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THE COHOMOLOGY OF KOSZUL-VINBERG ALGEBRA AND RELATED TOPICS

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Abstract

This short note is devoted to relationships between the cohomology theory of Koszul-Vinberg algebras and some related topics. Besides being useful for the study of some short exact sequences of KV algebras and of short exact sequences of modules over KV algebras, the so-called KV cohomology is useful to study the deformations of these KV algebras as well as to study the Dirac reductions of Poisson manifolds. We briefly discuss these items.

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1 Introduction

The cohomolgy theory of Koszul-Vinberg algebras (KV cohomology) has been initiated by Albert Nijenhuis [26] in order to study the deformations of locally flat manifolds [17]. Recently this pioneering work has been completed [20] while the same pioneering work was rediscovered using new concepts [5],[9]. In this paper we briefly discuss a few settings where the KV cohomology plays a useful role.
Firstly we point out the role of this cohomology in classical settings such as the classification of short exact sequences of Koszul-Vinberg algebras. Thus, regarding extensions and deformations of Koszul-Vinberg algebras the role of the second KV cohomology space and that of the third KV cohomology space are similar to those of the Chevalley-Eilenberg cohomology of Lie algebras (respectively is similar to the role of Hochschild cohomology of associative algebras). Nevertheless, this analogy does not hold for the extensions of modules over Koszul-Vinberg algebras.
The complex of superorder differential forms with values in vector bundles has been initiated by Koszul [16]. We give the definition of that complex. The KV cohomology of a locally flat manifold (M,∇) helps to relate the pioneering work of Nijenhuis to the cohomology of $T^\star M$-valued superorder differential forms.
The real KV cohomology of Lagrangian foliations is also related to Poisson manifolds and to their Dirac reductions. This relationship has been pointed out by J. Stasheff in a private communication to the author.

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2 KV Cochain Complexes

In this section we give relevant formulas which lead to the definition of so-called KV cochain complex of a Koszul-Vinberg algebra. The expression *KV cohomology* always means "cohomology of Koszul-Vinberg algebras" [20].

2.1 Koszul-Vinberg algebras and their modules

The ground field is the field $\mathbb{R}$ of real numbers. All vector spaces and all algebras are vector spaces and algebras over the field $\mathbb{R}$.

Definition 2.1. An algebra is a couple $A = (V, \mu)$ consisting of a vector space V and a V-valued bilinear map $\mu : V \times V \to V$.

Definition 2.2. The Koszul-Vinberg anomaly (or KV anomaly) KV_μ of an algebra A is the following tri-linear map
$KV_\mu : V \times V \times V \to V$
$KV_\mu(a,b,c) = a(bc) - (ab)c - b(ac) + (ba)c$ where ab stands for $\mu(a,b)$.
An algebra A whose KV anomaly vanishes identically is called Koszul-Vinberg algebra (or KV algebra).

Clearly associative algebras are Koszul-Vinberg algebras. However the origin of the notion of KV algebra is differential geometry. Thus the important examples of Koszul-Vinberg algebras come from differential geometry. For instance, let D be the covariant derivative of a torsion free linear connection in a smooth manifold M.Then the vector space $\mathcal{X}(M)$ of smooth vector fields endowed with multiplication $XY = D_X Y$ is a Koszul-Vinberg algebra iff the curvature tensor R_D vanishes identically.
Another example is the vector space $\mathcal{X}(F)$ of smooth vector fields which are tangent to a lagrangian foliation $\mathcal{F}$ in a symplectic manifold (M, ω). The multiplication of $\mathcal{X}(F)$ is defined by the formula

$$\omega(XY, Z) = X\omega(Y, Z) - \omega(Y, [X, Z]).$$

Of course every associative algebra is a Koszul-Vinberg algebra. The commutator of an algebra $\mathcal{A}$ is the skew symmetric algebra whose bracket is given by $[a,b] = ab - ba\ \forall a, b \in \mathcal{A}$. The commutator algebra of a Koszul-Vinberg algebra is a Lie algebra. The question to know whether a given Lie algebra is the commutator algebra of a Koszul-Vinberg algebra is a widely open problem. If $\mathcal{K}$ is a commutative field of characteristic zero then no semi-simple Lie algebra over $\mathcal{K}$ is the commutator algebra of a Koszul-Vinberg algebra.

