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LOWER SEMICONTINUOUS WITH LIPSCHITZ COEFFICIENTS

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Abstract

We are interested in integral functionals of the form

$$J(U,V) = \int_{\Omega} J\big(x,U(x),V(x)\big)dx,$$

where J is Carathéodory positive integrand, satisfying some growth condition of order $p \in]1,+\infty[$. We show that $\mathcal{A}(x,\partial)-$quasiconvexity of the integrand J with respect to the third variable is a necessary and sufficient condition of lower semicontinuity of J, where $\mathcal{A}(x,\partial)$ is a differential operator given by

$$\mathcal{A}(x,\partial) = \sum_{j=1}^{N} A^{(j)}(x)\partial_{x_j},$$

and the coefficients $A^{(j)}, j = 1,...,N$ are only Lipschitzian, i.e. $A^{(j)} \in W^{1,\infty}\big(\Omega;\mathbb{M}^{l\times d}\big)$ and satisfy the condition of *constant rank*. To this end, a framework of paradifferential calculus is needed to deal with the lower smoothness of the coefficients.

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1 Introduction

Minimization problems appear in many domains, as mechanics, electromagnetism and engineering, as means to compute the relaxed energy. Knowing that the relaxed problems is one of the pillars of calculus of variations (see [7, 9, 10]) and the existence of the minimum

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requires the lower semicontinuity of the energy. One of the minimization problems governed by a partial differential system has been recently studied by [11], who proved that $\mathcal{A}(\partial)$-quasiconvexity (see Definition 1.1 below) is a necessary and sufficient condition for lower semicontinuity of integral functionals, of the form

$$J(U,V) = \int_{\Omega} J\big(x, U(x), V(x)\big)\,dx,$$

where $J : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)$ is a normal integrand, $U_n \to U$ in measure and $V_n \rightharpoonup V$ in $L^p\big(\Omega;\mathbb{R}^d\big)$ such that $\mathcal{A}(\partial)V_n \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$, where $\mathcal{A}(\partial)$ is a differential operator with constant coefficients defined by

$$\mathcal{A}(\partial) = \sum_{j=1}^{N} A^{(j)}\partial_{x_j}, \quad A^{(j)} \in \mathbb{M}^{l\times d}$$

and satisfying the condition of constant rank, namely: there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank}\left(\sum_{j=1}^{N} A^{(j)}\xi_i\right) = r, \text{ for } \xi \in \mathbb{R}^N \backslash \{0\},$$

by making use of the compensated compactness Theory introduced by [15] and [20].

Let us recall the notion of $\mathcal{A}(\partial)-$ quasiconvexity for a first order differential operators of constant coefficients (see e.g.[10]):

Definition 1.1. A function $J : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{A}(\partial) -$ *quasiconvex* if

$$J(\xi) \leq \int_{\Omega} f\big(\xi + U(x)\big)dx$$

for all $\xi \in \mathbb{R}^d$ and all $U \in \mathcal{C}^{\infty}_{per}\big(Q;\mathbb{R}^d\big)$ such that $\mathcal{A}(\partial)U = 0$ and $\int_Q U(y)dy = 0$, where $\mathcal{C}^{\infty}_{per}\big(Q;\mathbb{R}^d\big)$ is the space of $\mathcal{C}^{\infty}$ functions on $\mathbb{R}^N$ and Q-periodic.

Among the models included in the framework of $\mathcal{A}(\partial)-$ quasiconvexity, we can cite for instance the following:

(1) [Divergence Free Fields] $\mathcal{A}(\partial)U = 0$ if and only if $\operatorname{div}U = 0$, where $U : \mathbb{R}^N \to \mathbb{R}^N$.

(2) [Maxwell's Equations] In magnetostatics, the magnetization $M : \mathbb{R}^3 \to \mathbb{R}^3$ and the induced magnetic field $J : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy the PDE constraints

$$\mathcal{A}(\partial)\begin{pmatrix} M \\ J \end{pmatrix} = \begin{pmatrix} \operatorname{div}(M+J) \\ \operatorname{curl}J \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For further examples, we can refer to [8, 11].

We emphasize that B. Dacorogna ([10] pp. 100-112) is the first one, who exploited the $\mathcal{A}(\partial)-$ quasiconvexity to investigate the lower semicontinuity of integral functional of the type

$$V \mapsto \int_{\Omega} J(V(x))dx$$

in the case, where the *kernel* of $\mathcal{A}$ contains the range of a suitable first order differential operator. The periodicity of the test function is needed to obtain the necessity of $\mathcal{A}-$quasiconvexity. To establish the sufficiency, we have to use the constant rank condition, for more details about this subject, we refer to [11].
Pedro Santos [18] has extended this framework to the variable coefficients case of the form:

$$\mathcal{A}(x,\partial) = \sum_{j=1}^{N} A^{(j)}(x)\partial_{x_j},$$

where the coefficients $A^{(j)} \in \mathcal{C}^{\infty}\left(\Omega;\mathbb{M}^{l\times d}\right) \cap W^{1,\infty},\ j=1,...,N,$ satisfy the condition of constant rank (CR), namely :

$$\operatorname{rank}\left(\sum_{j=1}^{N} A^{(j)}(x)\xi_i\right) = \text{const} \quad \text{for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}. \tag{CR}$$

He has adapted the notion of $\mathcal{A}(\partial)$-quasiconvexity to the variable coefficients case by freezing the coefficients at each point of Ω. Then, he made use of the pseudodifferential symbolic calculus to justify the composition of the differential operators with variable coefficients.
In the present paper, we will generalize some of the results of [18] in the case where the coefficients of the operator $\mathcal{A}(x,\partial)$ are only $W^{1,\infty}$. This will be done by using some special results from the paradifferential calculus introduced by [5, 13, 14].

Here are, now, the main results of this paper:

Theorem 1.2 (Necessary Condition for Lower Semicontinuity)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $p \in]1,+\infty[$ and let $J : \Omega \times \mathbb{R}^d \to [0,+\infty)$ be a continuous function such that the following conditions hold:*

(A1) *There exists $\omega_1 \in L^{\infty}_{loc}(\Omega)$ such that for all ξ_1,ξ_2 in $\mathbb{R}^d$ and a.e. $x \in \Omega$*

$$\left|J(x,\xi_1) - J(x,\xi_2)\right| \leq \omega_1(x)\left(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\right)|\xi_1 - \xi_2|.$$

(A2) *For any sequences $V_n \rightharpoonup V$ in $L^p\left(\Omega;\mathbb{R}^d\right)$ satisfying*

$$\mathcal{A}(x,\partial)V_n = \sum_{j=1}^{N} A^{(j)}(.)\partial_{x_j}V_n \to 0 \quad \left(W^{-1,p}(\Omega,\mathbb{R}^l)\right).$$

We assume that

$$\int_{\Omega} J(x,V(x))dx \leq \liminf_{n\to+\infty} \int_{\Omega} J(x,V_n(x))dx,$$

where $\mathcal{A}(x,\partial)$ satisfies the condition (CR). Then $J(x,.)$ is $\mathcal{A}(x,\partial)-$quasiconvex for all $x \in \Omega$.

Theorem 1.3 (Sufficient Condition for Lower Semicontinuity)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $p \in]1,+\infty[$ and let $J : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty[$ be a Carathéodory function, satisfying the following assumptions:*

(A3) *For some locally bounded function $\omega_2 : \Omega \times \mathbb{R}^m \to [0,+\infty[$, for all $\xi \in \mathbb{R}^d$, and for a.e. $x \in \Omega$*

$$0 \leq J(x,\zeta,\xi) \leq \omega_2(x,\zeta)\big(1+|\xi|^p\big).$$

(A4) *For a.e. $x \in \Omega$ and all $\zeta \in \mathbb{R}^m$, we assume that*

$$J(x,\zeta,.) \text{ is } \mathcal{A}(x,\partial)-\text{quasiconvex}.$$

Then for any sequence $(U_n)_{n\in\mathbb{N}}$ in $L^p\big(\Omega;\mathbb{R}^m\big)$ be such that U_n converging in measure to $U \in L^p\big(\Omega;\mathbb{R}^m\big)$ and any another sequence $(V_n)_{n\in\mathbb{N}}$ in $L^p\big(\Omega;\mathbb{R}^d\big)$ satisfying

$$V_n \rightharpoonup V \quad \big(L^p(\Omega;\mathbb{R}^d)\big), \quad \mathcal{A}(.,\partial)V_n \to 0 \quad \big(W^{-1,p}(\Omega;\mathbb{R}^l)\big),$$

we have:

$$\int_\Omega J\big(x,U(x),V(x)\big)dx \leq \liminf_{n\to+\infty} \int_\Omega J\big(x,U_n(x),V_n(x)\big)dx.$$

Let us describe how the paper is organized.
Section 2 contains four parts. Firstly, we begin by a notion of Young measures as well as some of its properties. Secondly, we give the essential results on the paradifferential calculus, which play a very important role in the case, where the coefficients of the differential operator $\mathcal{A}(x,\partial)$ are $W^{1,\infty}$. Thirdly, in order to make use of symbols acting only on spatial variables, we introduce the Littlewood-Paley decomposition, a paraproduct identity and a Lemma which characterizes the partial inverse symbols and the associated paradifferential operator. The fourth part is devoted to a few properties of the differential operator with constant coefficients. Section 3 gives in detail the proofs of Theorems 1.2 and 1.3. We end this paper by an appendix, where we give the proof of the Lemma 3.1, which forms the heart of the proof of the necessary condition.

