African Diaspora Journal of Mathematics
Volume 10, Number 1, pp. 49–54 (2010)
ISSN 1539-854X
www.african-j-math.org

ON SEQUENCES OF ZEROS AND ONES IN NON-ARCHIMEDEAN ANALYSIS - A FURTHER STUDY

P N NATARAJAN
Old No. 2/3, New No. 3/3, Second Main Road, R.A. Puram
Chennai 600 028, India
pinnangudinatarajan@gmail.com

Abstract

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. Sequences and infinite matrices have entries in K. Supplementing [4], we make a further study of sequences of 0's and 1's in K.

AMS Subject Classification: 40, 46.

Keywords: Non-archimedean field, non-archimedean full sets, hereditary class of sets.

1 Introduction

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. Entries of sequences and infinite matrices are in K. In [4], we made a study of sequences of 0's and 1's in K. In the present paper, we make a further study of sequences of 0's and 1's in K. Classes φ of subsets of the set $\mathbb{N}$ of positive integers called "non-archimedean full" are defined and studied. Characterizing conditions for a covering, hereditary class of subsets of $\mathbb{N}$ to be non-archimedean full in terms of the entries of an infinite matrix $A = (a_{nk})$ are obtained. In the main result, we obtain necessary and sufficient conditions for $c_A \supseteq \chi_\varphi$ in terms of the entries of the infinite matrix A, where φ is non-archimedean full. We then deduce Hahn's theorem, in the non-archimedean case, that an infinite matrix sums all bounded sequences in K if and only if it sums all sequences of 0's and 1's in K.

We recall the following, which is needed in the sequel. Given an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n,k = 1,2,\ldots$ and a sequence $x = \{x_k\}$, $x_k \in K$, $k = 1,2,\ldots$, by the A-transform of x, we mean the sequence $Ax = \{(Ax)_n\}$, where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k, \quad n = 1,2,\ldots,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty}(Ax)_n = \ell$, we say that $x = \{x_k\}$ is summable or A-summable to ℓ. c_A denotes the convergence field of A, i.e., the set of all sequences $x = \{x_k\}$ which are A-summable.

2 Non-archimedean Full Sets and Main Results

Definition 2.1. *A class of φ of subsets of $\mathbb{N}$ (the set of positive integers) is said to be "non-archimedean full" if*

(i) $\bigcup_{S\in\varphi} S = \mathbb{N}$ *(covering);*

(ii) If $S \subset T$ where $T \in \varphi$, then $S \in \varphi$ (hereditary); and

(iii) if $\{t_k\}$ is a sequence in K such that $\sup_{k\in S}|t_k| < \infty$ for every $S \in \varphi$, then $\sup_{k\geq 1}|t_k| < \infty$.

Example 2.1. $\varphi = 2^{\mathbb{N}}$ *is an example of a non-archimedean full class.*

Theorem 2.1. *Let φ be a class of subsets of $\mathbb{N}$ satisfying (i), (ii) of Definition 2.1. Then φ is non-archimedean full if and only if for any infinite matrix (a_{nk}) for which $\sup_{n\geq 1}\left(\sup_{k\in S}|a_{nk}|\right) <$ ∞ for every $S \in \varphi$, then $\sup_{n,k\geq 1}|a_{nk}| < \infty$.*

Proof. Necessity. Let φ be non-archimedean full. Suppose for some infinite matrix (a_{nk}), $\sup_{n\geq 1}\left(\sup_{k\in S}|a_{nk}|\right) < \infty$ for every $S \in \varphi$ but $\sup_{n,k\geq 1}|a_{nk}| = \infty$. We can now choose strictly increasing sequences $\{n(j)\}$, $\{k(j)\}$ of positive integers such that

$$M(j) = \sup_{k(j-1)<i\leq k(j)} |a_{n(j),i}| > \frac{1}{\rho^{2j}},$$

where, since K is non-trivially valued, $\pi \in K$ is such that $0 < \rho = |\pi| < 1$. Let $\mathbb{N}(j) = \{i/k(j-1) < i \leq k(j)\}$, $j = 1,2,\ldots$, $k(0) = 1$. Now, define

