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MATRIX MODELS, LARGE N LIMITS AND NONCOMMUTATIVE SOLITONS*

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Abstract. A survey of the interrelationships between matrix models and field theories on the noncommutative torus is presented. The discretization of noncommutative gauge theory by twisted reduced models is described along with a rigorous definition of the large N continuum limit. The regularization of arbitrary noncommutative field theories by means of matrix quantum mechanics and its connection to noncommutative solitons is also discussed.

1. Introduction

Two of the most novel aspects of noncommutative field theories (see [7, 15] and [22] for reviews) which are not seen in ordinary quantum field theories are the properties that (a) They can be regularized and analysed by means of matrix models, and (b) In some instances they admit novel soliton solutions with no commutative counterparts. Property (a) stems from the fact that noncommutative fields are most conveniently understood and analysed as operators acting on separable Hilbert spaces. Property (b) instead is due to the fact that noncommutative field theories behave in many respects like string theory, rather than conventional quantum field theory, and noncommutative solitons correspond to *D-branes* in open string field theory of tachyon dynamics (see [13] for a review). In this survey we will describe some aspects of the interrelationship between matrix models, noncommutative solitons and field theory on the noncommutative torus. The first half of the article deals with the finite-dimensional regularization of Yang-Mills theory on a two-dimensional noncommutative torus [1] and the precise definition of the large N limit of this matrix model [16]. The second half demonstrates that the two novel properties (a) and (b) above are in fact intimately related through a regularization of noncommutative field theory by means of matrix quantum mechanics

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which has a much simpler and tractable large N limit than its zero-dimensional counterpart [17].

2. Matrix Models and Gauge Theory on the Noncommutative Torus

We begin with an account of how the large N limit of a particular matrix model naturally leads to considerations of gauge theory on a noncommutative torus. This is the classic Connes-Douglas-Schwarz formalism [6] which was the original link between open string theory in background fields and noncommutative geometry through compactifications of Matrix Theory.

2.1. The IKKT Matrix Model

Consider the statistical mechanics of $d \geq 2$ complex $N \times N$ matrices $X_i \in \mathbb{M}_N(\mathbb{C})$, $i = 1, \ldots, d$ which is defined by the integral

$$Z = \int\limits_{\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}^d} \mathrm{d}X \ \mathrm{e}^{-S(X)} \tag{1}$$

where $\mathrm{d}X_i$ for each $i = 1, \ldots, d$ is the translationally invariant Haar measure on the Lie algebra $\mathbb{M}_N(\mathbb{C})$ and the action

$$S(X) = -\frac{1}{g^2} \sum_{i,j=1}^{d} \mathrm{Tr}\big[X_i \, , \, X_j\big]^2 \tag{2}$$

is a holomorphic function on the space $\mathbb{C}^d \times \mathbb{M}_N(\mathbb{C})$. The symbol Tr is the usual matrix trace and $g^2 > 0$ is a coupling constant. Note that the Lie algebra $\mathbb{M}_N(\mathbb{C})$ is also an associative $*$-algebra.

The zero-dimensional matrix model (1) is simply the dimensional reduction to a point of Yang-Mills gauge theory on $\mathbb{R}^d$ with structure group $\mathrm{GL}(N, \mathbb{C})$, i.e., we restrict the usual Yang-Mills action functional to *constant* gauge fields. It possesses two fundamental symmetries. Regard $X \in \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}^d$ with components $X_1, \ldots, X_d$ in a fixed basis of the vector space $\mathbb{C}^d$. Then the function (2) is invariant under the action of the complex orthogonal group $\mathrm{SO}(d, \mathbb{C})$ under which X transforms as a vector. This symmetry is just Lorentz invariance. The action is also invariant under the transformations

$$X_i \ \longmapsto \ U \, X_i \, U^{-1} \ , \qquad U \in \mathrm{GL}(N, \mathbb{C}) \ . \tag{3}$$

This symmetry is gauge invariance. We will restrict the integral (1) over a real slice of the space $\mathbb{C}^d \times \mathbb{M}_N(\mathbb{C})$ by truncating to hermitian $N \times N$ matrices $X_i \in \mathfrak{u}(N)$. Then the hermitian matrix integral (1) defines the IKKT *matrix model* [14]. The gauge symmetry (3) in this instance restricts to unitary matrices $U \in \mathrm{U}(N)$.

