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# HOMOLOGICAL DIMENSIONS OF THE AMALGAMATED DUPLICATION OF A RING ALONG A PURE IDEAL

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**Abstract**

The aim of this paper is to study the classical global and weak dimensions of the amalgamated duplication of a ring $R$ along a pure ideal $I$.

**AMS Subject Classification:** 62G05; 62G20.

**Keywords**: Amalgamated duplication of a ring along an ideal, pure ideal, (n,d)-rings and weak(n,d)-rings

## 1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unitary.
Let $R$ be a ring, and let $M$ be an $R$-module. As usual we use $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of $M$. We use also $gldim(R)$ and $wdim(R)$ to denote, respectively, the classical global and weak dimension of $R$.

For a nonnegative integer $n$, an $R$-module $M$ is $n$-presented if there is an exact sequence $F_n \to F_{n-1} \to ... \to F_0 \to M \to 0$ in which each $F_i$ is a finitely generated free $R$-module. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Set $\lambda_R(M) = \{n/M \text{ is } n\text{-presented } \}$ except if $M$ is not finitely generated. In this last case, we set $\lambda_R(M) = -1$. Not that $\lambda_R(M) \geq n$ is a way to express the fact that $M$ is $n$-presented.
Given nonnegative integers $n$ and $d$, a ring $R$ is called an $(n,d)$-ring if every $n$-presented $R$-module has projective dimension $\leq d$, and $R$ is called a weak $(n,d)$-ring if every $n$-presented cyclic $R$-module has projective dimension $\leq d$. For instance, the $(0,1)$-domains are the Dedekind domains, the $(1,1)$-domains are the Prüfer domains and the $(1,0)$-rings are the Von Neumann regular rings (see [1, 11, 12, 13, 14]). A commutative ring is called

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an $n$-Von Neumann regular ring if it is an $(n,0)$-ring. Thus, the 1-von Neumann regular rings are the von Neumann regular rings ([1, Theorem 1.3]) .

The amalgamated duplication of a ring $R$ along an ideal $I$ is a ring that is defined as the following subring with unit element $(1,1)$ of $R\times R$:

$$R\bowtie I=\{(r,r+i)/r\in R,i\in I\}$$

This construction has been studied, in the general case, and from the different point of view of pullbacks, by D'Anna and Fontana [6]. Also, in [5], they have considered the case of the amalgamated duplication of a ring, in not necessarily Noetherian setting, along a multiplicative canonical ideal in the sense of [10]. In [4], D'Anna has studied some properties of $R\bowtie I$, in order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi, in [16], have studied the diameter and girth of the zero-divisor of the ring $R\bowtie I$. Recently in [3], the authors study some homological properties of the rings $R\bowtie I$. Some references are [4, 5, 6, 16].

Let $M$ be an $R$-module, the idealization $R\propto M$ (also called the trivial extension), introduced by Nagata in 1956 (cf [17]) is defined as the $R$-module $R\oplus M$ with multiplication defined by $(r,m)(s,n)=(rs,rn+sm)$ (see [7, 9, 11, 12]).

When $I^2=0$, the new construction $R\bowtie I$ coincides with the idealization $R\propto I$. One main difference of this construction, with respect to idealization is that the ring $R\bowtie I$ can be a reduced ring (and, in fact, it is always reduced if $R$ is a domain).
The first purpose of this paper is to study the classical global and weak dimension of the amalgamated duplication of a ring $R$ along pure ideal $R$. Namely, we prove that if $I$ is a pure ideal of $R$, then $wdim(R\bowtie I)=wdim(R)$. Also, we prove that if $R$ is a coherent ring and $I$ is a finitely generated pure ideal of $R$, then $R\bowtie I$ is an $(1,d)$-ring provided the local ring $R_M$ is an $(1,d)$-ring for every maximal ideal $M$ of $R$. Finally, we give several examples of rings which are not weak $(n,d)$-rings (and so not $(n,d)$-rings) for each positive integers $n$ and $d$.

## 2 Main Results

Let $R$ be a commutative ring with identity element 1 and let $I$ be an ideal of $R$. We define $R\bowtie I=\{(r,s)/r,s\in R,s-r\in I\}$. It is easy to check that $R\bowtie I$ is a subring with unit element $(1,1)$, of $R\times R$ (with the usual componentwise operations) and that $R\bowtie I=\{(r,r+i)/r\in R,i\in I\}$.

