension of R. Given any S-module M, M_R is projective if and only if M_S is projective.*

Proof. $\Leftarrow$: See [8, Lemma 7.2.2].

$\Rightarrow$: If M_R is projective, then there is a R-module N such that $M \oplus N \cong nR$ for some cardinal number n. So $(M \oplus N) \otimes_R S \cong (nR) \otimes_R S$, that is, $M \otimes_R S \oplus N \otimes_R S \cong nS$. So $M \otimes_R S$ is projective as S-module. We can consider $M \otimes 1$ as an S-module as following definition: $(m \otimes 1)s = ms \otimes 1$. Thus $M \otimes 1 \cong M$ as S-modules. As R-module, $M \otimes 1$ has natural R-module construction as $(m \otimes 1)r = mr \otimes 1 = m \otimes r$. Clearly, $(M \otimes_R S)_R = (\bigoplus_{i=1}^{n} M \otimes a_i)_R$.

So $(M\otimes 1)_R|(M\otimes S)_R$. By the R-projectivity of S, $(M\otimes 1)_S|(M\otimes S)_S$. So M_S is projective. $\square$

Lemma 2.2. *Let S be an excellent extension of R, and let $A_R\cong B_R$. Given A_S, we can define S-module B such that $A_S\cong B_S$.*

Proof. Let $\alpha: A_R\to B_R$ and $\beta: B_R\to A_R$ be the isomorphisms. Define $bs=\alpha(\beta(b)s)$. It is easy to check that B is an S-module such that $A_S\cong B_S$. $\square$

For a positive integer s, recall that in [3, Page 275] a regular ring R is said to satisfy s-comparability if, for each pair of elements $x,\ y$ of R, either $xR\lesssim s(yR)$, or $yR\lesssim s(xR)$. A finitely generated projective R-module M satisfies s-comparability if, for each pair of direct summands A and B of M, $A\lesssim sB$ or $B\lesssim sA$. Recall that for a positive integer n a ring R has the n-unperforation property if $nA\lesssim nB$ implies that $A\lesssim B$ for any finitely generated projective R-modules A and B. A ring R has the unperforation property if it has n-unperforation property for any positive integer n.

Theorem 2.3. *Let S be an excellent extension of a regular ring R. If R has the n-unperforation property, then R satisfies s-comparability if and only if so does S.*

Proof. $\Rightarrow$: Let $x,\ y\in S$. xS and yS are finitely generated projective S-modules. By Lemma 2.1, $(xS)_R$ and $(yS)_R$ are finitely generated projective. Since R satisfies s-comparability, by [1, Proposition 2.1], finitely generated projective R-modules satisfy s-comparability. Thus $(xS)_R\lesssim s(yS)_R$ or $(yS)_R\lesssim s(xS)_R$. If $(xS)_R\lesssim s(yS)_R$. Let T be the direct summand of $s(yS)_R$ such that $(xS)_R\cong T_R$. Since xS is an S-module, We can consider T as an S-module such that $(xS)_S\cong T_S$ as xS is S-modules by Lemma 2.2. Since $T_R\mid s(yS)_R$, by the R-projectivity of S, $T_S\mid s(yS)_S$. Thus $(xS)_S\lesssim s(yS)_S$. Similarly, we have $(yS)_S\lesssim s(xS)_S$, if $(yS)_R\lesssim s(xS)_R$.

$\Leftarrow$: For any $x,\ y\in R$, $(xR)_R\lesssim R_R\lesssim nR_R\cong S_R$. So $(xR)\otimes_R S$ and $(yR)\otimes_R S$ are finitely generated projective S-modules. Since S satisfies s-comparability, $((xR)\otimes_R S)_S\lesssim s((yR)\otimes_R S)_S$ or $((yR)\otimes_R S)_S\lesssim s((xR)\otimes_R S)_S$. $((xR)\otimes_R S)_R\lesssim s((yR)\otimes_R S)_R$ or $((yR)\otimes_R S)_R\lesssim s((xR)\otimes_R S)_R$. Since S ia a free R-module with basis $\{a_1,\ \ldots,\ a_n\}$, we have $((xR)\otimes_R S)_R\cong \sum_{i=1}^{n}(xR)\otimes_R a_i)_R\cong n(xR)_R$. Similarly, $((yR)\otimes_R S)_R\cong n(yR)_R$. Thus, $n(xR)_R\lesssim s(n(yR))_R$ or $n(yR)_R\lesssim s(n(xR))_R$. By the hypothesis, we have $(xR)_R\lesssim s(yR)_R$ or $(yR)_R\lesssim s(xR)_R$. $\square$