This matrix model has many interesting applications which all require some formal $N \to \infty$ limit to be taken. Firstly, identify $\mathbb{M}_\infty(\mathbb{C})$ as the $*$-algebra of operators of the form $\mathcal{D} + A$ with $\mathcal{D}$ a first order differential operator with constant coefficients and A an operator of multiplication on a function which decreases rapidly at infinity in $\mathbb{R}^d$. The matrix model then reduces to ordinary gauge theory on $\mathbb{R}^d$. Secondly, let $\mathrm{C}^\infty(\Sigma)$ be the algebra of smooth complex-valued functions on a compact Riemann surface Σ equipped with a symplectic structure $\omega \in \mathcal{S}(\Sigma)$. Then one can construct a sequence of "quantization" maps $\sigma_N : \mathrm{C}^\infty(\Sigma) \to \mathbb{M}_N(\mathbb{C})$ such that $N\,[\sigma_N(f), \sigma_N(g)] \to \sigma_N(\{f,g\}_\omega)$ and $N \,\mathrm{Tr}\,\sigma_N(f) \to \int\limits_\Sigma \omega\, f$ in the limit $N \to \infty$. Let $X : \Sigma \to \mathbb{R}^d$ be an embedding of the surface in flat space. Then the extrema of the action $S(X) - \lambda\,\mathrm{vol}_\omega(\Sigma)$ regarded as a functional on $\mathrm{C}^\infty(\Sigma) \times \mathbb{C}^d \times \mathcal{S}(\Sigma)$ coincide with those of the Nambu-Goto action which computes the area of the embedded surface Σ. This connects the IKKT matrix model to the Green-Schwarz string [10]. Finally, the action (2) itself has a general set of classical vacuum states provided by configurations X_i with $[X_i, X_j] = \mathrm{i}\, B_{ij}\, \mathbb{1}$, where $B_{ij} = -B_{ji}$ are real constants. For $B_{ij} \neq 0$ these equations only have a solution when $N \to \infty$ whereby $\mathbb{M}_\infty(\mathbb{C})$ is regarded as the C^*-algebra $\mathcal{K}$ of compact operators acting on a separable Hilbert space.

These realizations all form part of the circumstantial evidence which has led to the conjecture [14] that the large N limit of the matrix model (1) provides a nonperturbative definition of Type IIB string theory. The vacua described above correspond to position coordinates of D-branes and these considerations immediately lead to noncommutative spacetime geometries [3, 24]. The rigorous definition of the large N limit here along with the precise meaning of convergence will be described later on.

2.2. Toroidal Compactification

We will now make the heuristic appearance of noncommutative geometry described above more precise [6]. Let us *compactify* the space $\mathbb{R}^d$ along the (12)-plane to $\mathbb{T}^2 \times \mathbb{R}^{d-2}$, where $\mathbb{T}^2$ is a square two-torus of sides R_1 and R_2. Since we interpret the $N \times N$ hermitian matrices X_i above as "coordinates" in $\mathbb{R}^d$, we would like to define the toroidal compactification of the IKKT matrix model.

This means that we should define a restriction of the action (2) to a subspace of $\mathfrak{u}(N)\otimes\mathbb{R}^d$ where an equivalence relation $X_i \sim X_i + 2\pi R_i \mathbb{1}$, $i = 1,2$ is satisfied, i.e., $S(X_i + 2\pi R_i \mathbb{1}) = S(X_i)$ for $i = 1,2$. Using the gauge symmetry (3), we define this equivalence relation as unitary gauge equivalence and hence consider the *quotient conditions*

$$\begin{aligned} X_i + 2\pi R_i \mathbb{1} &= U_i X_i U_i^{-1}, \qquad i = 1,2 \\ X_j &= U_i\, X_j\, U_i^{-1}, \quad \text{for all } j \neq i, \qquad j = 1,\dots,d, \quad i = 1,2 \end{aligned} \tag{4}$$

where U_1 and U_2 are unitary, $U_i^{-1} = U_i^\dagger$.

