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NON-NEGATIVITY OF THE SOLUTION OF THE BOLTZMANN EQUATION IN A CURVED SPACE-TIME

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Abstract

In this paper, we give some conditions that insure the non-negativity of the solution of the relativistic Boltzmann equation in a full curved space-time, generalizing thereby some previous results known in the flat Minkowski space-time and for the non-relativistic Boltzmann equation.

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1 Introduction

It is clear that the unknown of the Boltzmann equation is the distribution function (or the particle number density) f which is by definition a non-negative real valued function.

It therefore appears that when solving the Boltzmann equation, the non-negativity of f should be insured. In many cases the non-negativity of solutions is proven when proving the existence of solutions ([7, 9, 12]). But the proof of the non-negativity of solutions might be a problem due to technical tools used to prove existence. This is the case when global existence of solutions which are close to Maxwellian is studied ([15]). In order to obtain the non-negativity of the solutions f of the non-relativistic Boltzmann equation, Giraud [8] used an appropriate decomposition $f = M(1+F)$, and proved that for sufficiently small initial data F_0, there exists a solution F such that f is non-negative. Some other results on non-negativity of solutions of the non-relativistic Boltzmann equation have been found in 2001 by X. Lu and Y. Zhang [15]. They proved the non-negativity of mild solutions in the case of collision kernels with angular cut-off and for both soft and hard potentials under suitable assumptions. They further applied their results to derive the non-negativity of some previous known solutions by showing that those known solutions satisfy the so called

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suitable assumptions. Many other works on classical non-relativistic Boltzmann equation have been achieved by several authors who studied and proved non-negativity when proving existence of solutions ($[1,5,6,10,13,14]$). In contrast to the many works on the classical non-relativistic Boltzmann equation, the literature is relatively poor in the case of the relativistic Boltzmann equation (RBE). In this later case, M. Dudynski and Ekiel-Jezewska [7] proved simultaneously the existence and non-negativity of global mild solutions in the flat Minkowski space-time. N. Noutchegueme and E. Takou [12] obtained a global in time solution of the relativistic Boltzmann equation in the Robertson-Walker space-time by using a method that preserves the necessary non-negativity physical property of the solution. For the full curved space-time, D. Bancel and Y. Choquet Bruhat [3,4] proved the existence and uniqueness of a local strong solution of the Boltzmann equation; but the non-negativity of the solution was not proven. This paper aims at stating some suitable conditions under which the solution of the relativistic Boltzmann equation in a full curved space-time is non-negative.

The paper is organized as follows: section 2 is devoted to the relativistic Boltzmann equation in a full curved space-time with specified conditions on the collision kernel. In section 3, the main result is stated and proved. The paper also contains an appendix where the existence of particles paths for the relativistic Boltzmann equation is rigorously proved.

2 The relativistic Boltzmann equation in a curved space-time

2.1 Notations and preliminaries

In all what follows, unless otherwise is specified, Greek indexes vary from 0 to 3 and Latin ones from 1 to 3. The Einstein summation convention is used i.e., $A_\alpha B^\alpha = \sum A_\alpha B^\alpha$. A space-time (M,g) is considered where M is a four dimensional time oriented manifold and g is the metric tensor with signature $(-,+,+,+)$. (x^α) denote the local coordinates on M with $x^0 = t$ representing time and $x = \left(x^i\right)$ representing space. We consider the collisional evolution of massive relativistic particles in absence of electromagnetic field. The time-evolution of these particles is described by their distribution function f (also called the particles number density) which is a non-negative real-valued function of the position (x^α) and the momentum (p^α). The local coordinates on the phase space TM are denoted by (x^α, p^α). The following notations are used for convenience,

$$X = (t,x), \quad x = \left(x^1,x^2,x^3\right), \quad P = \left(p^0,p^1,p^2,p^3\right), \quad p = \left(p^1,p^2,p^3\right).$$

For two vector fields $U = (u^\alpha)$ and $V = (v^\alpha)$, denote by

$$\begin{aligned} &UV = g_{\alpha\beta}u^\alpha v^\beta, \quad uv = \sum_{i=1}^{3} u^i v^i, \quad |u| = \left[\sum_{i=1}^{3}\left(u^i\right)^2\right]^{\frac{1}{2}}, \\ &\langle U,V\rangle = \left|(UV)^2 - (UU)(VV)\right|^{\frac{1}{2}}. \end{aligned} \tag{2.1}$$

The rest mass $m > 0$ of the particles is normalized to unity so that the equation of the future sheet of the mass-shell $F_x(P)$ of the particles at (x^α) with momenta momenta P is given by

$$PP = -1, \quad p^0 > 0. \tag{2.2}$$

The condition $p^0 > 0$ means that particles eject towards the future. The equation (2.2) allows to express p^0 in terms of p and (x^α) through g. This gives

$$p^0 = -\frac{g_{0i}p^i + \left[\left(g_{0i}p^i\right)^2 - g_{00}\left(g_{ij}p^ip^j + 1\right)\right]^{\frac{1}{2}}}{g_{00}}. \tag{2.3}$$

Equation (2.3) shows that f is a function of (t,x,p). The invariant volume element on $F_x(P)$ is (see [3,4]),

$$dP = \frac{|g|^{\frac{1}{2}}}{p_0}dp,$$

where $dp = dp^1dp^2dp^3$ and $|g|$ stands for the absolute value of the determinant of $g = \left(g_{\alpha\beta}\right)$.