A regular ring R is said to satisfy almost comparability, if for $x,\ y\in R$ either $xR\lesssim_a yR$ or $yR\lesssim_a xR$. A finitely generated projective R-module M satisfies almost comparability, if for each pair of direct summands A and B of M, $A\lesssim_a B$ or $B\lesssim_a A$ [6].

Theorem 2.4. *Let S be an excellent extension of a regular ring R. If R has the n-unperforation property, then R satisfies almost comparability if and only if so does S.*

Proof. $\Rightarrow$: For any $x,\ y\in S$, since $(xS)_S$ and $(yS)_S$ are finitely generated projective S-modules, by Lemma 2.1, $(xS)_R$ and $(yS)_R$ are finitely generated projective R-modules. R satisfies almost comparability, by [6, Theorem 1.9], nR_R satisfies almost comparability for all positive integer n. Thus $(xS)_R \lesssim_a (yS)_R$ or $(yS)_R \lesssim_a (xS)_R$. Given any principal right ideal tS of S, which is cyclic projective S-module, it is finitely generated projective R-module. By [3, Proposition 2.6], there is a principal right ideal X of R such that $X \lesssim (tS)_R$. If $(xS)_R \lesssim_a (yS)_R$, then $(xS)_R \lesssim (yS)_R \oplus X \lesssim (yS)_R \oplus (tS)_R$. Since finitely generated submodule of projective module P is a direct summand of P, we have $(xS)_R \lesssim_{\oplus} (yS)_R \oplus (tS)_R$. By the R-projectivity of S and Lemma 2.2, we have $(xS)_S \lesssim_{\oplus} (yS)_S \oplus (tS)_S$, i.e., $(xS)_S \lesssim_a (yS)_S$. Similarly, if $(yS)_R \lesssim_a (xS)_R$, we have $(yS)_S \lesssim_a (xS)_S$.

$\Leftarrow$: For any $x,\ y \in R$, $(xR \otimes_R S)_S$, $(yR \otimes_R S)_S$ are finitely generated projective S-modules. S satisfies almost comparability, by [6, Theorem 1.9], nS_S satisfies almost comparability for all positive integer n. Thus $(xR \otimes_R S)_S \lesssim_a (yR \otimes_R S)_S$, or $(yR \otimes_R S)_S \lesssim_a (xR \otimes_R S)_S$. For any $z \in R$, if $(xR \otimes_R S)_S \lesssim_a (yR \otimes_R S)_S$, $(xR \otimes_R S)_S \lesssim (yR \otimes_R S)_S \oplus (zR \otimes_R S)_S$. So $(xR \otimes_R S)_R \lesssim (yR \otimes_R S)_R \oplus (zR \otimes_R S)_R$. It is easy to check that $(xR \otimes_R S)_R \cong n(xR)_R$, $(yR \otimes_R S)_R \cong n(yR)_R$ and $(zR \otimes_R S)_R \cong n(zR)_R$. Hence $n(xR)_R \lesssim n(yR)_R \oplus n(zR)_R$. By the hypothesis of n-unperforation property, $(xR)_R \lesssim (yR)_R \oplus (zR)_R$, i.e., $(xR)_R \lesssim_a (yR)_R$. Similarly, if $(yR \otimes_R S)_S \lesssim_a (xR \otimes_R S)_S$, we have $(yR)_R \lesssim_a (xR)_R$. □

A regular ring R satisfies weak comparability, if for each nonzero $x \in R$, ther is a positive integer $n = n(xR)$ such that $n(yR) \lesssim R$ impies that $yR \lesssim xR$. A finitely generated projective R-module M satisfies weak comparability, if for nonzero direct summand A of M, there is a positive integer $n = n(A)$ such that $nB \lesssim M$ impies that $B \lesssim A$ [5].