$$b_i = a_{n(j),i}\pi^j, \quad i \in \mathbb{N}(j), j = 1,2,\ldots$$

$$\begin{aligned}\sup_{i\in\mathbb{N}(j)} |b_i| &= \sup_{i\in\mathbb{N}(j)} |a_{n(j),i}|\rho^j \\ &= \rho^j M(j) \\ &> \rho^j \frac{1}{\rho^{2j}} \\ &= \frac{1}{\rho^j},\end{aligned}$$

so that

$$\sup_{i\geq 1}|b_i| = \infty,$$

since $\frac{1}{\rho^j} \to \infty$, $j \to \infty$, $\frac{1}{\rho} > 1$.

Since φ is non-archimedean full, there exists $S \in \varphi$ with $\sup_{i \in S} |b_i| = \infty$. Consequently, we have

$$\sup_{i \in S \cap \mathbb{N}(j)} |b_i| > 1 \text{ for infinitely many } j's,$$

for, otherwise, $\sup_{i \in S \cap \mathbb{N}(j)} |b_i| \leq 1,\ j = 1, 2, \ldots$ and so $\sup_{i \geq 1} |b_i| \leq 1$, a contradiction. Hence for these infinitely many j's,

$$\begin{aligned} \sup_{i \in S} |a_{n(j),i}| &\geq \sup_{i \in S \cap \mathbb{N}(j)} |a_{n(j),i}| \\ &= \sup_{i \in S \cap \mathbb{N}(j)} \frac{|b_i|}{\rho^j} \\ &> \frac{1}{\rho^j} \to \infty, j \to \infty, \text{ since } \frac{1}{\rho} > 1, \end{aligned}$$

contradicting the fact that $\sup_{n \geq 1} \left(\sup_{k \in S} |a_{nk}| \right) < \infty$ for every $S \in \varphi$.

Sufficiency. Let $\{t_k\}$ be any sequence in K such that $\sup_{k \in S} |t_k| < \infty$ for every $S \in \varphi$. Define the matrix (a_{nk}), where $a_{nk} = t_k$, $k = 1, 2, \ldots;\ n = 1, 2, \ldots$. Then $\sup_{n \geq 1} \left(\sup_{k \in S} |a_{nk}| \right) < \infty$ for every $S \in \varphi$. By hypothesis, $\sup_{n,k \geq 1} |a_{nk}| < \infty$. It now follows that $\sup_{k \geq 1} |t_k| < \infty$ and so φ is non-archimedean full. This completes the proof of the theorem. □

Corollary 2.1. *φ is a class of subsets of* $\mathbb{N}$ *satisfying (i) and (ii) of Definition 2.1. Then φ is non-archimedean full if and only if for any infinite matrix* (a_{nk}) *for which* $\sup_{n \geq 1} \left| \sum_{k \in S} a_{nk} \right| < \infty$ *for every* $S \in \varphi$*, then* $\sup_{n,k \geq 1} |a_{nk}| < \infty$.

Proof. Necessity. Let φ be non-archimedean full. Let (a_{nk}) be an infinite matrix for which $\sup_{n \geq 1} \left| \sum_{k \in S} a_{nk} \right| < \infty$ for every $S \in \varphi$. Let $S \in \varphi$ and $k_0 \in S$. Since φ is hereditary, $S' = S \backslash \{k_0\} \in \varphi$. So

$$\sup_{n \geq 1} \left| \sum_{k \in S} a_{nk} - \sum_{k \in S'} a_{nk} \right| < \infty,$$

i.e.,

$$\sup_{n \geq 1} |a_{nk_0}| < \infty,$$

for every $k_0 \in S$ and so $\sup_{n \geq 1} \left(\sup_{k \in S} |a_{nk}| \right) < \infty$, for every $S \in \varphi$. Since φ is non-archimedean full, it follows, from Theorem 2.1, that $\sup_{n,k \geq 1} |a_{nk}| < \infty$.