It is assumed that at a given position (x^α) only two particles collide instantaneously and that the collision only affects the momenta of the particles that change after the collision. (This is the so called instantaneous, binary and elastic scheme due to Lichnerowicz and Chernikov). Denote by (P,P_*) (resp. $\left(P^{'},P^{'}_*\right)$) the momenta of two colliding particles before (resp. after) collision. The relation between precollisional and postcollisional momenta is determined by the conservation law of momentum due to the elasticity of the collisions. This conservation law reads

$$P + P_* = P^{'} + P^{'}_*. \tag{2.4}$$

Equation (2.4) is equivalent to the system

$$p^0 + p^0_* = p^{'0} + p^{'0}_*, \quad p + p_* = p^{'} + p^{'}_*. \tag{2.5}$$

It is easy to show the following relations

$$PP_* = P^{'}P^{'}_*, \quad \langle P,P_*\rangle = \left\langle P^{'},P^{'}_*\right\rangle. \tag{2.6}$$

The relation

$$p^0 + p^0_* = p^{'0} + p^{'0}_*$$

is the conservation of energy and can be rewritten using (2.3) to give the following conservation law

$$e + e_* = e^{'} + e^{'}_*, \tag{2.7}$$

where e stands for the energy and is defined by

$$e = e(t,x,p) = \left[\left(g_{0i}p^i\right)^2 - g_{00}\left(g_{ij}p^ip^j + 1\right)\right]^{\frac{1}{2}}. \tag{2.8}$$

Remark. *The conservation law* (2.7) *is the generalization to the full curved space-time of analogous ones known in Minkowski and Robertson-Walker space-times (see* [7,12]*).*

For simplicity, denote by

$$f = f(t,x,p), \quad f_* = f(t,x,p_*), \quad f^{'} = f\left(t,x,p^{'}\right), \quad f^{'}_* = f\left(t,x,p^{'}_*\right). \tag{2.9}$$

2.2 The Boltzmann equation

In a curved space-time endowed with a metric g, the Cauchy problem for the Boltzmann equation is (see [3,4]),

$$\begin{aligned} &\tfrac{\partial f}{\partial t}+\tfrac{p^i}{p^0}\tfrac{\partial f}{\partial x^i}-\tfrac{\Gamma^i_{\alpha\beta}p^\alpha p^\beta}{p^0}\tfrac{\partial f}{\partial p^i}=\tfrac{C(f)}{p^0} \quad \text{in } \mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3, \\ &f(0,x,p)=f_0(x,p) \ \text{ on } \mathbb{R}^3\times\mathbb{R}^3. \end{aligned} \tag{2.10}$$

Here $\Gamma^i_{\alpha\beta}$ are the Christoffel symbols associated to the metric g, defined by

$$\Gamma^\nu_{\alpha\beta}=\frac{1}{2}g^{\nu\lambda}\left(\frac{\partial g_{\lambda\beta}}{\partial x^\alpha}+\frac{\partial g_{\lambda\alpha}}{\partial x^\beta}-\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right). \tag{2.11}$$

$C(f)(t,x,p)$ is the non-linear collision operator which describes the rate of change of f due to binary, instantaneous and elastic collisions. It is often formally written as the difference between the gain term $C^+(f)$ and the loss term $C^-(f)$ (see [4]),

$$C(f)(t,x,p)=C^+(f)(t,x,p)-C^-(f)(t,x,p),$$

where

$$\begin{aligned} C^+(f)(t,x,p)&=\textstyle\int_{\mathbb{R}^3}|g|^{\frac{1}{2}}\frac{1}{p_{*0}}\left[\int_{S^2}f'f'_*S(t,x,p,p_*,\omega)\,d\omega\right]dp_*, \\ C^-(f)(t,x,p)&=\textstyle\int_{\mathbb{R}^3}|g|^{\frac{1}{2}}\frac{1}{p_{*0}}\left[\int_{S^2}ff_*S(t,x,p,p_*,\omega)\,d\omega\right]dp_*. \end{aligned} \tag{2.13}$$

S^2 stands for the unit sphere in $\mathbb{R}^3$. $S(t,x,p,p_*,\omega)$ is the collision kernel (also called the collision cross-section) which is a non-negative real-valued regular function. It will later be assumed that $S(t,x,p,p_*,\omega)$ satisfies the following symmetry conditions for $s\in\mathbb{R}_+$, $x,p,p_*,p',p'_*\in\mathbb{R}^3$, $\omega\in S^2$

$$\frac{S(s,x,p,p_*,\omega)}{p^0(s,x,p)\,p_{*0}(s,x,p_*)}=\frac{S(s,x,p_*,p,\omega)}{p^0_*(s,x,p_*)\,p_0(s,x,p)}, \tag{2.14}$$

$$\frac{S\left(s,x,p',p'_*,\omega\right)}{p'^0(s,x,p')\,p'_{*0}(s,x,p'_*)}=\frac{\partial(p,p_*)}{\partial(p',p'_*)}\frac{S(s,x,p,p_*,\omega)}{p^0(s,x,p)\,p_{*0}(s,x,p_*)}. \tag{2.15}$$

This implies the relativistic analogue of the classical conservation laws for f (see (3.14)). The system (2.10) describes the time evolution of the distribution function $f=f(t,x,p)$ on the phase space TM. The Boltzmann equation is a first order partial differential equation with a non local source term. Different types of solutions can be considered, e.g. strong, weak, mild, renormalized solutions. In this paper, the non-negativity of mild solutions of (2.10) is considered. Those mild solution may be derived from the strong ones which are shown to exist (see [3,4]). The definition of a mild solution of (2.10) will be recalled later.