Taking the trace on both sides of the first equation in (2.2) shows that these conditions cannot be satisfied by finite-dimensional matrices unless $R_1 = R_2 = 0$. Thus we take the formal large N limit again and search for solutions to these equations in terms of operators X_j, U_i on a separable Hilbert space $\mathcal{H}$. Consistency of the quotient conditions requires that the object $U_1\, U_2\, U_1^{-1}\, U_2^{-1}$ commutes with *all* X_j, $j = 1,\dots,d$. A natural choice is then to set it equal to a scalar operator on $\mathcal{H}$, $U_1\, U_2\, U_1^{-1}\, U_2^{-1} = \lambda\, \mathbb{1}$. Unitarity restricts $\lambda = \mathrm{e}^{\mathrm{i}2\pi\theta}$ for some $\theta \in \mathbb{R}$. We thereby find that the unitary operators U_1 and U_2 obey the relation

$$U_1\, U_2 = \mathrm{e}^{\mathrm{i}2\pi\theta}\, U_2\, U_1 \ . \tag{5}$$

With the relation (5), U_1 and U_2 generate a noncommutative, associative unital $*$-algebra $\mathcal{A}_\theta$ with trace called the **noncommutative torus** [5]. A typical "smooth" element $f \in \mathcal{A}_\theta$ is of the form

$$f = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} f_{\boldsymbol{n}}\, U_1^{n_1}\, U_2^{n_2} \tag{6}$$

where $\{f_{\boldsymbol{n}}\} \in \mathcal{S}(\mathbb{Z}^2)$ is a Schwartz sequence. The trace on $\mathcal{A}_\theta$ is then defined by

$$\mathrm{Tr}(f) := \fint f = f_{\boldsymbol{0}} \ . \tag{7}$$

2.3. Solution Space

To determine the structure of the solutions X_i to the quotient conditions (2.2), let us momentarily set $\theta = 0$. In this case we can take $\mathcal{H} = \mathrm{L}^2(\mathbb{T}^2, E)$ to be the Hilbert space of square-integrable sections of a hermitian vector bundle $E \to \mathbb{T}^2$. The operators U_i, $i = 1,2$ may be represented on $\mathcal{H}$ as pointwise multiplication by the functions $\mathrm{e}^{\mathrm{i}\,\sigma_i}$ where $\sigma_i \in [0, 2\pi)$ are angular coordinates on the torus. Thus the U_i generate (via Fourier expansion) the commutative algebra of functions f

on a two-torus dual to $\mathbb{T}^2$. The first equation in (2.2) is then simply the Leibnitz rule for a connection on the bundle E given by $X_i = \mathrm{i}\,\nabla_i + A_i$, $i = 1,2$ and $X_j = A_j$ for $j > 2$, where ∇_1, ∇_2 specify a constant curvature connection and $A_i \in \mathrm{C}^\infty(\mathbb{T}^2, \mathrm{End}(E))$. This solution is unique up to unitary equivalence.

For $\theta \neq 0$, the quotient conditions imply that X_1, X_2 are connections on a *module* E over the algebra $\mathcal{A}_\theta$, while $X_j \in \mathrm{End}_{\mathcal{A}_\theta}(E)$ for all $j \neq 1,2$. To describe these solutions explicitly, fix $q \in \mathbb{Z}$ and $p \in \mathbb{Z}/q\,\mathbb{Z}$ with p, q coprime. Set

$$E = E_{p,q} := \mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^q \tag{8}$$

and let ∇_i, $i = 1,2$ be connections on $E_{p,q}$ of constant curvature

$$\big[\nabla_1\,,\,\nabla_2\big] = \frac{\mathrm{i}2\pi}{p - q\,\theta}\,\mathbb{1}\ . \tag{9}$$