Definition 2.3. A two-sided module over a KV algebra consists of a vector space W and of two bilinear maps
$A \times W \to W : (a, w) \to aw$, $W \times A \to W : (w, a) \to wa$
subject to the following requirements:
$a(bw) - (ab)w - b(aw) + (ba)w = 0$ and $a(wb) - (aw)b - w(ab) + (wa)b = 0\ \forall a, b \in A, w \in W$

Given a two-sided A-module W, $J(W)$ is the subspace of $w \in W$ such that $a(bw) - (ab)w = 0\ \forall a, b \in A$.

2.2 KV complex

Let $\mathbb{Z}$ be the group of integers and let W is a two-sided module over a KV algebra A. Then $C(A,W)$ is the $\mathbb{Z}$-graded vector space whose homogeneous subspaces $C^i(A,W)$ are defined as follows: $C^i(A,W) = 0$ whenever i is a negative integer, $C^0(A,W) = J(W)$ and $C^i(A,W) = Hom(A^{\otimes i}, W)$ whenever i is a positive integer.

One has the cochain complex $(C(A,W) = \sum_i C^i(A,W), \delta)$ where the coboundary operator δ is defined as follows:
$(\delta w)(a) = -aw + wa, \quad \forall w \in J(W)$.
If i is positive and if $f \in C^i(A,W)$ then $\delta f \in C^{i+1}(A,W)$ is defined by
$\delta f(a_1 \otimes ..a_{i+1}) = \sum_{j \leq i} (-1)^j [(a_j f)(a_1 \otimes ..\hat{a_j} \otimes ..a_{i+1}) + (f(a_1 \otimes ..\hat{a_j} \otimes ..a_i \otimes a_j))a_{i+1}]$

At the right side of the formula above $a_j f$ is defined by
$(a_j f)(b_1 \otimes ..b_i) = a_j(f(b_1 \otimes ..b_i)) - \sum_{1 \leq k \leq i} f(b_1 \otimes \ldots \otimes a_j b_k \otimes ..b_i)$

The cohomology space $H(A,W) = \oplus_i H^i(A,W)$ of the cochain complex $(C(A,W), \delta)$ is called W-valued KV cohomology of the KV algebra A.

Definition 2.4. The Maurer-Cartan function of a KV algebra A is the polynomial mapping P_A of $C^2(A,A)$ in $C^3(A,A)$ given by $P_A(\nu) = \delta\nu + KV_\nu$ where KV_ν is the Koszul-Vinberg anomaly of the algebra whose multiplication is $a.b = \nu(a,b)$

Every $\nu \in C^2(A,A)$ defines an algebra structure whose multiplication μ is defined by

$$\mu(a,b) = ab + \nu(a,b).$$

The KV anomaly KV_μ of μ is related to P_A by

$$KV_\mu(a,b,c) = P_A(a,b,c)$$

Let W be a vector space, the set of KV algebra structures in W is denoted by $KV(W)$. Once for all we fix a KV algebra structure $A = (W, \mu_o) \in KV(W)$. The set of zeros of P_A is denoted by $Z_A(C^2(A,A))$. The following is useful to algebraic geometry (as well as to combinatorial geometry) of the singular algebraic variety $KV(W)$.

Proposition 2.5. *[25] The map $\mu \to \nu = \mu - \mu_o$ induces a one to one map of $KV(W)$ onto $Z_A(C^2(A,A))$.*

This proposition is a straight consequence of the relationship between the KV anomaly and Maurer-Cartan polynomial function.

2.3 Deformations and Extensions of KV algebras.

In regard to the extensions of Koszul-Vinberg algebras as well as their deformations, the new cohomology theory is an alter ego of the Chevalley-Eilenberg cohomology theory of Lie algebras (respectively of the Hochschild cohomology theory of associative algebras)[11],[27]

.