Integrating (2.10) along the particles paths $t\mapsto(t,x+w,p+z)$, defined by

$$w^i(t,x,p)=\int_0^t\frac{P^i(s)}{P^0(s)}ds, \quad z^i(t,x,p)=-\int_0^t\frac{\Gamma^i_{\alpha\beta}P^\alpha(s)P(s)}{P^0(s)}ds, \tag{2.16}$$

leads to the following mild form of (2.10) on $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\begin{gathered} f(t, x+w, p+z) = f_0(x,p) + \textstyle\int_0^t \left(\frac{C(f)}{p^0}(s, x+w(s,x,p), p+z(s,x,p))\right) ds, \\ t \in \mathbb{R}_+,\ x, p \in \mathbb{R}^3. \end{gathered} \tag{2.17}$$

Denote by

$$f^{\#}(t,x,p) = f(t, x+w(t,x,p), p+z(t,x,p)).$$

Equation (2.17) then reads

$$f^{\#}(t,x,p) = f_0(x,p) + \int_0^t K(f)^{\#}(s,x,p)\, ds, \quad t \in \mathbb{R}_+,\ x, p \in \mathbb{R}^3, \tag{2.18}$$

where

$$K(f)(t,x,p) = \frac{C(f)}{p^0}(t,x,p). \tag{2.19}$$

The proof of the existence of particles paths defined by (2.16) is somewhat lengthy, requires some technical tools and is given in the appendix. We now give the definition of mild solutions of (2.10).

Definition. *A function f is called a mild solution of (2.10) on $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ with measurable initial value f_0 if f is measurable on $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ and for almost all (x,p) in $\mathbb{R}^3 \times \mathbb{R}^3$, $K^{\pm}(f)^{\#}(.,x,p)$ are in $L^1_{loc}(\mathbb{R}_+)$ and (2.18) holds for each $t \geq 0$.*

2.3 Assumptions on the collision kernel

Consider the following change of variables

$$(t,x,p) \to (t,y,v), \tag{2.20}$$

where

$$y(t,x,p) = x + w(t,x,p),\ v = p + z(t,x,p).$$

Here w and z are given by (2.16). Let B, D, α, J, l and β be defined for

$$(s,y,v,v_*,\omega) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$$

as follows

$$B(s,y,v,v_*,\omega) = \frac{S(s,y,v,v_*,\omega)}{v^0(s,y,v)\, v_{*0}(s,y,v_*)}, \tag{2.21}$$

$$D(s,y,v,v_*) = \int_{S^2} B(s,y,v,v_*,\omega)\, d\omega, \tag{2.22}$$

$$\alpha(s) = \underset{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3}{ess\sup} \left[\int_{\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} \frac{D(s,y,v,v_*)}{v^0(s,y,v)} |f_*|\, v^0(s,y,v_*)\, dv_*\right], \tag{2.23}$$

$$J(s,y,v) = \left|\frac{\partial(s,x,p)}{\partial(s,y,v)}\right|, \quad l(s,y,v) = \left[\left(p^0 J^{-1}\right)(s,y,v)\right]', \tag{2.24}$$

$$\beta(s) = \underset{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3}{ess\sup} \frac{|l(s,y,v)|\,J(s,y,v)}{v^0(s,y,v)}. \tag{2.25}$$

The assumptions $(2.14-15)$ can be rewritten as

$$\begin{array}{l} B(s,y,v,v_*,\omega) = B(s,y,v_*,v,\omega), \\ B\left(s,y,v',v'_*,\omega\right) = \frac{\partial(v,v_*)}{\partial\left(v',v'_*\right)} B(s,y,v,v_*,\omega). \end{array} \tag{2.26}$$

Moreover the functions α and β are assumed to satisfy the following conditions

$$\alpha \in L^1_{loc}(\mathbb{R}_+), \tag{2.27}$$

$$\beta \in L^1_{loc}(\mathbb{R}_+). \tag{2.28}$$

For any real number r, let r^+ and $\chi(r)$ be defined by

$$r^+ = \max(0,r), \tag{2.29}$$

$$\chi(r) = \begin{cases} 0 \text{ if } r \le 0, \\ 1 \text{ if } r > 0. \end{cases} \tag{2.30}$$

It is easy to check that

$$r^+ = r\chi(r), \tag{2.31}$$

$$f_0(x,p) \ge 0 \text{ implies } \left[-f^\#(t,x,p)\right]^+ = \int_0^t \left(\left[-f^\#(s,x,p)\right]^+\right)' ds, \tag{2.32}$$

$$\left(\left[-f^\#(s,x,p)\right]^+\right)' = -K(f)^\#(s,x,p)\,\chi\left(-f^\#(s,x,p)\right). \tag{2.33}$$

From (2.32) and (2.33) it follows that

$$\left[-f^\#(t,x,p)\right]^+ = -\int_0^t K(f)^\#(s,x,p)\,\chi\left(-f^\#(s,x,p)\right) ds. \tag{2.34}$$

Notice that the assumption (2.26) holds for the kernel used in the flat Minkowski space-time (see [2]).