The separable Hilbert space $\mathrm{L}^2(\mathbb{R})$ is the Schrödinger representation of the Heisenberg algebra (9), which by the Stone-von Neumann theorem is the unique irreducible module. The finite-dimensional Hilbert space $\mathbb{C}^q$ is the irreducible $q \times q$ representation of the Weyl algebra

$$\Gamma_1\,\Gamma_2 = \mathrm{e}^{\mathrm{i}2\pi p/q}\,\Gamma_2\,\Gamma_1 \tag{10}$$

which is uniquely solved (up to unitary equivalence) by $\mathrm{SU}(q)$ clock and shift matrices Γ_1 and Γ_2. Acting on (8), we then take

$$U_i = \exp\!\big(\tfrac{\mathrm{i}2\pi}{q}\,(p - q\,\theta)\,\nabla_i\big) \otimes \Gamma_i\ , \quad i = 1,2\ . \tag{11}$$

This construction stems from the property that projective modules over the algebra $\mathcal{A}_\theta$ are classified by the K-theory group $\mathrm{K}_0(\mathcal{A}_\theta) = \mathbb{Z} \oplus \mathbb{Z}$. If θ is an irrational number, then the trace (7) determines an isomorphism $\mathrm{Tr} : \mathrm{K}_0(\mathcal{A}_\theta) \to \mathbb{Z} + \mathbb{Z}\,\theta$ as ordered subgroups of $\mathbb{R}$ [5,17]. Stable modules are classified by the positive cone $\mathrm{K}_0^+(\mathcal{A}_\theta)$ defined by positivity of the Murray-von Neumann dimension

$$\dim(E_{p,q}) = \mathrm{Tr}_E(\mathsf{P}_{p,q}) = p - q\,\theta > 0 \tag{12}$$

where $\mathsf{P}_{p,q}$ is a hermitian projector ($\mathsf{P}_{p,q}^2 = \mathsf{P}_{p,q} = \mathsf{P}_{p,q}^\dagger$) such that $E_{p,q} = \mathsf{P}_{p,q}\mathcal{A}_\theta^N$ for some N. The finitely-generated projective module $E_{p,q}$ is called a **Heisenberg module**.

2.4. Noncommutative Gauge Theory

From the considerations above, we have thus shown that the general solutions of the quotient conditions (2.2) on a Heisenberg module $E = E_{p,q}$ are given by

$$\begin{aligned} X_i &= \mathrm{i}\,\nabla_i + A_i , \qquad i = 1,2 \\ X_j &= A_j , \qquad j > 2 \end{aligned} \tag{13}$$

where $A_i \in \mathrm{End}_{\mathcal{A}_\theta}(E_{p,q})$ are elements of the commutant of $\mathcal{A}_\theta$ in $E_{p,q}$, i.e., the set of operators on $E_{p,q}$ which commute with the irreducible representation (11). A straightforward computation shows that any such element admits an expansion

$$A_i = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} A_i(\boldsymbol{n})\; Z_1^{n_1}\, Z_2^{n_2} \tag{14}$$

where the endomorphisms Z_1, Z_2 are defined by

$$\begin{aligned} (Z_1 f_k)(s) &= \mathrm{e}^{\mathrm{i}2\pi s/q}\, f_{k-1}(s) \\ (Z_2 f_k)(s) &= \mathrm{e}^{\mathrm{i}2\pi k\, a/q}\, f_k\big(s + \tfrac{1}{p-q\,\theta}\big) \end{aligned} \tag{15}$$

for $(f_1, \dots, f_q) \in E_{p,q} = \mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^q$. We have chosen $a, b \in \mathbb{Z}$ to satisfy the first order Diophantine equation

$$a\,p + b\,q = 1\ . \tag{16}$$

One easily computes that Z_1, Z_2 generate another noncommutative torus $\mathcal{A}_{\theta'} \cong \mathrm{End}_{\mathcal{A}_\theta}(E_{p,q})$ since