2 Notations and Preliminaries

In this section, we fix some notations about the most used functional spaces and give some notions about Young measures and their properties, paradifferential calculus with Lipschitzian components, Littlewood-Paley decomposition and differential operators with constant coefficients.
Throughout this paper, Ω is an open bounded subset of $\mathbb{R}^N$, $Q = (-1,1)^N$ the unit cube centered at the origin, $Q_R(x_0) = x_0 + RQ$. A function $U \in L^p_{loc}\big(\mathbb{R}^N;\mathbb{R}^d\big)$ is said to be $Q-$*periodic* if $U(x+e_i) = U(x)$ for *a.e.* all $x \in \mathbb{R}^N$ and every $i = 1,...,N$, where $(e_1,...,e_N)$ is the canonical basis of $\mathbb{R}^N$. Let us recall that $L^p\big(Q;\mathbb{R}^d\big)$ is the closure of $C^\infty\big(Q;\mathbb{R}^d\big)$ in $L^p_{loc}\big(\mathbb{R}^N;\mathbb{R}^d\big)$. If $p \in]1,+\infty[$ then $W^{-1,p}\big(Q;\mathbb{R}^d\big)$ is the topological dual space of $W^{1,p'}\big(Q;\mathbb{R}^d\big)$, with $p' = \frac{p}{p-1}$. The dual of the closure of $C_c\big(\Omega;\mathbb{R}^d\big)$ is the set of $\mathbb{R}^d-$ valued Radon measures with finite mass $\mathcal{M}\big(\Omega;\mathbb{R}^d\big)$, through the duality

$$\langle \nu, J \rangle \stackrel{def}{=} \int_\Omega J(y)d\nu(y),\ \nu \in \mathcal{M}\big(\Omega, \mathbb{R}^d\big),\ J \in \mathcal{C}_c\big(\Omega; \mathbb{R}^d\big).$$

Also we denote by $\mathcal{L}_N$ the $N-$ dimensional Lebesgue measure on $\mathbb{R}^N$ and $\mathbb{M}^{l\times d}$ is the set of $l \times d$ real matrices. For a set B the characteristic function is denoted by χ_B.

2.1 Basic notions on Young measures

Definition 2.1. Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^1(\Omega)$. We say that $(y_n)_{n\in\mathbb{N}}$ is *equi-integrable*, if the following property hold. If $E \subset \Omega$ is a Borel set, then

$$\forall \varepsilon > 0,\ \exists \rho > 0 :\ \mathcal{L}_N(E) < \rho \Rightarrow \sup_{n\in\mathbb{N}} \int_E |y_n(x)|dx < \varepsilon.$$

As directly consequence of the previous definition:

▲ the equi-integrability is a necessary an sufficient condition for weak compactness in $L^1(\Omega)$ of the sequence $(y_n)_{n\in\mathbb{N}}$;

▲ $(y_n)_{n\in\mathbb{N}}$ is $p-$equi-integrable if $(|y_n|^p)_{n\in\mathbb{N}}$ is equi-integrable.

Definition 2.2. Let $J : \Omega \times \mathbb{R}^d \rightarrow$ be a function

(1) J is *normal integrand* if the two conditions are satisfied

(1.i) $x \mapsto J(x,V)$ is of Borel measurable ;

(1.ii) $V \mapsto J(x,V)$ is lower semicontinuous for all $x \in \Omega$.

(2) J is *Carathéodory* if J and $-J$ are normal integrands.

Now, we collect some important and useful properties for *Young measures*. For the detailed proofs, we refer to [3, 21].

Theorem 2.3. *Let $\Pi \subset \mathbb{R}^N$ be a measurable set of finite measure and let $(y_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions, $y_n : \Pi \rightarrow \mathbb{R}^d$. Then there exists a subsequence $(y_{n_k})_k$ and a weak * measurable map $\nu : \Pi \rightarrow \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ such that:*

(i) $\nu_x \geq 0, \|\nu_x\|_{\mathcal{M}} \leq 1$ *for a.e.* $x \in \Pi$*;*

(ii) *one has (i') $\|\nu\|_{\mathcal{M}} = 1$ for a.e. $x \in \Pi$ if and only if*

$$\lim_{C\rightarrow+\infty} \sup_{k\in\mathbb{N}} \mathcal{L}_N\big(\{x \in \mathbb{R}^N : |y_{n_k}(x)| \geq C\}\big) = 0; \tag{2.1}$$

(iii) *if Δ is a compact subset of $\mathbb{R}^d$ such that $d(y_{n_k}, \Delta) \rightarrow 0$ in measure, then*

$$supp\nu_x \subset \Delta \quad for\ a.e.\ x \in \Pi;$$

(iv) *if (i') holds, then in (iii) one may replace "if" by "if and only if";*

(v) *if $J : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand bounded from below then:*

$$\liminf_{k\to+\infty} \int_\Omega J(x, y_{n_k}(x))dx \geq \int_\Omega \langle \nu_x, J(x,.)\rangle dx,$$

where

$$\langle \nu_x, J(x,.)\rangle \stackrel{def}{=} \int_{\mathbb{R}^d} J(x,\xi)d\nu_x(\xi);$$

(vi) *if $J : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is Carathéodory function bounded from below*

and if (i') is satisfied, one has :

$$\lim_{k\to+\infty} \int_\Omega J(x, y_{n_k}(x))dx = \int_\Omega \langle \nu_x, J(x,.)\rangle dx;$$

if and only if $\left(J(., y_{n_k}(.))\right)$ is equi-integrable. In this case

$$J(., y_{n_k}(.)) \rightharpoonup \langle \nu_x, J(x,.)\rangle \quad in\ L^1(\Omega).$$

Then we have the following definition

Definition 2.4. (1) The map $\nu : \Pi \to \mathcal{M}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ is called the Young measure generated by the sequences $(y_{n_k})_{k\in\mathbb{N}}$;

(2) the Young measure ν is said to be homogeneous if there is Radon measure $\nu_0 \in \mathcal{M}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ such that $\nu_x = \nu_0$ for *a.e.* $x \in \Pi$.

Remark 2.5. (a) In the sense of Theorem 6.2 of [16], p. 97, if we take $g \equiv |.|^p$, the condition (2.1) holds if

$$\sup_{n\in\mathbb{N}} \int_\Pi |y_n|^p dx < +\infty$$

(b) As consequence of (vi), if $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence in L^p and J is continuous function in $\mathbb{R}^d$ such that $0 \leq J(\xi) \leq C(1+|\xi|^p)$ for some $C > 0$, then $J(y_n) \rightharpoonup \langle \nu_x, J(.)\rangle$ in L^p.

Proposition 2.6. *If (V_n) generates a Young measure ν and $U_n \to U$ a.e. in Ω, then the pair (U_n, V_n) generates the Young measure μ defined by*

$$\mu_x \stackrel{def}{=} \delta_{U(x)} \otimes \nu_x, \quad a.e. x \in \Omega.$$

2.2 Basic notions on paradifferential calculus

In this subsection, we want to briefly collect some definitions and results on paradifferential calculus: a more complete description of these notions can be found in original works of [5, 6, 13, 14]. We first recall the notion of the usual Sobolev spaces $H^{s,p}$. The key ingredient being to state a continuity theorem for this kind of operators in these spaces.
For any $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ we denote

$$\langle \xi \rangle^s = (1+|\xi|^2)^{s/2},$$

and the operator Λ^s by

$$\forall v \in \mathcal{S}(\mathbb{R}^N) \quad \mathcal{F}(\Lambda^s u)(\xi) = \langle \xi \rangle^{-s} \mathcal{F} u(\xi),$$

where $\mathcal{F}$ is the Fourier operator.

Definition 2.7. For any $s \in \mathbb{R}$ and $p \in]1,+\infty[$, we define the space $H^{s,p}(\mathbb{R}^N)$ to consist of tempered distributions u on $\mathbb{R}^N$ such that

$$\Lambda^{-s} u \in L^p(\mathbb{R}^N).$$

It is well known that if $s = k$ is a positive integer, $p \in]1,+\infty[$, the spaces $H^{s,p}(\mathbb{R}^N)$ coincide with the $W^{k,p}(\mathbb{R}^N)$, and in particular $H^{0,p}(\mathbb{R}^N)$ coincide with $L^p(\mathbb{R}^N)$. $H^{s,p}(\mathbb{R}^N)$ is a Banach space equipped with the obvious norm

$$\|u\|_{H^{s,p}} = \|\Lambda^{-s} u\|_{L^p}.$$

The dual space of $H^{s,p}(\mathbb{R}^N)$ coincide with $H^{-s,p'}(\mathbb{R}^N)$ where $p' = \frac{p}{p-1}$ (see [2, 19], for more details).