3 The main result

The main result is the following

Theorem. *Let f be a mild solution of the system* (2.10) *on* $\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$ *with a non-negative initial datum f_0. Assume* $(2.26-28)$. *Then f is non-negative on* $\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$.

Proof. The proof uses similar tools to those introduced by X. Lu & Y. Zhang [15] for the non-relativistic Boltzmann equation. It consists in three steps and uses the splitting of the function u defined on $\mathbb{R}_+$ as follows,

$$u(t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[-f^\#(t,x,p)\right]^+ \left(p^0 J^{-1}\right)^\# (t,x,p)\,dx\,dp. \tag{3.1}$$

It holds that

$$u(t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} [-f(t,y,v)]^+ p^0(t,y,v)\,dydv. \tag{3.1a}$$

Since $p^0(t,y,v)$ is positive, it is sufficient for proving the non-negativity of f on $\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$, to show that $u(t)=0$ for any $t\geq 0$.

Splitting of $u(t)$

From an integration by parts, using (2.33) and the non-negativity of f_0, it holds that

$$\begin{aligned}
&\Big[\big[-f^{\#}(s,x,p)\big]^+ \left(p^0J^{-1}\right)^{\#}(s,x,p)\Big]_0^t \\
&= \textstyle\int_0^t \left(\big[-f^{\#}(s,x,p)\big]^+\right)' \left(p^0J^{-1}\right)^{\#}(s,x,p)\,ds \\
&+ \textstyle\int_0^t \big[-f^{\#}(s,x,p)\big]^+ \left(\left(p^0J^{-1}\right)^{\#}(s,x,p)\right)' ds \\
&= -\textstyle\int_0^t K(f)^{\#}(s,x,p)\,\chi\left(-f^{\#}(s,x,p)\right)\left(p^0J^{-1}\right)^{\#}(s,x,p)\,ds \\
&+ \textstyle\int_0^t \big[-f^{\#}(s,x,p)\big]^+ \left(\left(p^0J^{-1}\right)^{\#}(s,x,p)\right)' ds.
\end{aligned} \tag{3.2}$$

Then by $(3.1-2)$ and the non-negativity of f_0 we get

$$u(t) = I_1(t) + I_2(t), \tag{3.3}$$

where

$$\begin{aligned}
I_1(t) &= -\textstyle\int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} K(f)^{\#}(s,x,p)\,\chi\left(-f^{\#}(s,x,p)\right)\left(p^0J^{-1}\right)^{\#}(s,x,p)\,dxdp\right) ds, \\
I_2(t) &= \textstyle\int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \big[-f^{\#}(s,x,p)\big]^+ \left(\left(p^0J^{-1}\right)^{\#}(s,x,p)\right)' dxdp\right) ds.
\end{aligned} \tag{3.4}$$

Treatment of $I_1(t)$

By the change of variables $(s,x,p)\to(s,y,v)$, it holds that

$$I_1(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} [-K(f)(s,y,v)]\,\chi(-f(s,y,v))\left[v^0J^{-1}\right](s,y,v)\,J(s,y,v)\,dydv\right) ds.$$

By (2.19) it follows that

$$I_1(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left\{\int_{\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} \left[\int_{S^2} \left(-f^{'}f^{'}_{*} + ff_*\right)\chi(-f)\,\frac{S}{v^0v_{*0}}d\omega\right] dv_*\right\} v^0 dydv\right) ds. \tag{3.5}$$

Here the following notations have been used for convenience

$$p^0(s,y,v) \equiv v^0(s,y,v), \quad p_{*0}(s,y,v_*) \equiv v_{*0}(s,y,v_*). \tag{3.6}$$

From lemma 2 established by X. Lu & Y. Zhang [15], it holds that

$$\left(-r^{'}r^{'}_{*} + rr_*\right)\chi(-r) \leq \left(-r^{'}r^{'}_{*}\right)^+ - (-rr_*)^+ + |r|(-r_*)^+, \tag{3.7}$$

for any real numbers $r, r_*, r^{'}, r^{'}_{*}$. (3.7) implies that

$$I_1(t) \leq I_{1,1}(t) + I_{1,2}(t), \tag{3.8}$$

where $I_{1,1}(t)$ and $I_{1,2}(t)$ are defined as follows,

$$I_{1,1}(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} \left[\int_{S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] Bd\omega \right] dv_* \right\} v^0 dydv \right) ds,$$
$$I_{1,2}(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} \left[\int_{S^2} \left[|f| (-f_*)^{+} \right] Bd\omega \right] dv_* \right\} v^0 dydv \right) ds.$$

It holds that

$$I_{1,1}(t) = \int_0^t \left\{ \int_{\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} \left[\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] Bv^0 dvdv_*d\omega \right] dy \right\} ds. \tag{3.9}$$