Theorem 2.5. *Let S be an excellent extension of a regular ring R. If R has the n-unperforation property, then R satisfies weak comparability if and only if so does S.*

Proof. $\Rightarrow$: We need to prove that for any nonzero $x \in S$, there is a positive integer $m = m((xS)_S)$ such that $m((yS)_S) \lesssim S$ implies that $(yS)_S \lesssim (xS)_S$. R satisfies weak comparability, by [5, Theorem 1.9], uR satisfies weak comparability for all positive integers u. Since $(xS)_S$ and $(yS)_S$ are finitely generated projective S-modules, by Lemma 1, $(xS)_R$ and $(yS)_R$ are finitely generated projective R-modules. Furthermore, for any $x \in R$, since $S = \oplus_{i=1}^{n} a_i R$, $(xS)_R = \sum_{i=1}^{n}(xa_i)R$, that is, $(xS)_R$ has at most n generated elements. Thus $(xS)_R$, $(yS)_R \lesssim nR_R$. By the weak comparability of nR_R, there is a positive integer $m_1 = m_1((xS)_R)$ such that $m_1((yS)_R) \lesssim nR$ implies that $(yS)_R \lesssim (xS)_R$. Let $m = m_1$. If $m((yS)_S) \lesssim S_S$, $m((yS)_R) \lesssim S_R \cong nR_R \lesssim nR_R$. Therefore, $(yS)_R \lesssim (xS)_R$. By the R-projectivity of S and Lemma 2.2, $(yS)_S \lesssim (xS)_S$.

$\Leftarrow$: We need to prove that for any nonzero $x \in R$, there is a positive integer $m = m((xR)_R)$ such that $m((yR)_R) \lesssim R$ implies that $(yR)_R \lesssim (xR)_R$. $(xR \otimes_R S)_S$ and $(yR \otimes_R S)_S$ are finitely generated projective S-modules. Furthermore, $(xR \otimes_R S)_S$, $(yR \otimes_R S)_S \lesssim S_S$ for all $y \in R$. Since S satisfies weak comparability, there is a positive integer $m_1 = m_1((xR \otimes_R S)_S)$ such that $m_1((yR \otimes_R S)_S) \lesssim S$ implies that $(yR \otimes_R S)_S \lesssim (xR \otimes_R S)_S$. Let $m = m_1$. If $m((yR)_R) \lesssim R_R$, then $m(yR \otimes_R S)_S \lesssim S_S$. By the above discussion, $(yR \otimes_R S)_S \lesssim (xR \otimes_R S)_S$. So $(yR \otimes_R S)_R \lesssim (xR \otimes_R S)_R$, i.e., $n(yR)_R \lesssim n(xR)_R$. By the hypothesis, $(yR)_R \lesssim (xR)_R$. □

Recall that a regular ring R is called Abelian provided all idempotents in R are central ($a \in R$ is central if $ax = xa$ for all $x \in R$). A ring is said to be strongly regular if for each $a \in R$ there exists $b \in R$ such that $a^2 b = a$. A ring is strongly regular if and only if it is Abelian regular [3, Theorem 3.5]. The index of a nilpotent element $x \in R$ is the least positive integer

n such that $x^n = 0$. Then index of R is supremum of the indices of all nilpotent elements of R. If it is finite, then R is said to have bounded index. It is well-known that an Abelian regular ring R has bounded index [3, Theorem 3.2], and a regular ring of bounded index is a regular ring whose primitive factor rings of R are Artinian [3, Theorem 7.2 and Theorem 6.2]. Since regular rings whose primitive factor are Artinian have the unperforation property [3, Proposition 6.11], we have

Corollary 2.6. *Let S be an excellent extension of a regular ring R. If R is a regular ring whose primitive factor rings of R are Artinian (particularly a regular ring of bounded index, or an Abelian regular ring), then*

(1) R satisfies s-comparability if and only if so does S.

(2) R satisfies almost comparability if and only if so does S.

(3) R satisfies weak comparability if and only if so does S.

Since $\aleph_0$-continuous regular rings (see the definition in [3, Page 173]) have the unperforation property [3, Theorem 14.30], we have

Corollary 2.7. *Let S be an excellent extension of an* $\aleph_0$*-continuous regular ring R. Then*

(1) R satisfies s-comparability if and only if so does S.

(2) R satisfies almost comparability if and only if so does S.

(3) R satisfies weak comparability if and only if so does S.

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