Definition 2.8. A paradifferential symbol of degree $m \in \mathbb{R}$ and regularity k, $\quad k \in \mathbb{N}$ is a function $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$, $(x,\xi) \mapsto a(x,\xi)$ such that a is $\mathcal{C}^\infty$ with respect to ξ and for all $\alpha \in \mathbb{N}^N$, there is a constant $C \overset{def}{=} C(\alpha)$ verifying

$$\forall \xi \in \mathbb{R}^N, \quad \|\partial_\xi^\alpha a(.,\xi)\|_{W^{k,\infty}} \leq C \langle \xi \rangle^{m-|\alpha|}.$$

The set of paradifferential symbols of degree m and regularity k is denoted by $\Gamma_k^m(\mathbb{R}^N)$. It is equipped with the obvious semi-norm.

Definition 2.9. Let $(\eta,\xi) \mapsto \chi(\eta,\xi)$ an admissible cut-off, that is, a smooth function satisfying, for two given real $0 < \varepsilon_1 < \varepsilon_2 < 1$

$$\chi(\eta,\xi) = \begin{cases} 1 & \text{if } |\eta| \leq \varepsilon_1 \langle \xi \rangle \\ 0 & \text{if } |\eta| \geq \varepsilon_2 \langle \xi \rangle. \end{cases}$$

Given a symbol $a \in \Gamma_k^m(\mathbb{R}^N)$, we define the paradifferential operator T_a^χ associated to the symbol a as follows :

$$T_a^\chi = Op(\sigma_a^\chi) \overset{def}{=} \sigma_a^\chi(x, D_x),$$

where the operator $Op(\sigma_a^\chi)$ acts on the Schwartz' class $\mathcal{S}(\mathbb{R}^N)$ by the usual formula

$$\forall u \in \mathcal{S}(\mathbb{R}^N),\ \forall x \in \mathbb{R}^N \quad Op(\sigma_a^\chi)u(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{i<x.\xi>}\sigma_a^\chi(x,\xi)\hat{u}(\xi)d\xi.$$

Here σ_a^χ is defined by :

$$\forall \eta \in \mathbb{R}^N \quad \sigma_a^\chi(.,\eta) = a(.,\eta) * G^\chi(.,\eta),$$

where $G^\chi(.,\eta) = \mathcal{F}^{-1}(\chi(.,\eta)$ is the inverse Fourier transform of $\xi \to \chi(\xi,\eta)$.

Remark 2.10. Recalling that the Hörmander's pseudodifferential symbols $S^m_{1,1}(\mathbb{R}^N)$ (see [12]) is defined by

$$S^m_{1,1}(\mathbb{R}^N) \stackrel{def}{=} \left\{ a \in C^\infty\left(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{C}\right) : \sup_{|\xi|\neq 0} \frac{|\partial_x^\beta \partial_\xi^\alpha a(.,\xi)|}{<\xi>^{(m-|\alpha|+|\beta|)/2}} < +\infty \right\},$$

the functions σ_a^χ belong to the $\Sigma_k^m(\varepsilon)$ class (see e.g [5]), the subclass of symbols $a \in S^m_{1,1}(\mathbb{R}^N)$ which have the partial Fourier transform of a with respect to the first variable supported in $|\eta| \leq \varepsilon|\xi|$.

Theorem 2.11. *For a given symbol $a \in \Gamma_k^m(\mathbb{R}^N)$, the paradifferential operator T_a^χ is a linear bounded operator from $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}(\mathbb{R}^N)$.*

By duality, the operators T_a^χ extend as a linear bounded operators from $\mathcal{S}'(\mathbb{R}^N)$ to $\mathcal{S}'(\mathbb{R}^N)$. One can then, define the adjoint operator $T_a^{\chi,*}$ by the formula

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^N) \quad \forall u \in \mathcal{S}'(\mathbb{R}^N) \quad \langle T_a^{\chi,*}u,\varphi\rangle = \langle u, T_a^{\chi,*}\varphi\rangle,$$

where the brackets denote for the bilinear duality $\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}(\mathbb{R}^N)$.

Let us now focus our attention only on the symbols $a \in \Gamma_k^m(\mathbb{R}^N)$ where $k = 0,1$. For a given paradifferential symbol $a \in \Gamma_1^m(\mathbb{R}^N)$, we denote by $\bar{a} \in \Gamma_1^m(\mathbb{R}^N)$ the complex conjugate of a. We can then identify the adjoint operator $T_a^{\chi,*}$ by the following lemma

Lemma 2.12. (1) *If we denote by $\sigma_a^{\chi,*}(x,\xi) \stackrel{def}{=} e^{-ix.\xi}(T_a^{\chi,*})e_\xi(x)$ where $e_\xi(x) = e^{ix.\xi}$, the pseudodifferential symbol associated to $T_a^{\chi,*}$, then $\sigma_a^{\chi,*} \in \Sigma_1^m$;*

(2) *there exists a symbol $r \in \Sigma_0^{m-1}$ such that*

$$T_a^{\chi,*} = T_{\bar{a}}^\chi + r(x,D_x).$$

Thanks to Lemma 2.12, we are now in position to apply the continuity Theorem of [6], in the $H^{s,p}$ spaces:

Theorem 2.13. *For a given $p \in]1,+\infty[$, $s \in \mathbb{R}$ and a symbol $a \in \Gamma_k^m(\mathbb{R}^N)$, the paradifferential operator T_a^χ is a linear bounded operators in the $H^{s,p}(\mathbb{R}^N)$ spaces, of order equal or less than m, that is : there exists a constant $C > 0$ such that*

$$\forall u \in H^{m+s,p}(\mathbb{R}^N) \quad \left\|T_a^\chi u\right\|_{H^{s,p}} \leq C\|u\|_{H^{s+m,p}}.$$

The next Theorem allows us to dispose of a very useful symbolic calculus, detailed as follows :

Theorem 2.14. (1) *For $a \in \Gamma_1^m(\mathbb{R}^N)$ and for two admissible cut-off χ_1 and χ_2 we have : $T_a^{\chi_1} - T_a^{\chi_2}$ is of order $\leq m-1$;*

(2) *let $a \in \Gamma_1^m(\mathbb{R}^N)$ and $b \in \Gamma_1^{m'}(\mathbb{R}^N)$. Then $ab \in \Gamma_1^{m+m'}(\mathbb{R}^N)$ and the operator $T_a^\chi \circ T_b^\chi - T_{ab}^\chi$ is of order $\leq m+m'-1$ for all admissible cut-off functions;*

(3) *let $a \in \Gamma_1^m(\mathbb{R}^N)$. Then the operator $T_a^{\chi,*} - T_{a^*}^\chi$ is of order $\leq m-1$ for all admissible cut-off functions χ. where $T_a^{\chi,*}$ denotes for the adjoint operator of T_a^χ.*

The part (1) of the Theorem allows us to fix without loss of generality the same cut-off function χ for all the paradifferential operators considered. So, we shall write T_a instead of T_a^χ.

2.3 Littlewood-Paley decomposition and paraproduct

Let us start with a classical dyadic decomposition of the full space (see for instance [9]).

Definition 2.15. There exist two radially functions $\chi \in C_c^\infty(\mathbb{R}^N); 0 \leq \chi \leq 1$ and $\psi \in C_c^\infty(\mathbb{R}^N \backslash \{0\})$ such that

(i) $\chi(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1.1; \quad \chi(\xi) = 0 \quad \text{for} \quad |\xi| \geq 1.9;$

(ii) $\chi(\xi) + \sum_{q \geq 0} \psi(2^{-q}\xi) = 1, \quad \dfrac{1}{2} \leq \chi^2(\xi) + \sum_{q \geq 0} \psi^2(2^{-q}\xi) \leq 1;$

(iii) $|k-k'| \geq 2 \Rightarrow \text{supp}\ \psi(2^{-k}.) \cap \text{supp}\ \psi(2^{-k'}.) = \emptyset;$

(iv) $k \geq 1 \Rightarrow \text{supp}\ \chi \cap \text{supp}\ \psi(2^{-k}.) = \emptyset.$

We set $h = \mathcal{F}^{-1}\chi$ and for $k \in \mathbb{Z}$

$$\chi_k(\xi) = \chi(2^{-k}\xi) \quad h_k = \mathcal{F}^{-1}\chi_k \quad \psi_k = \chi_k - \chi_{k-1}.$$

Introduce the operators S_k and Δ_k acting on $\mathcal{S}'$:

$$\begin{cases} S_k u = \mathcal{F}^{-1}\big(\chi(2^{-k}\xi)\hat{u}(\xi)\big) \overset{def}{=} \chi(2^{-k}D_x)u = h_k \star u \\ \Delta_k = S_k - S_{k-1} = (h_k - h_{k-1}) \star u. \end{cases}$$