By (2.26) and changing the role of v and v_* we have,

$$\begin{aligned}
&\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0(s,y,v)\, dvdv_*d\omega \\
&= \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0(s,y,v_*)\, dvdv_*d\omega.
\end{aligned} \tag{3.10}$$

Moreover, by the change of variables $(v,v_*) \longrightarrow \left(v^{'},v^{'}_{*}\right)$ and using (2.26) we have,

$$\begin{aligned}
&\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0(s,y,v)\, dvdv_*d\omega \\
&= \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[(-ff_*)^{+} - \left(-f^{'}f^{'}_{*}\right)^{+} \right] B\left(s,y,v^{'},v^{'}_{*},\omega\right) v^0\left(s,y,v^{'}\right) dv^{'}dv^{'}_{*}d\omega \\
&= -\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0\left(s,y,v^{'}\right) dvdv_*d\omega.
\end{aligned} \tag{3.11}$$

Finally, changing once more the role of v and v_*, using (2.22) in the last equality of (3.11), we have

$$\begin{aligned}
&\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0(s,y,v)\, dvdv_*d\omega \\
&= -\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0\left(s,y,v^{'}_{*}\right) dvdv_*d\omega.
\end{aligned} \tag{3.12}$$

It follows from $(3.10-12)$ that

$$\begin{aligned}
&4\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0(s,y,v)\, dvdv_*d\omega \\
&= \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega) \times \\
&\left[v^0(s,y,v) + v^0(s,y,v_*) - v^0\left(s,y,v^{'}\right) - v^0\left(s,y,v^{'}_{*}\right) \right] dvdv_*d\omega.
\end{aligned} \tag{3.13}$$

The conservation of the energy

$$v^0(s,y,v) + v^0(s,y,v_*) = v^0\left(s,y,v^{'}\right) + v^0\left(s,y,v^{'}_{*}\right)$$

therefore implies that,

$$\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \left[\left(-f^{'}f^{'}_{*}\right)^{+} - (-ff_*)^{+} \right] B(s,y,v,v_*,\omega)\, v^0(s,y,v)\, dvdv_*d\omega = 0. \tag{3.14}$$

From the definition of $I_{1,1}(t)$, (3.14) implies that

$$I_{1,1}(t) = 0. \tag{3.15}$$

The term $I_{1,2}(t)$ is now handled. Since $D(s,y,v,v_*) = \int_{S^2} B(s,y,v,v_*,\omega)\,d\omega$ it holds that,

$$I_{1,2}(t) = \int_0^t \left(\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} |f| (-f_*)^+ D(s,y,v,v_*,\omega)\, v^0(s,y,v)\, dv dv_* dy \right) ds. \tag{3.16}$$

By the change of variables $(v,v_*) \longrightarrow (v_*,v)$ and (2.26) we obtain,

$$I_{1,2}(t) = \int_0^t \left(\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} |f_*| (-f)^+ D(s,y,v,v_*,\omega)\, v^0(s,y,v_*)\, dv dv_* dy \right) ds,$$

and then

$$I_{1,2}(t) = \int_0^t \left(\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |g(s,y)|^{\frac{1}{2}} \frac{D(s,y,v,v_*)}{v^0(s,y,v)} |f_*|\, v^0(s,y,v_*) (-f)^+ v^0(s,y,v) dv dv_* dy \right) ds. \tag{3.17}$$

Due to the assumption (2.27), (3.17) implies that,

$$I_{1,2}(t) \leq \int_0^t \alpha(s) \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} (-f)^+ v^0(s,y,v) dv dy \right) ds. \tag{3.18}$$

Recall that, due to equalities $(3.1a)$ and (3.6), it holds that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} (-f)^+ v^0(s,y,v) dv dy = u(s).$$

So, inequality (3.18) gives

$$I_{1,2}(t) \leq \int_0^t \alpha(s)\, u(s) ds. \tag{3.19}$$

Finally, the relations (3.8), (3.15) and (3.19) imply that,

$$I_1(t) \leq \int_0^t \alpha(s)\, u(s) ds. \tag{3.20}$$

Treatment of $I_2(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[-f^{\#}(s,x,p)\right]^+ \times \left(\left(p^0 J^{-1}\right)^{\#}(s,x,p) \right)' dx dp \right) ds$

From (2.24), it holds that

$$I_2(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[-f^{\#}(s,x,p)\right]^+ l^{\#}(s,x,p)\, dx dp \right) ds.$$

By the change of variables $(s,x,p) \rightarrow (s,y,v)$, it follows that

$$I_2(t) = \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[-f(s,y,v)\right]^+ l(s,y,v)\, J(s,y,v)\, dy dv \right) ds.$$

Therefore, we get

$$I_2(t) \leq \int_0^t \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|l(s,y,v)|\, J(s,y,v)}{v^0(s,y,v)} \left[-f(s,y,v)\right]^+ v^0(s,y,v) dy dv \right) ds. \tag{3.22}$$

Using (3.22), assumption (2.28) implies that

$$I_2(t) \leq \int_0^t \beta(s)\, u(s) ds. \tag{3.23}$$

Combining the inequalities (3.20) and (3.23) together with equalities (3.3) and (3.4), the following final estimate of $u(t)$ is derived,

$$u(t) \leq \int_0^t [\alpha(s) + \beta(s)]\, u(s) ds. \tag{3.24}$$

By Gronwall's lemma, the inequality (3.24) implies that $u(t) = 0$. Thus the theorem is proved. □