It is easy to see that, if $\pi_i$ $(i=1,2)$ are the projections of $R\times R$ on $R$, then $\pi_i(R\bowtie I)=R$ and hence if $O_i=ker(\pi_i\backslash R\bowtie I)$, then $R\bowtie I/O_i\cong R$. Moreover $O_1=\{(0,i),i\in I\}$, $O_2=\{(i,0),i\in I\}$ and $O_1\cap O_2=(0)$.

Our first main result in this paper is given by the following Theorem:

**Theorem 2.1.** *Let $R$ be a ring and $I$ be a pure ideal of $R$. Then, $wdim(R\bowtie I)=wdim(R)$.*

To prove this Theorem we need some results.

**Lemma 2.2.** *[4, Proposition 7] Let R be a ring and let I be an ideal of R. Let P be a prime ideal of R and set:*

- $P_0 = \{(p, p+i)/p \in P, i \in I \cap P\}$,
- $P_1 = \{(p, p+i)/p \in P, i \in I\}$, *and*
- $P_2 = \{(p+i, p)/p \in P, i \in I\}$.

1. *If* $I \subseteq P$, *then* $P_0 = P_1 = P_2$ *and* $(R \bowtie I)_{P_0} \cong R_P \bowtie I_P$.
2. *If* $I \nsubseteq P$, *then* $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$ *and* $(R \bowtie I)_{P_1} \cong R_P \cong (R \bowtie I)_{P_2}$.

**Lemma 2.3.** *Let I be a non-zero flat ideal of a ring R. For any R-module M we have:*

1. $fd_R(M) = fd_{R\bowtie I}(M \otimes_R (R \bowtie I))$.
2. $pd_R(M) = pd_{R\bowtie I}(M \otimes_R (R \bowtie I)$.

*Proof.* Note that the $R$-module $R \bowtie I$ is faithfully flat since $I$ is flat.
Firstly suppose that $fd_R(M) \leq n$ (resp., $pd_R(M) \leq n$) and pick an $n$-step flat (resp., projective) resolution of $M$ over $R$ as follows:

$$(*) \quad 0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0.$$

Applying the functor $- \otimes_R R \bowtie I$ to $(*)$, we obtain the exact sequence of $(R \bowtie I)$-modules:

$$0 \to F_n \otimes_R (R \bowtie I) \to F_{n-1} \otimes_R (R \bowtie I) \to \ldots \to F_0 \otimes_R (R \bowtie I) \to M \otimes_R (R \bowtie I) \to 0$$

Thus $fd_{R\bowtie I}(M \otimes_R (R \bowtie I)) \leq n$ (resp., $pd_{R\bowtie I}(M \otimes_R (R \bowtie I)) \leq n$).

Conversely, suppose that $fd_{R\bowtie I}(M \otimes_R (R \bowtie I)) \leq n$ (resp., $pd_{R\bowtie I}(M \otimes_R (R \bowtie I)) \leq n$. Inspecting [2, page 118] and since $Tor_R^k(M, R \bowtie I) = 0$ for each $k \geq 1$, we conclude that for any $R$-module $N$ and each $k \geq 1$ we have:

**(1)** $Tor_R^k(M, N \otimes_R (R \bowtie I)) \cong Tor_{R\bowtie I}^k(M \otimes_R (R \bowtie I), N \otimes_R (R \bowtie I))$

**(2)** $Ext_R^k(M, N \otimes_R (R \bowtie I)) \cong Ext_{R\bowtie I}^k(M \otimes_R (R \bowtie I), N \otimes_R (R \bowtie I))$

On the other hand $Tor_R^k(M,N)$ and $Ext_R^k(M,N)$ are direct summands of $Tor_R^k(M, N \otimes_R (R \bowtie I))$ and $Ext_R^k(M, N \otimes_R (R \bowtie I))$ respectively. Then, we conclude that $fd_R(M) \leq n$ (resp., $pd_R(M) \leq n$) and this finish the proof of this result.

One direct consequence of this Lemma is:

**Corollary 2.4.** *Let I be a non-zero flat ideal of a ring R. Then:*

1. $wdim(R) \leq wdim(R \bowtie I)$.
2. $gldim(R) \leq gldim(R \bowtie I)$.