Observe that for every tempered distribution, the support of $\mathcal{F}(\Delta_k u)$ is contained in the ring $\{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, and u has the following Littlewood -Paley decomposition:

$$u = S_0 u + \sum_{k=1}^{\infty} \Delta_k u \quad (\mathcal{S}').$$

In particular we can characterize the spaces $H^{s,p}(\mathbb{R}^N)$

Proposition 2.16. *A tempered distribution $u \in H^{s,p}(\mathbb{R}^N)$ iff there exists a universal constant $C > 0$ such that*

$$\frac{1}{C}\|u\|_{H^{s,p}} \leq \Big\| \Big(\sum_{k\geq 1} \|2^{2ks}\Delta_k u\|^2_{L^p(\mathbb{R}^N)} \Big)^{\frac{1}{2}} \Big\|_{L^p} \leq C\|u\|_{H^{s,p}},$$

where we have used the identity $H^{s,p}(\mathbb{R}^N) = F^s_{p,2}(\mathbb{R}^N)$, with $F^s_{p_1,p_2}(\mathbb{R}^N)$ $(0 < p_1 < \infty)$ is the Triebel-Lizorkin space defined by

$$F^s_{p_1,p_2}(\mathbb{R}^N) = \Big\{ u \in \mathcal{S}^{'}(\mathbb{R}^N) : \|u\|_{F^s_{p_1,p_2}} \stackrel{def}{=} \|2^{sk} S_k u\|_{L^{p_1}(\ell_{p_2})} < \infty \Big\}.$$

For more details about the Besov-Triebel-Lizorkin spaces, we refer the reader to [17]. We recall also the very useful property of the space $W^{1,\infty}(\mathbb{R}^N)$ through the Littlewood -Paley decomposition:

Theorem 2.17. *If $u \in W^{1,\infty}(\mathbb{R}^N)$ then there exists $C > 0$ such that for every $k \in \mathbb{N}$*

$$\|\Delta_k u\|_{L^\infty} \leq C 2^{-k} \|u\|_{W^{1,\infty}}. \tag{2.2}$$

Let now, $i \in \mathbb{N}$ and

$$\Psi_i(\eta,\xi) = \sum_{k=1}^{\infty} \chi_{k-i}(\eta)\psi_k(\xi). \tag{2.3}$$

Then for $i \geq 3$, we can check that Ψ_i is an admissible function. A function $a : x \mapsto a(x) \in L^\infty(\mathbb{R}^N)$ can be seen as a symbol in $\Gamma^0_0(\mathbb{R}^N)$, independent of ξ. With Ψ_i given by (2.3) with $i = 3$, this leads to define the paradifferential operator T_a called *paraproduct operator with* a by:

$$T_a u = S_{-3} a S_0 u + \sum_{k=1}^{\infty} S_{k-3}\Delta_k u. \tag{2.4}$$

Proposition 2.18. *For all $a \in L^\infty(\mathbb{R}^N)$, T_a defined by (2.4) is an operator of order ≤ 0.*

In the sequel, we will consider symbols and operators acting on functions with matrix values $\mathbb{M}^{l\times d}$. Nevertheless, we will make a slight abuse of notation in that we do not refer to this fact in the used norms.
Given a collection $\big(A^{(j)}\big)_{1\leq j\leq N}$ of functions in $W^{1,\infty}\big(\Omega;\mathbb{M}^{l\times d}\big)$, we denote by $\mathcal{A}(x,\xi)$ for $x \in \Omega$ and $\xi \in \mathbb{R}^N$ the matrix symbol

$$\mathcal{A}(x,\xi) = \sum_{j=1}^{N} A^{(j)}(x)\xi_j.$$

We fix a cut-off function $\delta \in C^\infty_0\big(\mathbb{R}^N;[0,1]\big)$ such that $\delta \equiv 1$ for some compact neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$. We set now for $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}$

$$\mathcal{A}_\delta(x,\xi) = \sum_{j=1}^{N} \delta(x) A^{(j)}(x)\xi_j. \tag{2.5}$$

The symbol $\mathcal{A}_\delta$ is positively homogeneous of degree 1 in ξ and it is easy to check that $x \mapsto \mathcal{A}_\delta(x,.) \in W^{1,\infty}\big(\mathbb{R}^N; \mathbb{M}^{l\times d}\big)$. Thanks to the compactness of S^{N-1} we get that

$$\forall \alpha \in \mathbb{N}^N \;\; \exists C_\alpha > 0 : \|\partial_\xi^\alpha \mathcal{A}_\delta(.,\xi)\|_{W^{1,\infty}} \leq C_\alpha < \xi >^{1-|\alpha|},$$

so that $\mathcal{A}_\delta$ belongs to the paradifferential class of symbols $\Gamma_1^1(\mathbb{R}^N)$, (see Definition 2.8). For $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}$, let us consider $P_\delta(x,\xi)$ the orthogonal projection onto $\ker \mathcal{A}_\delta(x,\xi)$: $P_\delta(x,\xi) : \mathbb{R}^d \to \mathbb{R}^d$

$$P_\delta(x,\xi)V = \begin{cases} 0 & \text{if } V \in \big(\ker \mathcal{A}_\delta(x,\xi)\big)^\perp \\ V & \text{if } V \in \ker \mathcal{A}_\delta(x,\xi). \end{cases} \tag{2.6}$$

In the particular case where $l = d$, the symbol $P_\delta(x,\xi)$ may be represented by a Dunford integral:

$$\forall x \in \mathbb{R}^N \quad \forall \xi \in \mathbb{R}^N \backslash \{0\} \quad P_\delta(x,\xi) = \frac{1}{2i\pi} \int_\gamma (zI - \mathcal{A}_\delta(x,\xi))^{-1} dz, \tag{2.7}$$

where γ is a closed path enclosing the roots of $zI - \mathcal{A}_\delta$.

For $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \backslash \{0\}$, let us introduce $Q_\delta(x,\xi) : \mathbb{R}^m \to \mathbb{R}^d$ the partial inverse of $\mathcal{A}_\delta$ implicitly defined by the equations:

$$\begin{aligned} Q_\delta(x,\xi)W &= 0 \quad \text{for all } W \in \big(\mathcal{A}_\delta(x,\xi)\big)^\perp \\ Q_\delta(x,\xi)\mathcal{A}_\delta(x,\xi) &= I_d - P_\delta(x,\xi). \end{aligned} \tag{2.8}$$

The properties of P_δ and Q_δ are given by the following Lemma:

Lemma 2.19. *Let P_δ and Q_δ defined by (2.6) and (2.8). Then we have:*

(1) $P_\delta \in \Gamma_0^0(\mathbb{R}^N)$;

(2) *the symbol* $Q_\delta \in \Gamma_1^{-1}(\mathbb{R}^N)$;

(3) *the operator*

$$\mathcal{R} = T_{Q_\delta} \circ T_{\mathcal{A}_\delta} - \big(I_d - T_{P_\delta}\big)$$

is of order ≤ -1, where I_d is the identity operator. In particular there exists a positive constant C such that for any $U \in L^p(\mathbb{R}^N)$

$$\|U - T_{P_\delta}U\|_{L^p} \leq C\big(\|T_{\mathcal{A}_\delta}U\|_{W^{-1,p}} + \|U\|_{W^{-1,p}}\big), \tag{2.9}$$

$$\|T_{\mathcal{A}_\delta}T_{P_\delta}U\|_{W^{-1,p}} \leq C\|U\|_{W^{-1,p}}. \tag{2.10}$$

Proof. For (1), thanks to the assumption (CR) satisfied by the matrix $\mathcal{A}(x,\xi)$ which is positively homogeneous of degree 1, we can deduce that the mapping $(x,\xi) \mapsto P_\delta(x,\xi)$ is positively homogeneous of degree 0 in ξ and inherit clearly the regularity $W^{1,\infty}$ of $\mathcal{A}_\delta$, so $P_\delta \in \Gamma_1^0(\mathbb{R}^N)$. Then, the associated operator T_{P_δ} is of degree ≤ 0.
Concerning (2), since the symbols $\mathcal{A}_\delta$ and P_δ are positively homogeneous of degree 1 and

0 in ξ respectively, it a simple routine to check that the symbol Q_δ defined by (2.8) is positively homogeneous of degree -1 and , thanks to the regularity $W^{1,\infty}$ of A_δ , it can be seen like a symbol in $\Gamma_1^{-1}(\mathbb{R}^N)$.

For (3), according to the symbolic calculus of Theorem 2.14, the operator $\mathcal{R}$ is of order ≤ -1. For the estimate (2.9), we have for every $U \in L^p(\mathbb{R}^N)$

$$U - T_{P_\delta}U = T_{Q_\delta} \circ T_{\mathcal{A}_\delta}U - \mathcal{R}U.$$

Since Q_δ and $\mathcal{R}$ are of order ≤ -1. Therefore

$$\begin{aligned} \|U - T_{P_\delta}U\|_{L^p} &= \|T_{Q_\delta} \circ T_{\mathcal{A}_\delta}U + \mathcal{R}U\|_{L^p} \\ &\leq C\big(\|T_{\mathcal{A}_\delta}U\|_{W^{-1,p}} + \|U\|_{W^{-1,p}}\big), \end{aligned}$$

which achieves the proof of (2.9).
For (2.10), using the fact that

$$\mathcal{A}_\delta(x,\xi)P_\delta(x,\xi) \equiv 0,$$

we see, with the help of the symbolic calculus that $T_{\mathcal{A}_\delta}T_{P_\delta}$ is an operator of order ≤ 0, and the proof of Lemma 2.19 is complete. □

2.4 Operators with constant coefficients

We present some results about operators with constant coefficients, namely

$$\mathcal{A}(\partial)V = \sum_{j=1}^{N} A^{(j)}\partial_{x_j}V$$

which satisfies the condition (CR).