Appendix: Existence of characteristics with respect to the relativistic Boltzmann equation

Consider the Boltzmann equation

$$p^0 \frac{\partial f}{\partial t} + p^i \frac{\partial f}{\partial x^i} - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = C(f). \tag{A.1}$$

The characteristic system associated to equation $(A.1)$ is

$$\frac{dt}{ds} = p^0, \quad \frac{dx^i}{ds} = p^i, \quad \frac{dp^i}{ds} = -\Gamma^i_{\alpha\beta} p^\alpha p^\beta, \quad \frac{df}{ds} = C(f). \tag{A.2}$$

The last equation of system $(A.2)$ is automatically verified if the first three hold together with equation $(A.1)$. We can therefore focus on the following system

$$\frac{dt}{ds} = p^0, \quad \frac{dx^i}{ds} = p^i, \quad \frac{dp^i}{ds} = -\Gamma^i_{\alpha\beta} p^\alpha p^\beta. \tag{A.3}$$

The solutions of system $(A.3)$ are called particles paths or trajectories. Note that p^0 is known if $g_{\alpha\beta}$ and p^i are known since

$$p^0 = -\frac{g_{0i} p^i + \left[\left(g_{0i} p^i \right)^2 - g_{00} \left(g_{ij} p^i p^j + 1 \right) \right]^{\frac{1}{2}}}{g_{00}}.$$

Moreover, under suitable regularity assumptions on the metric g, p^0, $\frac{p^i}{p^0}$, $\frac{\Gamma^i_{\alpha\beta} p^\alpha p^\beta}{p^0}$ are C^∞ functions of x^α and p^i. Furthermore equation $\frac{dt}{ds} = p^0$ $\left(\text{with } p^0 > 0\right)$ shows that t can be taken as an increasing parameter along the particles paths. The differential system $(A.3)$ can be written as follows,

$$\frac{dt}{d\tau} = 1, \ \frac{dx^i}{d\tau} = \frac{p^i}{p^0}, \ \frac{dp^i}{d\tau} = -\frac{\Gamma^i_{\alpha\beta} p^\alpha p^\beta}{p^0}. \tag{A.4}$$

Setting $z = \left(t, x^i, p^i\right)$ and $Y = \left(1, \frac{p^i}{p^0}, -\frac{\Gamma^i_{\alpha\beta} p^\alpha p^\beta}{p^0}\right)$, system $(A.4)$ takes the following form

$$\frac{dz}{d\tau} = Y(z). \tag{A.5}$$

As $\frac{p^i}{p^0}$ and $\frac{\Gamma^i_{\alpha\beta}p^\alpha p^\beta}{p^0}$ are C^∞ functions of z, the vector field $Y(z)$ is locally lipschitz. The local existence of particles paths therefore follows i.e., for any $z_0 = \left(t_0, x^i_0, p^i_0\right)$, there exists $\varepsilon(z_0) > 0$ such that equation $(A.5)$ has, for $|\tau| < \varepsilon(z_0)$, one and only one solution z satisfying $z(0) = z_0$. According to the system $(A.4)$, the solution $z(\tau)$ is given by

$$z(\tau) = \left(t(\tau, z_0), x^i(\tau, z_0), p^i(\tau, z_0)\right),$$

where

$$\begin{aligned} t(\tau, z_0) &= \tau + t_0, \\ x^i(\tau, z_0) &= x^i_0 + \int_0^\tau \frac{p^i}{p^0}(s)\, ds, \\ p^i(\tau, z_0) &= p^i_0 - \int_0^\tau \frac{\Gamma^i_{\alpha\beta}p^\alpha p^\beta}{p^0}(s)\, ds. \end{aligned} \tag{A.6}$$

To prove the global existence of particles paths, it is enough to show that any solution of the Cauchy problem

$$\frac{dz}{d\tau} = Y(z), \quad z(0) = z_0, \tag{A.7}$$

is contained in a fixed ball. Let T be a given real number such that $T > 0$. It is clear that $t(\tau) = \tau + t_0$ is bounded for $0 \le \tau \le T - t_0$.

The next step is to prove that $x^i(\tau) = x^i_0 + \int_0^\tau \frac{p^i}{p^0}(s)\, ds$ is bounded.

Let x_0 and x_t be the projections of the points

$$y_0 = \left(x^i_0, p^i_0\right)$$

and

$$y_t = \left(x^i_t, p^i_t\right)$$

respectively. The distance $d(x_0, x_t)$ is defined as the length of Γ where $\Gamma : \tau \mapsto x^i(\tau)$ is the projection of the flow of the vector field Y passing through $z_0 = \left(t_0, x^i_0, p^i_0\right)$ and $z_t = \left(t, x^i_t, p^i_t\right)$. This implies that

$$d(x_0, x_t) = \int_{t_0}^{t} \left[g_{ij}(x(\tau)) \frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]^{\frac{1}{2}} d\tau. \tag{A.8}$$

According to the system $(A.4)$, along Γ it holds

$$g_{ij}(x(\tau)) \frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = g_{ij}(x(\tau)) \frac{p^i}{p^0}\frac{p^j}{p^0}. \tag{A.9}$$