Let us endow a two-sided A-module W with the structure of trivial KV algebra

$$ww' = 0, \quad \forall w, w' \in W.$$

Then we have:

Theorem 2.6. *The set Ext(A,W) of the equivalence classes of extensions of the KV algbra A by the trivial KV algebra W is isomorphic to the cohomology space $H^2(A,W)$.*

The theorem we just stated is a classical result in the theory of Lie algebras [6],(resp in theory of associative algebras [12]) In regard to short exact sequences of algebras with the trivial kernel, the KV cohomology of Koszul-Vinberg algebras works as the Chevalley-Eilenberg cohomology of Lie algebras (resp as the Hochschild cohomology of associative algebras).

Indeed, let V and W be two modules over the same Lie algebra g. Then the vector space $L(W,V)$ of linear mappings from W to V is a g-module under the action defined by

$$af(w) = a(f(w)) - f(aw)$$

$\forall f \in L(W,V), a \in g, w \in W$. It is well known that there exists a one to one correspondence between the set $Ext_g(W,V)$ of extension classes of the g-module W by the g-module V and the Chevelley-Eilenberg cohomology space $H^1(g, L(W,V))$.

Mutatis mutandis one has the same result by replacing modules over Lie algebras and Chevalley-Eilenberg cohomology of Lie algebras by modules over associative algebras and Hochschild cohomology of associative algebras.
Regarding extensions of modules over the same Koszul-Vinberg algebra A, the relationship between short exact sequences of KV modules over A, namely

$$0 \to W \to T \to V \to 0$$

and the KV cohomology space $H^1(A, L(W,V))$ doesn't walk as one would expect. However there exists a one to one correspondence between the set of equivalence classes of those extensions and the term $E_1^{1,1}$ of a canonical spectral sequence of KV complexes, [20].

Indeed let W be a two-sided module over a KV algebra A. Let $B = A \triangleright W$ be the semi-direct product of A with W. Thus B is a KV algebra structure in $A \oplus W$ whose multiplication is given by

$$(a,w)(a',w') = (aa', aw' + wa').$$

Every two-sided module V over A is a two-sided module over B under the following actions of B

$$(a,w)v = av$$

and

$$v(a,w) = va$$

$\forall a \in A, v \in V, w \in W$.

Now one considers the KV complex $C(B,V)$ of V-valued cochains of B. Then the pair (W,B) yields an alter ego $E_r = (E_r^{p,q})$ of the classical Hochschild-Serre spectral sequence whose term $E_1^{1,1}$ is isomorphic to the set $Ext_A(W,V)$ of equivalence classes of extension of the module W by the module V.
Roughly speaking the KV complex $C(B,V)$ is endowed with the filtration

$$F^jC(B,V) = \oplus_q F^jC(B,V) \cap C^q(B,V)$$

Where $f \in F^jC(B,V) \cap C^q(B,V)$ if and only if $f(b_1,..,b_q) = 0$ whenever the subset $(b_1,..,b_q)$ contains more than $q-j$ elements of the ideal $W \subset B$.
The subspaces $F^jC(B,V)$ have the following properties.
(i) $F^{j+1}C(B,V) \subset F^jC(B,V)$,
(ii) $\delta(F^jC(B,V)) \subset F^jC(B,V)$.
Thus we get the filtered cochain complex $(C(B,V), \delta, F^jC(B,V))$ whose spectral sequence is used above.

It must be noticed that $E_1^{1,1}$ is quite different from the cohomology space $H^1(A, L(W,V))$.

3 Hyperbolicity and Completeness of Locally Flat Manifolds

Definition 3.1. A locally flat manifold is a manifold M endowed with a torsion free linear connection D whose curvature tensor vanishes identically.

Definition 3.2. A locally flat manifold (M,D) whose universal covering $(\tilde{M},\tilde{D})$ is diffeomorphic to a convex domain that does not contain any straight line is called hyperbolic locally flat manifold.

The hyperbolicity is closely related to Hessian Riemannian metric of Koszul type, (see [29]). Kozsul has proved that compact hyperbolic locally flat manifolds always admit non trivial deformations, [17]. Behind this non rigidity property there exists a non vanishing theorem of the cohomology of Koszul-Vinberg algebras. This has been performed in [20].