$$Z_1\, Z_2 = \mathrm{e}^{-\mathrm{i}2\pi\theta'}\, Z_2\, Z_1 \tag{17}$$

where

$$\theta' = \frac{a\,\theta + b}{p - q\,\theta} \tag{18}$$

lies in the $\mathrm{SL}(2,\mathbb{Z})$ Möbius orbit of $\theta \in \mathbb{R}$. This means that $\mathcal{A}_{\theta'}$ is *Morita equivalent* to the algebra $\mathcal{A}_\theta$, with $E_{p,q}$ the equivalence bimodule. Finally, setting the $j > 2$ components to 0 we find by substituting (14) into the matrix model action (2) the functional

$$S(X) = \mathrm{YM}(A) := -\tfrac{1}{g^2}\, \mathrm{Tr}_E \big[\mathrm{i}\,\nabla_1 + A_1\, ,\, \mathrm{i}\,\nabla_2 + A_2\big]^2\ . \tag{19}$$

The noncommutative field theory defined by this action is just Yang-Mills theory on the Heisenberg module $E = E_{p,q}$.

3. Matrix Models and Gauge Theory on the Fuzzy Torus

In the previous Section we began with a matrix model given by a perfectly well-defined integral over the finite-dimensional space $\mathfrak{u}(N) \otimes \mathbb{R}^d$. However, for the toroidal compactification of the model it was necessary to pass to a formal limit whereby the matrix rank $N \to \infty$ and rewrite objects in terms of (compact) operators on a separable Hilbert space. We would now like to understand the origin of this large N limit better and to make it more rigorous. We will examine how and to what extent the noncommutative torus algebra $\mathcal{A}_\theta$ admits representations in terms of finite-dimensional matrix algebras. This will unveil precisely how a non-perturbative regularization of noncommutative gauge theory can be obtained and how the large N continuum limit which removes the regulator N must be taken.

3.1. The Eguchi-Kawai Model

Let us begin by sketching the basic idea behind the finite-dimensional approximations that we shall construct. If $\theta = \frac{M}{N}$ is a rational number with $M, N \in \mathbb{N}$ coprime, then there exists a surjective algebra $*$-morphism $\pi : \mathcal{A}_{M/N} \to \mathbb{M}_N(\mathbb{C})$ defined on generators by

$$\pi(U_i) = \Gamma_i, \qquad i = 1, 2 \tag{20}$$

where Γ_1 and Γ_2 obey the Weyl algebra $\Gamma_1\,\Gamma_2 = \mathrm{e}^{\mathrm{i}2\pi M/N}\,\Gamma_2\,\Gamma_1$, and thus generate the finite-dimensional matrix algebra $\mathbb{M}_N(\mathbb{C})$. In this context the associative $*$-algebra generated by the Weyl algebra is sometimes called the **fuzzy torus**. Since any irrational number θ can be written as the limit of a sequence of rational numbers, we can anticipate that some sort of limiting procedure also works at the level of the corresponding algebras. This issue will be addressed in the next section.

A compact version of the IKKT matrix model (1,2) can be defined by exponentiating everything from the Lie algebra to the Lie group. Hence we define the matrix integral

$$\mathrm{Z} = \int\limits_{\mathrm{U}(N)\otimes(\mathbb{S}^1)^d} \mathrm{d}U\ \mathrm{e}^{-\mathrm{S}(U)} \tag{21}$$

where $\mathrm{d}U_i$ for each $i = 1, \ldots, d$ is the left-right invariant Haar measure on the

$N \times N$ unitary group $\mathrm{U}(N)$ and

$$\mathrm{S}(U) = -\frac{1}{g^2} \sum_{1 \le i \ne j \le d} \mathrm{Tr}\big(U_i\, U_j\, U_i^\dagger\, U_j^\dagger\big) \ . \tag{22}$$

This unitary matrix model is called the *Eguchi-Kawai model* [8]. For "infinitesimal" values $U_i = 1\!\!1 + \mathrm{i}\, X_i$, $X_i \in \mathfrak{u}(N)$ it reduces to the IKKT matrix model. It is the reduction of $\mathrm{U}(N)$ Wilson lattice gauge theory on $\mathbb{Z}^d$ to a single plaquette. The matrix model possesses the gauge symmetry $U_i \mapsto \Omega\, U_i\, \Omega^{-1}$ with $\Omega \in \mathrm{U}(N)$.