The hyperbolicity assumptions made by Y. Choquet-Bruhat and D. Bancel on g in [3,4] insure that

$$\exists c > 0 : g_{ij} \frac{p^i}{p^0} \frac{p^j}{p^0} \leq c. \tag{A.10}$$

The hyperbolicity assumptions mentioned above are the following,

$$\begin{array}{c} \exists a, b > 0 : a\,|\xi|^2 \leq g_{ij}\xi^i\xi^j \leq b\,|\xi|^2, \\ -g_{00} \geq a, -g^{00} \geq a, \end{array} \tag{A.11}$$

where

$$|\xi|^2 = \sum_{i=1}^{3} \left(\xi^i\right)^2. \tag{A.12}$$

Recall that the mass-shell on which the momenta of the particles are located has the equation

$$g_{\alpha\beta} p^\alpha p^\beta = -1, \quad p^0 > 0. \tag{A.13}$$

A lengthy calculation shows that equation $(A.13)$ is equivalent to

$$g_{ij}\left(w^i + l^i\right)\left(w^j + l^j\right) = -\frac{1}{g^{00}} - \left(\frac{1}{p^0}\right)^2, \quad p^0 > 0, \tag{A.14}$$

where

$$w^i = \frac{p^i}{p^0}, \quad l^i = -\frac{g^{0i}}{g^{00}}. \tag{A.15}$$

From assumptions $(A.11)$ and equation $(A.14)$ one deduces that

$$a\,|w+l|^2 \leq g_{ij}\left(w^i + l^i\right)\left(w^j + l^j\right) \leq -\frac{1}{g^{00}} \leq \frac{1}{a}. \tag{A.16}$$

Therefore

$$|w+l| \leq \frac{1}{a}. \tag{A.17}$$

From the inequality $|w| - |l| \leq |w+l|$, it follows due to inequality $(A.17)$ that

$$|w| \leq |l| + \frac{1}{a}. \tag{A.18}$$

The relations $(A.11)$ and $(A.15)$ imply that

$$\left|l^i\right| \leq \frac{1}{a}\left|g^{0i}\right|. \tag{A.19}$$

If we assume that g is uniformly bounded as it is done in $[3,4]$, then l is bounded and so is w. Now using assumptions $(A.11)$ it follows that

$$g_{ij} \frac{p^i}{p^0} \frac{p^j}{p^0} = g_{ij} w^i w^j \leq b\,|w|^2.$$

So as w is bounded, the inequality $(A.10)$ is obtained. Coming back to equality $(A.8)$ and using inequality $(A.10)$ it holds that

$$d\left(x_0, x_t\right) \leq \sqrt{c}\left(t-t_0\right) \leq \sqrt{c}\left(T-t_0\right), \text{ for } 0 \leq t \leq T. \tag{A.20}$$

From this it follows that $x^i(\tau)$ is bounded for $0 \leq \tau \leq T-t_0$.

The last step is to prove that $p^i(\tau)$ is bounded in $\mathbb{R}^3$.

To achieve this task, it is sufficient to show that

$$\exists c>0 : g_{ij} p^i p^j \leq c. \tag{A.21}$$

Setting

$$v(\tau)=\left(g_{ij} p^i p^j+1\right)\left(\tau, z_0\right),$$

it holds that

$$\frac{dv}{d\tau}=2 g_{ij} p^i \frac{dp^j}{d\tau}+p^i p^j \frac{dg_{ij}}{d\tau}=2 g_{ij} p^i \frac{dp^j}{d\tau}+p^i p^j \frac{\partial g_{ij}}{\partial x^\alpha} \frac{dx^\alpha}{d\tau}. \tag{A.22}$$

By $(A.4)$ and $(A.22)$, we gain

$$\frac{dv}{d\tau}=2 g_{ij} p^i\left(-\frac{\Gamma_{\alpha\beta}^j p^\alpha p^\beta}{p^0}\right)+p^i p^j\left(\frac{\partial g_{ij}}{\partial x^0}+\frac{\partial g_{ij}}{\partial x^k} \frac{p^k}{p^0}\right). \tag{A.23}$$

Denote for simplicity,

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}=g_{\alpha\beta,\gamma}.$$

The relation

$$\Gamma_{\alpha\beta}^j=\frac{1}{2} g^{\lambda j}\left(g_{\alpha\lambda,\beta}+g_{\beta\lambda,\alpha}-g_{\alpha\beta,\lambda}\right),$$

implies that

$$2 g_{ij} \Gamma_{\alpha\beta}^j=g_{\alpha i,\beta}+g_{\beta i,\alpha}-g_{\alpha\beta,i}. \tag{A.24}$$

It follows that

$$\begin{aligned} 2 g_{ij} p^i \Gamma_{\alpha\beta}^j p^\alpha p^\beta &= p^i p^\alpha p^\beta\left(g_{\alpha i,\beta}+g_{\beta i,\alpha}-g_{\alpha\beta,i}\right) \\ &= 2 p^i p^\alpha p^\beta g_{\alpha i,\beta}-p^i p^\alpha p^\beta g_{\alpha\beta,i}. \end{aligned} \tag{A.25}$$