Following [11], we associate to $\mathcal{A}(\partial)$, the following continuous projection[1]

$$S : L^p\big(\mathcal{T}_N;\mathbb{R}^d\big) \to L^p\big(\mathcal{T}_N;\mathbb{R}^d\big),$$

where $\mathcal{T}_N$ is the N torus defined by

$$\mathcal{T}_N = \big\{(e^{2\pi i x_1},\ldots,e^{2\pi i x_N}) \in \mathbb{C}^N : (x_1,\ldots,x_N) \in \mathbb{R}^N\big\}.$$

The space $L^p\big(\mathcal{T}_N;\mathbb{R}^d\big)$ is identified with $L^p\big(Q;\mathbb{R}^d\big)$. The properties of S can be summarized in the following lemma

Lemma 2.20. *We assume that (CR) assumption holds. Then for $p \in]1,+\infty[$, we have*

(i) $S \circ SV = SV$, *and* $\mathcal{A}(\partial)\big(SV\big) = 0$ *for* $V \in L^p\big(\mathcal{T}_N;\mathbb{R}^d\big)$;

(ii) $\|V - SV\|_{L^p} \leq C_p\|\mathcal{A}(\partial)(V)\|_{W^{-1,p}}$ *for all* $V \in L^p\big(\mathcal{T}_N;\mathbb{R}^d\big)$ *such that* $\int_{\mathcal{T}_N} V(x)dx = 0$ *and for some* $C_p > 0$;

(iii) *if* $(V_n)_{n\in\mathbb{N}}$ *is bounded sequence in* $L^p\big(\mathcal{T}_N;\mathbb{R}^d\big)$ *and* $p-$*equi-integrable. Then* $\big(|SV_n|\big)_{n\in\mathbb{N}}$ *is also* $p-$*equi-integrable.*

[1] For the construction of S, we make use of the Fourier multipliers associated to the orthogonal projection operator $P(\xi) : \mathbb{R}^d \to \mathbb{R}^d$, with $\xi \in \mathbb{R}^N$ onto $\ker \mathcal{A}(\xi)$, where $\mathcal{A}(\xi)$ is the symbol of $\mathcal{A}(\partial)$.

The next lemma gives an important result, called Decomposition lemma

Lemma 2.21. *Let $p \in]1,+\infty[$ and let $(U_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p\left(\mathcal{T}_N;\mathbb{R}^d\right)$ which satisfies the following properties:*

$$\mathcal{A}(\partial)U_n \to 0 \quad \left(W^{-1,p}\right),\, U_n \rightharpoonup U \quad \left(L^p\right),$$

and generates the Young measure ν. Then there exists a $p-$equi-integrable sequence $(V_n)_{n\in\mathbb{N}}$ in $L^p\left(\Omega;\mathbb{R}^d\right) \cap \ker\mathcal{A}$ which generates ν and is such that

$$\mathcal{A}(\partial)V_n = 0,\, \int_\Omega V_n(x)dx = \int_\Omega U(x)dx,\;\; \|V_n - U_n\|_{L^q} \to 0 \quad \forall q \in [1,p).$$

For the detailed proof of the previous lemmas, we refer the reader to the paper of [11].

3 Proof of the main result

3.1 Proof of Theorem1.2

Proof. We follow the main lines of the proofs of [11] and [18]. However, in order to deal with the limited regularity of the coefficients of the operator $\mathcal{A}(x,\partial)$, we use some features of the paradifferential calculus recalled in section 2.2. Throughout this proof we denote by C a generic constant whose value may vary from line to line.
We fix $x_0 \in \Omega, \kappa \in \mathbb{R}^d$ and $p \in]1,+\infty[$. Let $R > 0$ be such that $Q_{2R}(x_0) \Subset \Omega$ and $W \in \mathcal{C}^\infty\left(\mathbb{R}^N;\mathbb{R}^d\right)$ be a $Q-$ periodic function, satisfying

$$\int_Q W(y)dy = 0 \quad \text{ and } \quad \mathcal{A}(x_0,\partial)W \overset{def}{=} \sum_{j=1}^N A^{(j)}(x_0)\partial_{x_j}W = 0. \tag{3.1}$$

Let $\varepsilon > 0$, the uniform continuity of the function J on compact sets implies that there exists a positive integer n_0 such that

$$\begin{aligned} &\forall n \in \mathbb{N} : n \geq n_0,\, \forall x,x' \in \overline{Q_R(x_0)},\;\; \forall \xi \in \overline{Q_{|\kappa|+\|w\|_{L^\infty}}(0)} \\ &|x-x'| < \frac{1}{n} \Rightarrow |J(x,\xi) - J(x',\xi)| < \varepsilon. \end{aligned} \tag{3.2}$$

We first decompose the cube $Q_R(x_0)$ as follows: by the Vitali covering Theorem, up to sets of measure zero, we know that there exists a finite sequence $\{x_s\}_{1\leq s\leq n^N} \subset \Omega$ such that

$$Q_R(x_0) = \bigcup_{s=1}^{n^N} Q_{\frac{R}{n}}(x_s).$$

We choose a cut-off function $\varphi \in \mathcal{C}_0^\infty\left(Q_R(x_0)\right)$ such that $0 \leq \varphi \leq 1$ and $\mathcal{L}_N\left\{Q_R(x_0) \cap \{\varphi \neq 1\}\right\} < \varepsilon R^N$. We define the sequence $(W_m)_m$ by:

$$W_m(x) = \varphi(x)W^*\left(\frac{mn(x-x_s)}{R}\right)\chi_{Q_{\frac{R}{n}}(x_s)}, \tag{3.3}$$

where $W^*(y) = W\big(y + (1,\ldots,1)\big)$.
Since W^* is $Q-$ periodic, then in virtue of Lemma A.1. p.249 of [4] and (3.1), we get when $m \to +\infty$

$$W^*\left(\frac{mn(x-x_s)}{R}\right) \rightharpoonup 0 \quad \Big(L^p\big(Q_{\frac{R}{n}}(x_s);\mathbb{R}^d\big)\Big).$$

Hence $W_m \rightharpoonup 0$ in L^p and from the compact embedding $L^p \hookrightarrow W^{-1,p}$, one has

$$W_m \to 0 \quad \big(W^{-1,p}\big).$$

Furthermore, the sequence $\{W_m\}_m$ is uniformly bounded in L^a for all $a \in [1,+\infty]$. For any $x \in \mathbb{R}^N$ we set

$$B_\delta^{(j)}(x) = \delta(x)A^{(j)}(x) - A^{(j)}(x_0) \quad 1 \le j \le N.$$

The functions $B_\delta^{(j)}$ are in $W^{1,\infty}$. The symbol $\mathcal{A}_\delta$ defined in (2.5) becomes

$$\mathcal{A}_\delta(x,\xi) = \sum_{j=1}^N B_\delta^{(j)}(x)\xi_j + \sum_{j=1}^N A^j(x_0)\xi_j,$$

so the paradifferential operator associated to $\mathcal{A}_\delta$ acting on W_m is

$$\begin{aligned} T_{\mathcal{A}_\delta}W_m &= \sum_{j=1}^N T_{B_\delta^{(j)}}\partial_{x_j}W_m + \sum_{j=1}^N A^{(j)}(x_0)\partial_{x_j}W_m, \\ &\overset{def}{=} \mathcal{B}_1 W_m + \mathcal{A}(x_0,\partial)W_m. \end{aligned} \tag{3.4}$$

In order to treat these two sums we state the following lemma

Lemma 3.1. *Let $\mathcal{B}_1$ and $\mathcal{A}(x_0,\partial)$ as above. Then following properties hold:*

(1) $\mathcal{A}(x_0,\partial)W_m \to 0 \quad \big(W^{-1,p}\big)$.