3.2. Compact Quotient Conditions

In the Eguchi-Kawai model there is a perfectly well-defined finite-dimensional version of the quotient conditions for toroidal compactification which are obtained via exponentiation of the constraints (2.2) [1]. They are given by

$$\begin{aligned} &\Omega_i U_i \Omega_i^{-1} = \mathrm{e}^{\mathrm{i}2\pi r_i/N}\, U_i, \qquad r_i \in \mathbb{Z}, \qquad i = 1,2 \\ &\Omega_i U_j \Omega_i^{-1} = U_j, \quad \text{for all} \quad j \ne i, \qquad j = 1,\dots,d, \qquad i = 1,2 \end{aligned} \tag{23}$$

where $\Omega_1, \Omega_2 \in \mathrm{U}(N)$. Taking the trace of both sides of the first equation in (23) now only requires that $\mathrm{Tr}(U_i) = 0$ for $i = 1,2$. This truncates the matrix integral (21) to traceless unitary matrices, but the equations are still consistent for finite-dimensional matrices. Similarly to the non-compact case, the consistency condition generated by the quotient conditions (23) can be chosen to be given by $\Omega_1\, \Omega_2 = \mathrm{e}^{\mathrm{i}2\pi l/N}\, \Omega_2\, \Omega_1$ for some $l \in \mathbb{N}$.

We can solve these quotient conditions by introducing discrete versions of the gauge connections described before acting on finite-dimensional modules over the matrix algebra $\mathbb{M}_N(\mathbb{C})$. Let $N = M\, q$, $M = m\, n\, q$, and represent the noncommutative torus algebra $\mathcal{A}_\theta$ in $\mathbb{M}_M(\mathbb{C}) \otimes \mathbb{M}_q(\mathbb{C}) \cong \mathbb{M}_N(\mathbb{C})$ through

$$\Omega_1 = \big(\Gamma_2\big)^m \otimes \big(\tilde\Gamma_1^\dagger\big)^p \ , \qquad \Omega_2 = \big(\Gamma_1\big)^m \otimes \tilde\Gamma_2^\dagger \tag{24}$$

where $\Gamma_1\, \Gamma_2 = \mathrm{e}^{\mathrm{i}2\pi/M}\, \Gamma_2\, \Gamma_1$ and $\tilde\Gamma_1\, \tilde\Gamma_2 = \mathrm{e}^{\mathrm{i}2\pi/q}\, \tilde\Gamma_2\, \tilde\Gamma_1$. Then one has $\Omega_1\, \Omega_2 = \mathrm{e}^{\mathrm{i}2\pi\theta}\, \Omega_2\, \Omega_1$ with

$$\theta = \frac{p}{q} - \frac{m}{n\, q} \ . \tag{25}$$

The commutant of $\mathcal{A}_\theta$ in $\mathbb{C}^M \otimes \mathbb{C}^q \cong \mathbb{C}^N$ is easily seen to be generated by the matrices [1]

$$Z_1 = \big(\Gamma_2\big)^n \otimes \tilde\Gamma_1^\dagger \ , \qquad Z_2 = \big(\Gamma_1^\dagger\big)^n \otimes \big(\tilde\Gamma_2\big)^a \tag{26}$$

which obey $Z_1\, Z_2 = \mathrm{e}^{\mathrm{i}2\pi\theta'}\, Z_2\, Z_1$ and thereby generate another noncommutative torus representation of $\mathcal{A}_{\theta'}$ in $\mathbb{M}_N(\mathbb{C})$ with

$$\theta' = \frac{n}{m\,q} - \frac{a}{q} = \frac{a\,\theta + b}{p - q\,\theta}\ . \tag{27}$$

3.3. Discrete Noncommutative Gauge Theory

We can introduce a fixed discrete gauge "connection" on the module $\mathbb{C}^N$ by setting $r_1 = r_2 = m\,q$ and defining

$$D_1 = \Gamma_1^\dagger \otimes 1\!\!1_q\ , \qquad D_2 = \Gamma_2 \otimes 1\!\!1_q\ . \tag{28}$$