**Proof of Theorem** 2.1 The inequality $wdim(R) \leq wdim(R \bowtie I)$ holds directly from Corollary 2.4 since $I$ is pure and then flat. So, only the other inequality need a proof.
Using [7, Theorem 1.3.14] we have:

$$(\top) \quad wdim(R \bowtie I) = sup\{wdim((R \bowtie I)_M) | M \text{ is a maximal ideal of } R \bowtie I\}.$$

Let $M$ be an arbitrary maximal ideal of $R \bowtie I$ and set $m := M \cap R$. Then necessarily $M \in \{M_1, M_2\}$ where $M_1 = \{(r, r+i)/r \in m, i \in I\}$ and $M_2 = \{(r+i, r)/r \in m, i \in I\}$ (by [6, Theorem 3.5]). On the other hand, $I_m \in \{0, R_m\}$ since $I$ is pure and $m$ is maximal in R (by [7, Theorem 1.2.15]). Then, testing all cases of Lemma 2.3, we resume two cases;

1. $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.
2. $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ or $I \subseteq m$.

Hence, we have $wdim((R \bowtie I)_M) = wdim(R_m) \leq wdim(R)$. So, the desired inequality follows from the equality $(\top)$.

**Corollary 2.5.** *Let $I$ be a finitely generated pure ideal of a ring $R$. Then $R$ is a semihereditary ring if, and only if, $R \bowtie I$ is a semihereditary ring.*

*Proof.* Follows immediately from Theorem 2.1 and [3, Theorem 3.1].

Recall that a ring $R$ is called Gaussian if $c(fg) = c(f)c(g)$ for every polynomials $f, g \in R[X]$, where $c(f)$ is the content of $f$, that is, the ideal of $R$ generated by the coefficients of $f$. See for instance [8].

**Corollary 2.6.** *Let $R$ be a reduced ring and let $I$ be a pure ideal of $R$. Then $R$ is Gaussian if, and only if, $R \bowtie I$ is Gaussian.*

*Proof.* Follows immediately from Theorem 2.1 , [8, Theorem 2.2] and [6, Theorem 3.5(a)(vi)].

By the fact that every ideal over a Von Neumann regular ring is pure, we conclude from Theorem 2.1 the following Corollary which have already proved in [3] with different methods.

**Corollary 2.7.** *Let $R$ be a ring and let $I$ be an ideal of $R$. If $R$ is a Von Neumann regular ring, then so is $R \bowtie I$.*

If the ring $R$ is Noetherian the global and weak dimensions coincide. Hence, Theorem 2.1 can be writing as follows:

**Corollary 2.8.** *If $I$ is a pure ideal of a Noetherian ring $R$, then $gdim(R \bowtie I) = gdim(R)$.*

A simple example of Theorem 2.1 is given by introducing the notion of the trace of modules. Recall that if $M$ is an $R$-module, the trace of $M$, $Tr(M)$, is the sum of all images of morphisms $M \rightarrow R_R$ (see [15]). Clearly $Tr(M)$ is an ideal of $R$.

**Example 2.9.** If $M$ is a projective module over a ring $R$, then $wdim(R \bowtie Tr(M)) = wdim(R)$.

*Proof.* Clear since $Tr(M)$ is a pure ideal whenever $M$ is projective (by [19, pp. 269-270]).

Now, we study the transfer of an $(1, d)$-property.

**Theorem 2.10.** *Let $R$ be a coherent ring such that for every maximal ideal $m$ of $R$ the local ring $R_m$ is an $(1,d)$-ring, and let $I$ be a finitely generated pure ideal of $R$. Then $R \bowtie I$ is an $(1,d)$-ring.*

*Proof.* Using [1, Theorem 3.2] and [3, Theorem 3.1], we have to prove that for any maximal ideal $M$ of $R \bowtie I$, the ring $(R \bowtie I)_M$ is an $(1,d)$-ring. So, let $M$ be such ideal and set $m := M \cap R$. From the proof of Theorem 2.1, we have two possible cases:

1. $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.
2. $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ or $I \subseteq m$.

So, by the hypothesis conditions, $(R \bowtie I)_M$ is an $(1,d)$-ring since $R_m$ is it, as desired.

By the fact that every ideal over a semisimple ring is pure we conclude from Theorem 2.10 the following Corollary.