Let (M,∇) be a locally flat manifold whose universal covering is denoted by $(\tilde{M},\tilde{\nabla})$. We fix once for all $x_o \in M$. Up to diffeomorphism, $\tilde{M}$ is the set $[([0,1],O),(M,x_o)]$ of homotopy classes of smooth mappings of $([0,1],0)$ in (M,x_o). Set $W = T_{x_o}M$. Given a smooth mapping

$$c : ([0,1],0) \to (M,x_o)$$

let $\tau(t) : W \to T_{c(t)}M$ be the parallel transport along c. The development mapping of $c(t)$ is denoted by $Q(c)$. It is defined by

$$Q(c) = \int_0^1 \tau^{-1}(t)(\frac{dc(t)}{dt})dt$$

Since the curvature tensor of ∇ vanishes identically $Q(c)$ depends only on the homotopy class of $c(t)$. Thus one gets a map Q of $\tilde{M}$ in W.

Definition 3.3. A locally flat manifold (M,∇) is complete if the development mapping Q is a diffeomorphism onto W.

The problem to know whether a given locally flat manifold (M,∇) is complete a widely open problem.
Let $\pi(M)$ be the fundamental group of M at x_o. Given $[\gamma] \in \pi(M)$ and $[c] \in [([O,1],0),(M,x_o)]$ one has

$$Q([c\gamma]) = \tau([\gamma])Q([c]) + Q([\gamma])$$

So the mapping

$$[\gamma] \in \pi(M) \to (Q([\gamma]), \tau([\gamma]))$$

is an affine representation of the group $\pi(M)$ in the affine space W. The linear part $\tau(\pi(M)) \subset Gl(W)$ is called the linear holonomy of (M,∇). It is denoted by $h(M,\nabla)$. The conjecture of Markus is the following statement

Theorem 3.4. *Let (M,∇) be a compact connected locally flat manifold. If the linear holonomy group $h(M,\nabla)$ is unimodular then (M,∇) is complete*

This conjecture has been proved when additional particular assumptions are made. For instance the conjecture of Markus walks if the group $h(M,\nabla)$ is nilpotent [10]. It walks also if $h(M,\nabla)$ is Lorentzian [4].
Set $m = dim(M)$. When $h(M,\nabla)$ is unimodular the m^{th} real KV cohomology space of (M,∇) contains a cohomology class which is an obstruction to the completeness of (M,∇) [20].
An example of compact non complete locally flat manifold is the Hopf manifold $\Gamma\backslash R^n$ where

$$\Gamma = \left\{\lambda^j\right\}_{j\in\mathbb{Z}},$$

$\lambda > 1$ is a positive real number.

4 Relation to the pioneering work of Nijenhuis

In his pioneering attempt (with collaboration of Koszul), [26],[18] Nijenhuis is concerned with only left modules. I plan to revisit this pioneering work and to show that the cochain complex of Nijenhuis is actually a subcomplex of a cochain complex of a Koszul-Vinberg algebra.

Let W be a left module over a KV algebra A. This is a module over the Lie algebra A_L whose bracket is defined by

$$[a,b] = ab - ba$$

$\forall a,b \in A$. The vector space $L(A,W)$ of linear maps from A to W is a A_L-module.
The complex initiated by Nijenhuis is nothing but the Chevalley-Eilenberg complex $C_{ce}(A_L, L(A,W))$.

Its j^{th} cohomology space is denoted by $H^j_{ce}(A_L, L(A,W))$. We recall that Nijenhuis defines the j^{th} W-valued cohomology space $H^j_N(A,W)$ of the KV algebra A to be $H^{j-1}_{ce}(A_L, L(A,W))$. The graded vector space $C_N(A,W) = \oplus_j Hom(\wedge^j A, L(A,W))$ is preserved by the coboundary operator of the KV complex $C(A,W)$. So $C_N(A,W)$ is a KV subcomplex. Since W is a left module the j^{th} KV cohomology space $H^j C_N(A,W)$ coincides with $H^{j-1}_{ce}(A_L, Hom(A,W))$ [20].