Sufficiency. Let (a_{nk}) be an infinite matrix such that $\sup_{n\geq 1}\left(\sum_{k\in S}|a_{nk}|\right) < \infty$ for every $S\in\varphi$. Then,

$$\sup_{n\geq 1}\left|\sum_{k\in S} a_{nk}\right| \leq \sup_{n\geq 1}\left(\sup_{k\in S}|a_{nk}|\right) < \infty,$$

for every $S\in\varphi$. By hypothesis, $\sup_{n,k\geq 1}|a_{nk}| < \infty$ and so φ is non-archimedean full, using Theorem 2.1, completing the proof. □

The following result is worthwhile to record.

Theorem 2.2. *There is no minimal non-archimedean full class.*

Proof. Let S_0 be any infinite subset of a non-archimedean full class φ and $\Delta = \{S\in\varphi/S_0 \not\subseteq S\}$. Then $\Delta \subsetneqq \varphi$ and Δ satisfies (i) and (ii) of Definition 2.1. Let $\{t_k\}$ be a sequence in K such that $\sup_{k\geq 1}|t_k| = \infty$. Since φ is non-archimedean full, there exists $W\in\varphi$ such that $\sup_{k\in W}|t_k| = \infty$. So $\sup_{k\in W\backslash S_0}|t_k| = \infty$ or $\sup_{k\in W\cap S_0}|t_k| = \infty$. In the first case, if $T = W\backslash S_0$, then $T\in\Delta$ and $\sup_{k\in T}|t_k| = \infty$. In the second case, take $T = S_0\backslash\{s\}$, where $s\in S_0$. Then $T\in\Delta$ and $\sup_{k\in T}|t_k| \geq \sup_{k\in W\cap S_0}|t_k| = \infty$. In view of Definition 2.1, Δ is non-archimedean full, where $\Delta \subsetneqq \varphi$. Thus there is no minimal non-archimedean full class. □

We define $\chi_\varphi = \{\chi_S/S\in\varphi\}$, where χ_S denotes the characteristic function of the subset S of $\mathbb{N}$.

As an application to matrix summability, we have the following result.

Theorem 2.3. *Let φ be a non-archimedean full class and $A = (a_{nk})$ be any infinite matrix. Then $c_A \supseteq \chi_\varphi$ if and only if*

(i) $\lim_{k\to\infty} a_{nk} = 0$, $n = 1,2,\ldots$;

(ii) $\lim_{n\to\infty}\sup_{k\in S}|a_{n+1,k} - a_{nk}| = 0$ *for every* $S\in\varphi$.

Proof. Necessity. Let $c_A \supseteq \chi_\varphi$. It is clear that (i) holds. So

$$\lim_{k\to\infty}(a_{n+1,k} - a_{nk}) = 0.$$

Suppose (ii) does not hold. We use the "sliding hump method" to arrive at a contradiction. We can now choose $\varepsilon > 0$, $S\in\varphi$ and two strictly increasing sequences $\{n(i)\}$, $\{k(i)\}$ of positive integers such that

$$\sup_{k\in S}|a_{n(i)+1,k} - a_{n(i),k}| > \varepsilon;$$

$$\sup_{1\leq k\leq k(i-1)}|a_{n(i)+1,k} - a_{n(i),k}| < \frac{\varepsilon}{8};$$

and

$$\sup_{k>k(i)} |a_{n(i)+1,k} - a_{n(i),k}| < \frac{\varepsilon}{8}.$$

In view of the above inequalities, there exists $k(n(i)) \in S$, $k(i-1) < k(n(i)) \leq k(i)$ such that