**Corollary 2.11.** *Let $R$ be a ring and let $I$ be an ideal of $R$. If $R$ is a semisimple ring, then so is $R \bowtie I$.*

Now, we give a wide class of rings which are not weak $(n,d)$-rings (and so not $(n,d)$-rings) for each positive integers $n$ and $d$.

**Theorem 2.12.** *Let $R$ be a ring and let $I$ be a proper ideal of $R$ which satisfies the following condition:*

1. *$R_m$ is a domain for every maximal ideal $m$ of $R$.*
2. *$I_m$ is a principal proper ideal of $R_m$ for every maximal ideal $m$ of $R$.*

*Then, $wdim(R \bowtie I)(= gldim(R \bowtie I)) = \infty$.*

*Proof.* Let $m$ be a maximal ideal of $R$ such that $I \subseteq m \subsetneq R$. By Lemma 2.3, $R_m \bowtie I_m = (R \bowtie I)_M$ where $M = \{(p, p+i) | p \in m, i \in I\}$. From [3, Theorem 2.13] and by the hypothesis conditions, we have $wdim(R \bowtie I)_M = wdim(R_m \bowtie I_m) = \infty$. Then, the desired result follows from [7, Theorem 1.3.14].

The following example shows that the condition "$I_m$ is a principal proper ideal of $R_m$ for every maximal ideal $m$ in $R$" is necessary in Theorem 2.12.

**Example 2.13.** Let $R$ be a Von Neumann regular ring and let $I$ be a proper ideal of $R$. Then $wdim(R \bowtie I) = 0$ since $(R \bowtie I)$ is a Von Neumann regular ring, and $I_m$ is not a proper ideal of $R_m$ since $R_m$ is a field.

# References

[1] D. L. Costa, Parameterising families of non-Noetherian rings. *Comm. Algebra.* **22**(1994), 3997-4011

[2] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press. Princeton (1956).

[3] M. Chhiti and N. Mahdou, *Some homological properties of an amalgamated duplication of a ring along an ideal*, Submitted for publication. Available from math .AC/0903.2240 V1 12 mar 2009

[4] M. D'Anna, A construction of Gorenstein rings. *J. Algebra.* **306** (2006), no. 2, 507-519.

[5] M. D'Anna and M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal. *Ark. Mat.* **45** (2007), no. 2, 241-252.

[6] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties. *J. Algebra Appl.* **6** (2007), no. 3, 443-459.

[7] S. Glaz, *Commutative Coherent Rings*, Springer-Verlag, Lecture Notes in Mathematics, 1371 (1989).

[8] S. Glaz, The weak Dimenssions of Gaussian rings. *Proc.Amer. Maths. Soc.* **133** (2005),2507-2513.

[9] J. A. Huckaba, *Commutative Coherent Rings with Zero Divizors*. Marcel Dekker, New York Basel, (1988).

[10] W. Heinzer, J. Huckaba and I. Papick, m-canonical ideals in integral domains. *Comm.Algebra.* **26**(1998), 3021-3043.

[11] S. Kabbaj and N. Mahdou, Trivial Extensions Defined by coherent-like condition. *Comm.Algebra.* **32** (10) (2004), 3937-3953

[12] S. Kabbaj and N. Mahdou, Trivial extensions of local rings and a conjecture of Costa. *Lecture Notes in Pure and Appl. Math.*, Vol.231, Marcel Dekker, New York, (2003), 301-312.

[13] N. Mahdou, On Costa's conjecture. *Comm. Algebra.* **29** (7) (2001), 2775-2785.

[14] N. Mahdou, On 2-Von Neumann regular rings. *Comm. Algebra.* **33** (10) (2005), 3489-3496.

[15] WM. McGovern, G. Puninski, P. Rothmaler, When every projective module is a direct sum of finitely generated modules. *J. Algebra.* **315** (2007). 454-481.

[16] H. R. Maimani and S. Yassemi, Zero-divisor graphs of amalgamated duplication of a ring along an ideal. *J. Pure Appl. Algebra.* **212** (1) (2008), 168-174.

[17] M. Nagata, *Local Rings*. Interscience, New york, (1962).

[18] G. Puninski and P. Rothmaler, When every finitely generated flat module is projective. *J. Algebra.* **277** (2004), 542-558.

[19] W. V. Vasconcelas, Finiteness in projective ideals. *J. Algebra.* **25**(1973). 269-278.