Now let E be a vector bundle over a smooth manifold M and let $\Gamma(E)$ be the vector space of smooth sections of E. For every non negative integer p we consider the vector space $\Omega^p(M,E)$ of E-valued differential p-forms defined in M. Thus, let $\theta \in \Omega^p(M,E)$. To vector smooth fields $X_1,..,X_p$ it assigns the smooth section $\omega(X_1,..,X_p) \in \Gamma(E)$.

Definition 4.1. [16] Let k be a non negative integer. A E-valued p-form ω is of order $\leq k$ if at every point $x \in M$ the value $\omega(X_1,..,X_p)(x)$ depends on the k^{th} jets $j^k_x(X_1),..,j^k_x(X_p)$

The space of E-valued differential forms of order $\leq k$ is denoted by $\Omega_{[k]}(M,E)$. The vector space

$$\Omega_{[\infty]}(M,E) = \cup_k \Omega_{[k]}(M,E)$$

is the vector space of superorder E-valued differential forms in M.
Now, suppose E is a flat vector bundle. This means that E is equipped with a connection

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

whose curvature tensor R_∇ vanishes identically. Therefore, $\Gamma(E)$ is a module over the Lie algebra $\Gamma(TM)$. The action of $\Gamma(TM)$ in $\Gamma(E)$ is defined by the connection

$$\nabla : \ (X,s) \in \Gamma(TM) \times \Gamma(E) \mapsto \nabla_X(s) \in \Gamma(E).$$

The vector space

$$\Omega_{[\infty]}(M,E) = \oplus_p \Omega^p_{[\infty]}(M,E)$$

is a cochain complex whose coboundary operator

$$\delta : \Omega^p_{[\infty]}(M,E) \to \Omega^{p+1}_{[\infty]}(M,E)$$

is defined as follows

$$\begin{aligned}\delta\omega(X_1,..,X_{p+1}) &= \sum_j (-1)^{j+1} \nabla_{X_j}(\omega(..,\hat{X}_j,..,X_{p+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],..\hat{X}_i,..\hat{X}_j,..,X_{p+1}).\end{aligned}$$

Here are two examples of canonical superorder cohomology class [16]
(1) Take $E = \wedge^m T^\star M$ where $m = dim(M)$. Of course, the Lie derivative L_X endows E with a flat connection. Let $v \in \Gamma(E)$ be a volume form. The divergence form div(X) is defined by

$$L_X(v) = div(X)v.$$

It is of order ≤ 0.
(2) Take $E = \Omega^2(M, TM)$, the space of ordinary TM-valued differential 2-forms. Every linear connection ∇ defines a E-valued differential 1-form

$$\omega(X) = L_X \nabla.$$

Roughly speaking $L_X\nabla$ is a TM-valued 2-form defined as follows

$$\omega(X) : (Y,Z) \to [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X,Z]$$

Thus, $X \mapsto \omega(X)$ is of order ≤ 2. It is a cocycle whose (Chevalley-Eilenberg) cohomology class $[\omega] \in H^1(\Gamma(TM), \Omega^2(M,TM))$ does not vanish and doesn't depend on the choice of the linear connection ∇. The reader is referred to [16] for more details.

Actually the KV cohomology helps to connect the pioneering complex of Nijenhuis, namely $C_N(A,W)$, with the cohomology of superorder differential forms (see [16]). This relationship is studied in [23].

5 Other Relevant Relations

5.1 An alternative Cohomology Theory of the Category of Associative Algebras.

Every associative algebra A is a Koszul-Vinberg algebra and a two-sided module V over an associative algebra A is a KV module over A.
Let V be a module over an associative algebra A. The classical V-valued Hochschild complex $HC(A,V)$of A is the graded vector space $\oplus_q HC^q(A,V)$.
If q is a negative integer then $HC^q(A,V) = 0$;
$HC^0(A,V) = V$. If q is a positive integer then $HC^q(A,V) = Hom(A^{\otimes q}, V)$.
The coboundary operator
$d : HC^qC(A,V) \to HC^{q+1}(A,V)$ is defined as it follows.
Given $v \in V$ and $b \in B$ one sets