(2) *There exists a positive constant C depending on the $W^{1,\infty}$ norm of the components of $\mathcal{A}(x,\partial)$ such that for all $m \in \mathbb{N}$*

$$\|\mathcal{B}_1(W_m)\|_{W^{-1,p}} \le C\left(\sum_{j=1}^N \left(\int_{Q_{2R}(x_0)} |A^{(j)}(x) - A^{(j)}(x_0)|^p dx\right)^{1/p} + \|W_m\|_{L^p}\right). \tag{3.5}$$

We postpone the proof of the lemma to the Appendix and continue the proof of the Theorem 1.2:
We introduce the projector P_δ defined in (2.6) and for any $m \in \mathbb{N}$, we define

$$\Theta_m = T_{P_\delta}W_m.$$

As T_{P_δ} is an operator of order ≤ 0, according to Theorem 2.13 the sequence $\{\Theta_m\}_m$ is uniformly bounded in L^p. Taking into account the weak convergence of the sequence

$(W_m)_m$ in L^p, we can extract a subsequence still denoted $\{\Theta_m\}_m$ weakly convergent to 0 in L^p. Therefore, we deduce

$$\mathcal{A}_\delta(x_0,\partial)\Theta_m \to 0 \quad \big(W^{-1,p}\big).$$

In order to deal with local estimates, we need to localize the sequence $\{W_m\}_m$. Consider a cut-off function $\gamma \in \mathcal{C}_0^\infty\big(Q_{2R}(x_0)\big)$ such that $0 \le \gamma \le 1$, $\gamma \equiv 1$ in $Q_R(x_0)$ and set:

$$\tilde{\Theta}_m = \gamma\Theta_m.$$

We obviously check that

$$\tilde{\Theta}_m \rightharpoonup 0 \qquad \big(L^p\big), \quad \mathcal{A}_\delta(x_0,\partial)\tilde{\Theta}_m \to 0 \quad (W^{-1,p}),$$

so that we apply assumption (A2), it follows

$$\int_\Omega J(x,\kappa)dx \le \liminf_{m\to+\infty} \int_\Omega J(x,\kappa+\tilde{\Theta}_m(x))dx. \tag{3.6}$$

On the other hand, applying assumption (A1) for $\xi_1 = \tilde{\Theta}_m(x)$ and $\xi_2 = W_m(x)$, we get the following estimate

$$\begin{aligned} & \left|\int_\Omega J(x,\kappa+\tilde{\Theta}_m(x))dx - \int_\Omega J(x,\kappa+W_m(x))dx\right| \\ \le \quad & C\int_{Q_{2R}(x_0)} |\tilde{\Theta}_m(x)-W_m(x)|\big(1+|\tilde{\Theta}_m(x)|^{p-1}+|W_m(x)|^{p-1}\big)dx. \end{aligned} \tag{3.7}$$

By making use repeatedly of the Hölder inequality, (3.7) becomes

$$\begin{aligned} & \left|\int_\Omega J(x,\kappa+\tilde{\Theta}_m(x))dx - \int_\Omega J(x,\kappa+W_m(x))dx\right| \\ \le \quad & C\bigg(\int_\Omega |\tilde{\Theta}_m(x)-W_m(x)|^p dx\bigg)^{1/p}\Bigg(R^{N/2} + \bigg(\int_{Q_R(x_0)} |\tilde{\Theta}_m(x)|^{p'}dx\bigg)^{1/p'} \\ & + \bigg(\int_{Q_R(x_0)} |W_m(x)|^{p'}dx\bigg)^{1/p'}\Bigg). \end{aligned} \tag{3.8}$$

The function $\tilde{\Theta}_m - W_m$ is supported in $Q_{2R}(x_0)$, we thus have in virtue of estimate (2.9)

$$\bigg(\int_\Omega |\tilde{\Theta}_m(x)-W_m(x)|^p dx\bigg)^{1/p} \le \|W_m - T_{P_\delta}W_m\|_{L^p} \le C\Big(\|T_{\mathcal{A}_\delta}W_m\|_{W^{-1,p}} + \|W_m\|_{W^{-1,p}}\Big).$$

This implies

$$\begin{aligned}
&\left| \int_\Omega J(x,\kappa+\tilde{\Theta}_m(x))dx - \int_\Omega J(x,\kappa+W_m(x))dx \right| \qquad (3.9)\\
\leq\ & \left(\|\mathcal{B}_1(W_m)\|_{W^{-1,p}} + \|W_m\|_{W^{-1,p}} \right) \left(R^{N/p'} + \left(\int_{Q_R(x_0)} |W_m(x)|^{p'} dx \right)^{1/p'} \right)\\
\leq\ & CR^{N/p'} \left(\|\mathcal{B}_1(W_m)\|_{W^{-1,p}} + \|W_m\|_{W^{-1,p}} \right).
\end{aligned}$$

Letting $m \to +\infty$ in (3.9) and applying (3.6), we shall have

$$\begin{aligned}
\int_{Q_R(x_0)} J(x,\kappa)dx \ \leq\ & \limsup_{m\to+\infty} \int_{Q_R(x_0)} J(x,\kappa+W_m(x))dx \qquad (3.10)\\
& + CR^{N/2} \lim_{m\to+\infty} \|\mathcal{B}_1(W_m)\|_{W^{-1,p}}.
\end{aligned}$$

In the last line we have used that W_m is supported in $Q_R(x_0)$.
Let us now turn to the first term of right-hand side of (3.10). The uniform continuity (3.2) and the construction of the sequence $\{W_m\}_m$ in (3.3) give

$$\int_{Q_R(x_0)} J(x,\kappa+W_m(x))dx \ = \ \sum_{s=1}^{n^N} \int_{Q_{\frac{R}{n}}(x_s)} J\left(x, \kappa + W^*\left(mn\frac{x-x_s}{R} \right) \right) dx + M\varepsilon R^N,$$

where

$$M \stackrel{def}{=} \sup\left\{ |J(x,z)| : x \in \overline{Q_R(x_0)}, \quad |z| \leq |\kappa| + \|w\|_{L^\infty} \right\}.$$

Another application of (3.2) yields

$$\int_{Q_R(x_0)} J(x,\kappa+W_m(x))dx \ \leq \ \sum_{s=1}^{n^N} \int_{Q_{\frac{R}{n}}(x_s)} J\left(x_s, \kappa + W^*\left(mn\frac{x-x_s}{R} \right) \right) dx + (1+M)\varepsilon R^N.$$

After an appropriate change of variables which transforms $Q_{\frac{R}{n}}(x_s)$ into Q we get

$$\int_{Q_R(x_0)} J(x,\kappa+W_m(x))dx \leq \sum_{s=1}^{n^N} \frac{R^N}{n^N} \int_Q J(x_s,\kappa+W(my))dy + (1+2M)\varepsilon R^N.$$

Thus

$$\begin{aligned}
&\limsup_{m\to+\infty} \int_{Q_R(x_0)} J(x_s,\kappa+W_m(x))dx \qquad (3.11)\\
&\leq \sum_{s=1}^{n^N} \int_{Q_{\frac{R}{n}}(x_s)} \left(\int_Q J(x_s,\kappa+W(y))dy \right) dx + 2(1+M)\varepsilon R^N,\\
&\leq \int_{Q_R(x_0)} \left(\int_Q J(x,\kappa+W(y))dy \right) dx + 2(1+M)\varepsilon R^N.
\end{aligned}$$

Plugging (3.11) in (3.10) and dividing by R^N we get

$$\begin{aligned}\frac{1}{R^N}\int_{Q_R(x_0)} J(x,\kappa)dx &\leq \frac{1}{R^N}\int_{Q_R(x_0)}\left(\int_Q J(x,\kappa+W(y))dy\right)dx \\ &+\frac{C}{R^N}\lim_{m\to+\infty}\|\mathcal{B}_1(W_m)\|_{W^{-1,p}}+O(\varepsilon).\end{aligned} \tag{3.12}$$

From Lemma 3.1, (3.12) becomes

$$\begin{aligned}\frac{1}{R^N}\int_{Q_R(x_0)} J(x,\kappa)dx &\leq \frac{1}{R^N}\int_{Q_R(x_0)}\left(\int_Q J(x,\kappa+W(y))dy\right)dx \\ &+\frac{C}{R^N}\sum_{j=1}^{N}\left(\int_{Q_{2R}(x_0)}|A^{(j)}(x)-A^{(j)}(x_0)|^p dx\right)^{1/p}+O(\varepsilon).\end{aligned}$$

When $R\to 0$ respectively $\varepsilon\to 0$, we get in view of Lebesgue Theorem

$$J(x_0,\kappa)\leq\int_Q J(x_0,\kappa+W(y))dy,$$

which proves that $\xi\mapsto J(x_0,\xi)$ is $\mathcal{A}(x_0,\partial)$-quasiconvex. □

3.2 Proof of Theorem 1.3

Proof. Let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ be two sequences, such that $U_n\to U$ in measure, $V_n\rightharpoonup V$ in $L^p(\Omega;\mathbb{R}^d)$ and $\mathcal{A}(x,\partial)V_n\to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^l)$. Our aim is to show that

$$\liminf_{n\to+\infty}\int_\Omega J\big(x,U_n(x),V_n(x)\big)dx\leq\int_\Omega J\big(x,U(x),V(x)\big)dx.$$

Without loss of generality, we can assume up to a subsequence that

$$\liminf_{n\to+\infty}\int_\Omega J\big(x,U_n(x),V_n(x)\big)dx=\lim_{n\to+\infty}\int_\Omega J\big(x,U_n(x),V_n(x)\big)dx.$$

Since $(V_n)_n$ is weakly convergent in L^p, then it is bounded in L^p, so there exists another subsequence, not relabeled that generates the Young measure $\nu=(\nu_x)_{x\in\Omega}$ (see Theorem 6.2. p. 97 of [16]).
According to the Proposition 2.6, the pair $\big(U_n,V_n\big)$ generates the Young measure $(\mu_x)_{x\in\Omega}$ defined by: for every $x\in\Omega$; $\mu_x=\delta_{U(x)}\otimes\nu_x$, where $\delta_{U(.)}$ is the Dirac mass at $U(.)$. Furthermore, by assumption (A3), we get in view of (v) of Theorem 2.3 the following

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega J\big(x,U_n(x),V_n(x)\big)dx &\geq \int_\Omega\int_{\mathbb{R}^m\times\mathbb{R}^d} J(x,\zeta,\xi)d\mu_x(\zeta,\xi)dx \\ &= \int_\Omega\int_{\mathbb{R}^d} J(x,U(x),\xi)d\nu_x(\xi)dx.\end{aligned}$$