$$|a_{n(i)+1,k(n(i))} - a_{n(i),k(n(i))}| > \varepsilon.$$

Define $x = \{x_k\}$, where

$$x_k = \begin{cases} 1, & \text{if } k = k(n(i)); \\ 0, & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned}
&(Ax)_{n(i)+1} - (Ax)_{n(i)} \\
&= \sum_{k=1}^{\infty} \{a_{n(i)+1,k} - a_{n(i),k}\} x_k \\
&= \sum_{k=1}^{k(i-1)} \{a_{n(i)+1,k} - a_{n(i),k}\} x_k + \sum_{k=k(i-1)+1}^{k(i)} \{a_{n(i)+1,k} - a_{n(i),k}\} x_k \\
&\quad + \sum_{k=k(i)+1}^{\infty} \{a_{n(i)+1,k} - a_{n(i),k}\} x_k \\
&= \sum_{k=1}^{k(i-1)} \{a_{n(i)+1,k} - a_{n(i),k}\} x_k + \{a_{n(i)+1,k(n(i))} - a_{n(i),k(n(i))}\} \\
&\quad + \sum_{k=k(i)+1}^{\infty} \{a_{n(i)+1,k} - a_{n(i),k}\} x_k,
\end{aligned}$$

so that

$$\begin{aligned}
\varepsilon &< |a_{n(i)+1,k(n(i))} - a_{n(i),k(n(i))}| \\
&\leq \max\left[|(Ax)_{n(i)+1} - (Ax)_{n(i)}|, \frac{\varepsilon}{8}, \frac{\varepsilon}{8} \right],
\end{aligned}$$

which implies that

$$|(Ax)_{n(i)+1} - (Ax)_{n(i)}| > \varepsilon, i = 1, 2, \ldots.$$

Thus $x \notin c_A$. Note, however, that $x \in \chi_\varphi$. Consequently $\chi_\varphi \not\subseteq c_A$, a contradiction. Consequently (ii) holds.

Sufficiency. Let (i) and (ii) hold. In view of (i), $\sum_{k\in S} a_{nk}$ converges for every $S \in \varphi$. Now,

$$\begin{aligned}
\left| \sum_{k\in S} a_{n+1,k} - \sum_{k\in S} a_{nk} \right| &= \left| \sum_{k\in S} \{a_{n+1,k} - a_{nk}\} \right| \\
&\leq \sup_{k\in S} |a_{n+1,k} - a_{nk}| \\
&\to 0, n \to \infty, \text{ using } (ii),
\end{aligned}$$

which implies that $\lim_{n\to\infty} \sum_{k\in S} a_{nk}$ exists for every $S \in \varphi$. Thus $c_A \supseteq \chi_\varphi$, completing the proof of the theorem. □

Corollary 2.2 (Hahn's theorem for the non-archimedean case)**.** *An infinite matrix* $A = (a_{nk})$ *sums all bounded sequences if and only if it sums all sequences of 0's and 1's.*

Proof. Leaving the trivial part of the result, suppose A sums all sequences of 0's and 1's, i.e., $c_A \supseteq \chi_\varphi$, where $\varphi = 2^{\mathbb{N}}$. Since $\mathbb{N} \in \varphi$,

$$\lim_{k\to\infty} a_{nk} = 0, \quad n = 1, 2, \ldots.$$

Also

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} |a_{n+1,k} - a_{nk}| = 0$$

i.e.,

$$\lim_{n\to\infty} \sup_{k\geq 1} |a_{n+1,k} - a_{nk}| = 0.$$

In view of Theorem 2 of [3], it follows that A sums all bounded sequences. □

References

[1] G. Bachman, *Introduction to p-adic numbers and valuation theory*, Academic Press, 1964.

[2] H. Hahn, Über Folgen linearer operationen, *Monatsh. Math.* **32** (1922), 3–88.

[3] P.N. Natarajan, The Steinhaus theorem for Toeplitz matrices in non-archimedean fields, *Comment. Math. Prace Mat.* **20** (1978), 417–422.

[4] P.N. Natarajan, On certain spaces containing the space of Cauchy sequences, *J. Orissa Math. Soc.* **9** (1990), 1–9.

[5] J.J. Sember and A.R. Freedman, On summing sequences of 0's and 1's, *Rocky Mountain J. Math.* **11** (1981), 419–425.