The relations $(A.24)$ and $(A.25)$ imply that

$$\frac{dv}{d\tau}=-\frac{2}{p^0} p^i p^\alpha p^\beta g_{\alpha i,\beta}+\frac{1}{p^0} p^i p^\alpha p^\beta g_{\alpha\beta,i}+p^i p^j\left(g_{ij,0}+\frac{p^k}{p^0} g_{ij,k}\right). \tag{A.26}$$

The reduction of equation $(A.26)$ gives

$$\frac{dv}{d\tau}=-2 p^0 p^i g_{0i,0}+p^0 p^i g_{00,i}-p^i p^j g_{ij,0}. \tag{A.27}$$

Now set

$$\theta(\tau) = \left[\left(g_{0i}p^i\right)^2 - g_{00}v\right](\tau) = g_{0i}g_{0j}p^ip^j - g_{00}v. \tag{A.28}$$

It holds that

$$\begin{aligned}\tfrac{d\theta}{d\tau} &= 2g_{0i}p^i\left[g_{0j}\tfrac{dp^j}{d\tau} + p^jg_{0j,\alpha}\tfrac{dx^\alpha}{d\tau}\right] - g_{00}\tfrac{dv}{d\tau} - vg_{00,\alpha}\tfrac{dx^\alpha}{d\tau}\\ &= 2g_{0i}p^i\left[g_{0j}\left(-\tfrac{\Gamma^j_{\alpha\beta}p^\alpha p^\beta}{p^0}\right) + p^jg_{0j,0} + p^jg_{0j,k}\left(\tfrac{p^k}{p^0}\right)\right]\\ &-g_{00}\left(-2p^0p^ig_{0i,0} + p^0p^ig_{00,i} - p^ip^jg_{ij,0}\right) - vg_{00,0} - vg_{00,k}\left(\tfrac{p^k}{p^0}\right).\end{aligned} \tag{A.29}$$

A straightforward and lengthy calculation shows that

$$\begin{aligned}\tfrac{d\theta}{d\tau} &= -p^0p^i\left(g_{0i}g_{00,0} - 2g_{00}g_{0i,0} + g_{00}g_{00,i}\right)\\ &+p^ip^j\left(2g_{0i}g_{0j,0} + g_{00}g_{ij,0} + \tfrac{p^k}{p^0}g_{0i}g_{jk,0} - 2g_{0i}g_{00,j}\right)\\ &-vg_{00,0} - \tfrac{p^i}{p^0}vg_{00,i}.\end{aligned} \tag{A.30}$$

Using the fact that $w = \left(\frac{p^i}{p^0}\right)$ is bounded, we have

$$\exists c_0 > 0 : \left|\frac{p^i}{p^0}\right| \leq c_0. \tag{A.31}$$

Due to assumptions $(A.11)$ and the definition of v, it holds that

$$a|p|^2 \leq g_{ij}p^ip^j \leq v.$$

From the relation

$$v = -\frac{1}{g_{00}}\left(-g_{00}v\right) \leq \frac{1}{a}\theta.$$

it follows that

$$\left|p^i\right| \leq \frac{1}{a}\sqrt{\theta}. \tag{A.32}$$

Since

$$p^0 = -\frac{g_{0i}p^i + \sqrt{\theta}}{g_{00}},$$

it follows from assumptions $(A.11)$ that

$$\left|p^0\right| \leq \frac{1}{a}\left(\left|g_{0i}p^i\right| + \sqrt{\theta}\right).$$

So, if g is bounded, then

$$\exists c_g > 0 : \left|p^0\right| \leq c_g\sqrt{\theta}, \tag{A.33}$$

where c_g is a constant depending on g. Due to the presence of first derivatives of g in equation $(A.30)$, we assume that those derivatives are also uniformly bounded. Doing this, we derive the following inequality from the relations $(A.30)$, $(A.31)$, $(A.32)$ and $(A.33)$

$$\frac{d\theta}{d\tau} \leq C\theta, \tag{A.34}$$

where C is a constant depending on g and the first derivatives of g. Inequality $(A.34)$ implies that

$$\theta(\tau) \leq \theta(0) + \int_0^\tau C\theta(s)\,ds. \tag{A.35}$$

Thus Gronwall's lemma implies that

$$[\theta(\tau)]^{\frac{1}{2}} \leq [\theta(0)]^{\frac{1}{2}} \exp\left[\frac{1}{2}\int_0^\tau Cds\right] = [\theta(0)]^{\frac{1}{2}} \exp\left[\frac{1}{2}C\tau\right]. \tag{A.36}$$

Finally θ satisfies the following inequality for $0 \leq \tau \leq T - t_0$, $0 \leq t_0 < T$,

$$\theta(\tau) \leq [K(T)]^2, \tag{A.37}$$

where $K(T) = [\theta(0)]^{\frac{1}{2}} \exp\left[\frac{1}{2}CT\right]$. Coming back to the definitions of

$$v(\tau) = \left(g_{ij}p^i p^j + 1\right)(\tau),$$

and

$$\theta(\tau) = \left[\left(g_{0i}p^i\right)^2 - g_{00}v\right](\tau),$$

we get

$$\left(g_{ij}p^i p^j\right)(\tau) < v(\tau) \leq \frac{1}{a}\theta(\tau) \leq \frac{[K(T)]^2}{a},$$

which is the desired inequality $(A.21)$.

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