In order to apply Lemma 2.21, we need to truncate the sequence $(V_n)_{n\in\mathbb{N}}$ by a family of functions $T_h:\mathbb{R}^d\to\mathbb{R}^d$ taking the form

$$T_h(y)\overset{def}{=}\begin{cases} y & \text{if } |y|\le h\\ h\dfrac{y}{|y|} & \text{if } |y|>h.\end{cases}$$

Using the fact that $(V_n)_{n\in\mathbb{N}}$ generates $\nu=(\nu_x)_{x\in\Omega}$, we shall have

$$\begin{aligned}\lim_{h\to+\infty}\lim_{n\to+\infty}\int_\Omega |T_h(V_n(x))|^p dx &= \lim_{h\to+\infty}\int_\Omega \langle \nu_x, |T_h(.)|^p\rangle dx\\ &= \int_\Omega \langle \nu_x, |y|^p\rangle dx<\infty.\end{aligned}$$

This implies that $(T_h\circ V_n)_{(h,n)\in\mathbb{N}^2}$ generates the Young measure ν and satisfies the same properties that $(V_n)_{n\in\mathbb{N}}$. Hence, in view of Lemma 2.21 there exists a $p-$equi-integrable subsequence $\widetilde{V}_n=T_{h_n}\circ V_n$ still generates the Young measures ν and

$$\begin{aligned}&\mathcal{A}(x,\partial)\widetilde{V}_n\to 0\quad \big(W^{-1,p}\big),\ \lim_{n\to+\infty}\int_\Omega |\widetilde{V}_n(x)|^p dx=\int_\Omega\langle \nu_x,|y|^p\rangle dx \qquad (3.13)\\ &\lim_{n\to+\infty}\|\widetilde{V}_n-T_h\circ V_n\|_{L^q}=0\ \text{ with }\ q\in]1,p[.\end{aligned}$$

Let $x_0\in\Omega$ be a Lebesgue point of $x\mapsto\langle\nu_x,|z|^p\rangle$ and $x\mapsto|V(x)|^p$. Let Σ be a countable subset in $\mathcal{C}_0\big(\mathbb{R}^d;\mathbb{R}^d\big)^2$, then for every $\phi\in\Sigma$ we get

$$\lim_{R\to 0^+}\int_Q |\langle\nu_{x_0+Ry},\phi\rangle-\langle\nu_{x_0},\phi\rangle|dx=0. \qquad (3.14)$$

On the other hand, define the sequence $W_{n,R}$ in $L^p\big(Q;\mathbb{R}^d\big)$ by $W_{n,R}(y)\overset{def}{=}\widetilde{V}_n(x_0+Ry)$ for $y\in Q$. In view of (3.13), we have

$$\begin{aligned}&\lim_{n\to+\infty}\sum_{j=1}^N A^{(j)}(x_0+Ry)\partial_{y_j}W_{n,R}(.)=0 \text{ in }\quad \big(W^{-1,q}\big) \qquad (3.15)\\ &\lim_{R\to 0^+}\lim_{n\to+\infty}\int_Q |W_{n,R}(y)|^p dy=\langle\nu_{x_0},|y|^p\rangle.\end{aligned}$$

Furthermore, for every $\psi\in L^{q'}$

$$\lim_{R\to 0^+}\lim_{n\to+\infty}\int_Q\big(W_{n,R}(y)-V(x_0)\big)\cdot\psi(y)dy=0.$$

Now, from (vi) of Theorem 2.3, it follows

$$\phi\circ W_{n,R}\overset{*}{\rightharpoonup}\langle\nu_{x_0},\phi\rangle\quad \text{in } L^\infty.$$

[2] $\mathcal{C}_0\big(\mathbb{R}^d;\mathbb{R}^d\big)=\{J\in\mathcal{C}\big(\mathbb{R}^d;\mathbb{R}^d\big):\lim_{\xi\to+\infty}J(\xi)=0\}$;

Consequently, for every $\varphi \in \mathcal{C}_c(Q;\mathbb{R}^d)^3$ and $\phi \in \Sigma$ (for which (3.14) holds)

$$\lim_{R\to 0^+}\lim_{n\to+\infty}\int_Q \varphi(y)\cdot\phi\big(W_{n,R}(x_0+Ry)\big)dy = \langle \nu_{x_0},\phi\rangle \int_Q \varphi(y)dy.$$

A diagonal argument ensures that there exists a map $(R,n)\mapsto R_n$, with $R_n \to 0^+$ when $n\to+\infty$ and sequence $(\widetilde{W}_n)_{n\in\mathbb{N}}$ in $L^p(Q;\mathbb{R}^d)$ defined for every $y\in Q$ by $\widetilde{W}_n(y) \stackrel{def}{=} W_{n,R_n}(x_0+R_n y)$ such that $\widetilde{W}_n \rightharpoonup V(x_0)$ in L^p,

$$\sum_{j=1}^N A^{(j)}(x_0+R_n y)\partial_{y_j}\widetilde{W}_n(y) \to 0 \quad \big(W^{-1,q}\big) \tag{3.16}$$

and, for every $(\theta,\phi)\in\Gamma\times\Sigma$, one has

$$\lim_{n\to+\infty}\int_Q \theta(y)\cdot\phi(\widetilde{W}_n(y))dy = \langle \nu_{x_0},\phi\rangle\int_Q \theta(y)dy, \tag{3.17}$$

where Γ is a countable dense subset in $L^1(Q)$. In addition, as $(\widetilde{V}_n)_{n\in\mathbb{N}}$ is $p-$ equi-integrable we have

$$\lim_{n\to+\infty}\int_Q \big|\widetilde{W}_n(y)\big|^p dy = \langle \nu_{x_0}, |y|^p\rangle.$$

Thus, we deduce that the sequence $(\widetilde{W}_n)_{n\in\mathbb{N}}$ is $p-$equi- integrable and generates ν_{x_0}. Other important properties of $(\widetilde{W}_n)_{n\in\mathbb{N}}$ are collected in the following proposition:

Proposition 3.2. *Let $(\widetilde{W}_n)_{n\in\mathbb{N}}$ be the sequence defined as above. Then we have*

(P1) $\mathcal{A}(x_0,\partial)\widetilde{W}_n \to 0 \quad \big(W^{-1,q}\big)$;

(P2) *for a.e. $x\in\Omega$ there exists a sequence $(W_n)_{n\in\mathbb{N}}$ in L^p, $Q-$ periodic, and such that the following assertions hold:*

- (P2.i) *$(W_n)_{n\in\mathbb{N}}$ is $p-$equi- integrable and generates the homogeneous Young measure ν_{x_0};*
- (P2.ii) $\int_Q W_n(y)dy = V(x_0)$.

Proof of Proposition 3.2. First of all, let us mention that up to a mollifier sequence, we can show that, if ρ_R stands for the dilatation operator defined by $\rho_R\varphi(.) \stackrel{def}{=} \varphi(R.)$, then $\partial_{y_j}\rho_R$ may be identified with $R\rho_R\partial_{y_j}$ in $W^{1,\infty}$.
Thus, denoting by τ_{-x_0} the translation operator : $\tau_{-x_0}\varphi(.) = \varphi(x_0+.)$, for (P1), we have

$$\begin{aligned}
\mathcal{A}(x_0,\partial)\widetilde{W}_n(y) &= \sum_{j=1}^N \partial_{y_j}\big[\big(A^{(j)}(x_0)-\rho_{R_n}\big(\tau_{-x_0}A^{(j)}\big)(y)\big)\widetilde{W}_n(y)\big] \\
&\quad + R_n\sum_{j=1}^N \partial_{y_j}\big[\rho_{R_n}\big(\tau_{-x_0}A^{(j)}\big)(y)\big]\widetilde{W}_n(y) + \sum_{j=1}^N \rho_{R_n}\big(\tau_{-x_0}A^{(j)}\big)(y)\partial_{y_j}\widetilde{W}_n(y) \\
&\stackrel{def}{=} \mathbb{I}_{n,1}+\mathbb{I}_{n,2}+\mathbb{I}_{n,3},
\end{aligned}$$

[3] $\mathcal{C}_c(Q;\mathbb{R}^d) = \{J\in \mathcal{C}(Q;\mathbb{R}^d) : \mathrm{supp}J \text{ is compact}\}$.

Since the coefficients $A^{(j)}$ $j=1,...,N$ are continuous and $(\widetilde{W}_n)_{n\in\mathbb{N}}$ is $q-$equi- integrable, this yields $\mathbb{I}_{n,1} \longrightarrow 0$ in $W^{-1,q}$.

For the second member of right-hand side, using the fact that $(\widetilde{W}_n)_{n\in\mathbb{N}}$ is bounded and $R_n \to 0^+$, we get $\mathbb{I}_{n,2} \longrightarrow 0$ in $W^{-1,q}$.