$$dv(b) = bv - vb.$$

Given $f \in HC^q(A,V)$ and $b_1 \otimes b_2 .. \otimes b_{q+1}$ one sets

$$\begin{aligned} df(b_1 \otimes b_2 .. \otimes b_{q+1}) &= b_1 f(b_2 \otimes .. \otimes b_{q+1}) + \sum_{k \leq q} q(-1)^k f(b_1 .. \otimes b_k b_{k+1} \otimes .. b_{q+1}) \\ &+ (-1)^{q+1} f(b_1 \otimes ... \otimes b_q) b_{q+1}. \end{aligned}$$

The q^{th} cohomology space of the complex above is denoted by $HH^q(A,V)$.
Now let us regard A as KV algebra and V as KV module over A. The KV complex $C_{KV}(A,V)$ differs from the Hochschild complex $HC(A,V)$. Indeed

$$C^0_{KV}(A,V) = J(V).$$

Let $q > 1$ and let $f \in C^q_{KV}(A,V)$. Then

$$\delta f(b_1, .., b_{q+1}) = \sum_{k \leq q} (-1)^k ((b_k f)(.., \hat{b_k}, .. b_{q+1}) + (f(.., \hat{b_k}, .., b_q, b_k)) b_{q+1}).$$

Now it is easy to check that for $q = 0$ and $q = 1$ one has

$$HH^q(A,V) = H^q_{KV}(A,V),$$

For $q = 2$ one has the inclusion map

$$HH^2(A,V) \subset H^2_{KV}(A,V).$$

So, beside its classical Hochschild cohomology functor $HH^\star(A,V)$ the category of associative algebras admits also the KV cohomology functor $H_{KV}(A,V)$. The two functors differ each from other, [20]. Ther arises the problem to know what are the relationships between $HH^q(A,V)$ and $H^q_{KV}(A,V)$ for $q > 2$. [Nijenhuis, private communication].

To end this subsection, we briefly discuss a canonical relationship between the real KV cohomology of a locally flat manifold (M,∇) and the de Rham cohomology of the manifold M. As before the vector space $\Gamma(TM)$ of smooth vector fields is a KV algebra whose multiplication is defined by

$$XY = \nabla_X Y.$$

Of course, the vector space $\Gamma(T^\star M)$ is nothing but the space of smooth differential forms in M. It is a left module over the KV algebra $\Gamma(TM)$. Given $X,Y \in \Gamma(TM)$ and $\theta \in \Gamma(T^\star M)$ the left action of $\Gamma(TM)$ $X\theta$ is defined by

$$(X\theta)(Y) = L_X(\theta(Y)) - \theta(\nabla_X Y).$$

We may consider the KV complex of superorder cochains $C(\Gamma(TM),\Gamma(T^\star M))$. It contains the Nijenhuis subcomplex

$$C_N(\Gamma(TM),C^\infty(M,\mathbb{R})) = \oplus_q Hom_{\mathbb{R}}(\wedge^q \Gamma(TM),\Gamma(T^\star M)).$$

Now if one restricts to cochains of order ≤ 0 the complex of Nijenhuis is nothing but the $T^\star M$-valued differential forms. In other words

$$C_N(\Gamma(TM),\Gamma(T^\star M)) = \Omega_{[0]}(M,T^\star M).$$

We already known that the graded vector space

$$\Omega_{[0]}(M,T^\star M) = \oplus_q \Omega^q_{[0]}(M,T^\star M)$$

is a KV complexe with

$$H_N^{q-1}(M,T^\star M) = H^q_{KV}(\Gamma(TM),C^\infty(M,\mathbb{R})).$$

Now denote by p the canonical projection of $\Omega^q_{[0]}(M,T^\star M)$ onto the vector space $\Omega(M,R)$ of ordinary differential q-forms. Let

$$d : \Omega^q(M,\mathbb{R}) \to \Omega^{q+1}(M,\mathbb{R})$$

be the de Rham coboundary operator and let

$$\delta : \Omega^q_{[0]}(M, T^\star) \to \Omega^{q+1}_{[0]}(M, T^\star M)$$

be the KV coboundary operator. It is easy to check that $\forall \theta \in \Omega^q_{[0]}(M, T^\star M)$

$$d(p(\theta)) = (-1)^q p(\delta(\theta)).$$

We set

$$K = \oplus_q \left\{ kerp : \Omega^q_{[0]}(M, T^\star M) \to \Omega^{q+1}_{[0]}(M, \mathbb{R}) \right\}.$$

This yields a short exact sequence of cochain complexes

$$0 \to K^q \to \Omega^q_{[0]}(M, T^\star M) \to \Omega^{q+1}(M, \mathbb{R}) \to 0$$

So, one get a long exact cohomology sequence connecting cohomology of the KV complex $(\Omega_{[0]}(M, T^\star M), \delta)$ to de Rham cohomology of the manifold M

$$\to H^q(K) \to H^q_{KV}(\Gamma(TM), C^\infty(M, R)) \to H^q_{dR}(M, \mathbb{R}) \to H^{q+1}(K) \to$$

We recall that

$$H^{q-1}_N(\Omega_{[0]}(M, T^\star M)) = H^q_{KV}(\Gamma(TM), C^\infty(M, \mathbb{R}))$$

and

$$H^{q-1}_N(\Omega_{[0]}(M, T^\star M)) = H^{q-1}_{CE}(\Gamma(TM), \Omega^1_{[0]}(M, \mathbb{R})).$$

5.2 KV Cohomology, Poisson Structures and their Dirac Reductions.

Let A be an associative commutative algebra. Then the skew symmetric part Π_θ of every two dimensional cocycle θ of the Hochschild complex $HC^\star(A, A)$ is a quasi-Poisson cocycle in the following sense.
(i) For every $a \in A$ the linear map Π_a defined by $\Pi_a(b) = \Pi(a, b)$ is a derivation of the associative algebra A.
Of course the bilinear mapping $\theta \in C^2(A, A)$ defines an algebra structure in the vector space $|A|$. Suppose that the Koszul-Vinberg anomaly KV_θ of the algebra $(|A|, \theta)$ vanishes identically. Then,
(ii) given $a, b, c \in A$ one has

$$a\Pi_\theta(b, c) - \Pi_\theta(ab, c) + \Pi_\theta(a, bc) - \Pi_\theta(a, b)c = 0$$

An important example of Koszul-Vinberg algebra is

$$g = A_L \oplus C^\infty(M)$$

where A_L is the space of vector fields which are tangent to a lagrangian foliation L in a symplectic manifold (M, ω). The $C^\infty(M)$-valued KV cohomology of g generates Poisson tensors which preserve the space of first integrals of L. This yields Dirac reductions along the leaves of L, (see [3]).
Consequently every smooth (resp. holomorphic) Poisson structure is the Dirac reduction along the fibers of the smooth (resp. holomorphic) cotangent fiber bundle [22].

Consider the associative algebra $C^\infty(M)$ of real valued smooth functions in manifolds M. Then every two dimensional cohomology class
$[\theta] \in HH^2(C^\infty(M), C^\infty(M))$ contains one and only one quasi-Poisson cocycle [15]. This result has its analogue in the KV cohomolgy space $H_{KV}(C^\infty(M, \mathbb{R}), C^\infty(M, \mathbb{R}))$ [22].

5.3 Misceleneous

The KV cohomology may be used to study the coexistence of some geometrical data in the same manifold. For instance one can use the KV cohomology to prove the following statements.
(1) The only compact symplectic solvmanifolds $\Gamma\backslash G$ (with G completely solvable) that admit kahlerian metrics are the flat tori $\mathbb{Z}^n\backslash\mathbb{R}^n$.
Concerning nilmanifolds, this statement has been proved by C. Benson and C. Gordon [1] and by Dusa McDuff [8].
(2) Suppose a Kahlerian manifold (M,ω,J) is homogeneous under the action of a completely solvable Lie group. Then M is the total space of a fiber bundle over a bounded domain whose fibers are simply connected homogeneous kahlerian manifolds, [13] See also [14], [7],[24].

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