The third member $\mathbb{I}_{n,3}$ goes to 0 is directly consequence of (3.16), and (P1) is proved. Concerning (P2), let us observe that for all $\varphi \in C_0^\infty\big(Q;[0,1]\big)$

$$\mathcal{A}(x_0,\partial)(\varphi\widetilde{W}_n) = \varphi\mathcal{A}(x_0,\partial)(\widetilde{W}_n) + \sum_{j=1}^{N} A^{(j)}(\widetilde{W}_n)\partial_{y_j}\varphi \to 0 \quad \big(W^{-1,p}\big),$$

where we have used (P1) and the compact embedding $L^p \hookrightarrow W^{-1,p}$.
Under this remark we may consider a sequence of smooth cut-off functions $\varphi_s \in C_0^\infty\big(Q;[0,1]\big)$ with $\varphi_s \nearrow 1$, and such that, setting $\hat{W}_{s,n}(y) \overset{def}{=} \big(\varphi_s\widetilde{W}_n\big)(y)$, with $y \in Q$. We have $\hat{W}_{s,n} \in L^p\big(Q;\mathbb{R}^d\big)$, and for all $(\theta,\phi) \in \Gamma\times\Sigma$ we obtain

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\int_Q \theta(y)\cdot\phi(\hat{W}_{s,n}(y))dy = \langle \nu_{x_0},\phi\rangle \cdot \int_Q \theta(y)dy.$$

Using the fact $\big(\widetilde{W}_n\big)_{n\in\mathbb{N}}$ is $p-$equi- integrable, we infer that $\big(\hat{W}_n\big)_{n\in\mathbb{N}}$ is $p-$equi- integrable and generates the homogeneous measure ν_{x_0}. Moreover, from (P1) and the compact embedding $L^p \hookrightarrow W^{-1,p}$, we find

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\mathcal{A}(x_0,\partial)\hat{W}_{s,n} = 0 \quad \big(W^{-1,q}\big).$$

We apply once again a diagonal argument, we shall obtain a new sequence, denoted $(\bar{W}_n)_{n\in\mathbb{N}}$ which is $p-$equi- integrable, generates ν_{x_0} and satisfies

$$\bar{W}_n \rightharpoonup V(x_0) \quad \big(L^p\big), \quad \mathcal{A}(x_0,\partial)\bar{W}_n \to 0 \quad \big(W^{-1,p}\big).$$

Now, we take

$$W_n \overset{def}{=} S\left[\bar{W}_n - V(x_0) - \int_Q \big(\bar{W}_n - V(x_0)\big)dx\right] + V(x_0),$$

where S is the operator defined in Lemma 2.20. The sequence $(W_n)_{n\in\mathbb{N}}$ remains belonging to $L^p, Q-$ periodic, $p-$equi-integrable and generates the homogeneous measure ν_{x_0}. Moreover we have

$$W_n \rightharpoonup V(x_0), \quad \int_Q W_n(y)dy = V(x_0), \quad \mathcal{A}(x_0,\partial)W_n = 0. \tag{3.17}$$

Which achieves the proof of (P2.i), (P2.ii) and (P.2). □

Coming back to the proof of Theorem 1.3. The assumption (A4) of $\mathcal{A}(x,\partial)-$ quasiconvexity of J with respect to the third variable, (v) of Theorem 2.3 and (3.17) yield

$$
\begin{aligned}
\langle \nu_{x_0}, J\rangle &= \int_{\mathbb{R}^d} J\big(x_0, U(x_0), \xi\big)\, d\nu_{x_0}(\xi) \\
&= \lim_{n\to+\infty} \int_Q J\big(x_0, U(x_0), W_n(y)\big)\, dy \geq J(x_0, U(x_0), V(x_0)).
\end{aligned}
$$

Now, the proof is completed. □

4 Appendix

Proof of Lemma 3.1. The first statement of the Lemma follows as in [18] from the weak convergence of the sequence $\{W_m\}_m$.
For the second one, recall that, for any $j \in \{1, \cdots, N\}$

$$B_\delta^{(j)}(x) = A_\delta^{(j)}(x) - A^{(j)}(x_0) \in W^{1,\infty}.$$

Introducing the paradifferential operator $T_{B_\delta^{(j)}}$ we write

$$T_{B_\delta^{(j)}} \partial_{x_r} W_m = \partial_{x_r} T_{B_\delta^{(j)}} W_m + [T_{B_\delta^{(j)}}, \partial_{x_r}] W_m,$$

where $[T_{B_\delta^{(j)}}, \partial_{x_r}]$ denotes for the commutator of the two operators.
Since $\partial_{x_r} B_\delta^{(j)} \in L^\infty$, it is easy to check in (2.4) that we can identify between $[T_{B_\delta^{(j)}}, \partial_{x_r}]$ and $T_{\partial_{x_r} B_\delta^{(j)}}$, if we observe that ∂_{x_r} commutes with the operators S_k and Δ_k, thanks to the convolution properties. Therefore, by the continuity of $\partial_{x_r} : L^p \to W^{-1,p}$, it is enough to estimate the L^p norm of $T_{B_\delta^{(j)}} W_m$.
We introduce a cut-off function $\gamma \in \mathcal{C}_0^\infty\big(Q_{2R}(x_0)\big)$ such that $0 \leq \gamma \leq 1$ and $\gamma \equiv 1$ in $Q_R(x_0)$ and write:

$$T_{B_\delta^{(j)}} W_m = T_{\gamma B_\delta^{(j)}} W_m + T_{(1-\gamma) B_\delta^{(j)}} W_m. \tag{4.1}$$

We aim to estimate the L^p norm of each part of the expression (4.1). We will denote by C_j a generic constant depending of the index j and the L^∞ norm of γ whose value may vary from line to line.
Concerning the second term of the right-hand side, the L^p continuity of the paradifferential operator $T_{(1-\gamma) B_\delta^{(j)}}$ on L^p, yields

$$\|T_{(1-\gamma) B_\delta^{(j)}} W_m\|_{L^p} \leq C_j \|B_\delta^{(j)}\|_{W^{1,\infty}} \|W_m\|_{L^p}. \tag{4.2}$$

It remains to deal with the L^p norm of $T_{\gamma B_\delta^{(j)}} W_m$: starting with the definition of the paraproduct (2.4), and taking into account that both of $\gamma B_\delta^{(j)}$ and W_m are in $L^\infty \cap L^p$, supported in $Q_{2R}(x_0)$, we make use of the Bony's decomposition of the expression $\gamma B_\delta^{(j)} W_m$ and write:

$$\gamma B_\delta^{(j)} W_m = T_{\gamma B_\delta^{(j)}} W_m + T_{W_m} \gamma B_\delta^{(j)} + \mathcal{R}(\gamma B_\delta^{(j)}, W_m), \tag{4.3}$$

where $\mathcal{R}(\gamma B_\delta^{(j)}, W_m) = \sum_{|k-q|\le 3} \Delta_k \gamma B_\delta^{(j)} \Delta_q W_m$.

For any $k \in \mathbb{N}^*$ we set $v_k(m) \overset{def}{=} \sum_{|k-q|\le 3} \Delta_k \gamma B_\delta^{(j)} \Delta_q W_m$. In virtue of Theorem 2.17, we have

$$\|v_k(m)\|_{L^p} \le 2^{-k} \|B_\delta^{(j)}\|_{W^{1,\infty}} \sum_{q\ge 1} \|\Delta_q W_m\|_{L^p} \le 2^{-k} \|B_\delta^{(j)}\|_{W^{1,\infty}} \|W_m\|_{L^p}.$$

Summing up over all of k's, we obtain

$$\|\mathcal{R}(\gamma B_\delta^{(j)}, W_m)\|_{L^p} \le C \|B_\delta^{(j)}\|_{W^{1,\infty}} \|W_m\|_{L^p}. \tag{4.4}$$

Let us observe that the left-hand side of (4.3) satisfies:

$$\|\gamma B_\delta^{(j)} W_m\|_{L^p} \le C_j \|B_\delta^{(j)}\|_{L^\infty} \|W_m\|_{L^p}. \tag{4.5}$$

On the other hand, the L^p continuity of the paradifferential operator T_{W_m} on L^p spaces and the fact $\mathrm{supp}\big(\gamma B_\delta^{(j)}\big) \subset Q_{2R}(x_0)$, implies that

$$\|T_{W_m} \gamma B_\delta^{(j)}\|_{L^p} \le C_j \|W_m\|_{L^\infty} \|B_\delta^{(j)}\|_{L^p} \le C_j \|W_m\|_{L^\infty} \left(\int_{Q_{2R}(x_0)} |A^{(j)}(x) - A^{(j)}(x_0)|^p dx \right)^{1/p}. \tag{4.6}$$

Coming back to (4.3), we infer from the estimates (4.4), (4.5) and (4.6) that

$$\|T_{\gamma B_\delta^{(j)}} W_m\|_{L^p} \le C_j \left(\|W_m\|_{L^\infty} \left(\int_{Q_{2R}(x_0)} |A^{(j)}(x) - A^{(j)}(x_0)|^p dx \right)^{1/p} + \|B_\delta^{(j)}\|_{L^\infty} \|W_m\|_{L^p} \right). \tag{4.7}$$

Taking into account the uniform boundedness of $\{W_m\}_m$ in L^∞ and summing up over $j \in \{1, \cdots, N\}$ in (4.1), (4.2) and (4.7) yields the conclusion of Lemma 3.1.

□

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