It has constant curvature given by

$$D_1 D_2 = \exp\left(\tfrac{\mathrm{i}2\pi q}{p - q\theta} r_1 r_2\right) D_2 D_1. \tag{29}$$

Setting $U_j = 1\!\!1_N$ for $j > 2$, the most general solutions to the compact quotient conditions (23) are then given by

$$U_i = \tilde{U}_i\, D_i, \qquad i = 1, 2 \tag{30}$$

where $\tilde{U}_i \in \mathcal{A}_{\theta'}$ are elements of the commutant of $\mathcal{A}_\theta$, i.e., $\Omega_j\, \tilde{U}_i\, \Omega_j^\dagger = \tilde{U}_i$ for $i, j = 1, 2$. Substituting (30) into the matrix model action (22) thereby leads to a discrete version of Yang-Mills gauge theory with action

$$\mathrm{S}\big(\tilde{U}\, D\big) = \mathrm{W}\big(\tilde{U}\big) := -\tfrac{2}{g^2}\, \Re\, \mathrm{Tr}\left[\mathrm{e}^{-\mathrm{i}2\pi/M}\ \tilde{U}_1 \big(D_1\, \tilde{U}_2\, D_1^\dagger\big)\big(D_2\, \tilde{U}_1^\dagger\, D_2^\dagger\big)\tilde{U}_2^\dagger\right]. \tag{31}$$

The exact solution of this matrix model for $n = 1$ is given in [19].

A basis for the solution space $\mathcal{A}_{\theta'}$ is provided by the matrices

$$J_{\boldsymbol{m}} = (Z_2)^{m_1}\, (Z_1)^{m_2}\, \mathrm{e}^{\mathrm{i}\pi\theta'\, m_1\, m_2} \tag{32}$$

with $\boldsymbol{m} \in (\mathbb{Z}/m\,q\,\mathbb{Z})^2$. In terms of this basis we may expand the discrete gauge fields as

$$\tilde{U}_i = \frac{1}{(m\,q)^2}\, \sum_{\boldsymbol{x}} \mathcal{U}_i(\boldsymbol{x})\, \sum_{\boldsymbol{m}} J_{\boldsymbol{m}}\, \mathrm{e}^{-\frac{\mathrm{i}2\pi}{m\,q}\, \boldsymbol{m}\wedge\boldsymbol{x}} \tag{33}$$

where $\boldsymbol{x} = (x_1, x_2)$ with $x_i = 0, 1, \dots, m\,q - 1$. This maps the discrete gauge theory defined by the action (31) onto a noncommutative version of Wilson lattice gauge theory on $\mathbb{T}^2 \cap \mathbb{Z}^2$ [1,2].

4. Large N Limit

We will now describe precisely the large N limit of the finite-dimensional approximation of Section 3 which leads back to the original noncommutative gauge theory constructed in Section 2. We will focus on how to do this at a purely algebraic level, and then discuss some of the important topological consequences of the construction.

4.1. AF-Algebras

The idea behind the construction of this section is to take the "large N limit" by embedding the noncommutative torus algebra $\mathcal{A}_\theta$ into an *approximately finite-dimensional* (AF) algebra

$$A_\infty = \lim_{\substack{\longrightarrow \\ n\in\mathbb{N}_0}} A_n \tag{34}$$

defined as the norm closure of the inductive limit of an inductive system

$$A_0 \xrightarrow{\rho_1} A_1 \xrightarrow{\rho_2} A_2 \xrightarrow{\rho_3} \cdots A_n \xrightarrow{\rho_{n+1}} \cdots \tag{35}$$

where each A_n is a finite-dimensional C^*-algebra with the usual operator norm $\| - \|_{A_n}$, and ρ_n are injective $*$-morphisms. The inductive limit (34) is a C^*-algebra with norm given by

$$\big\|(f_n)_{n\in\mathbb{N}_0}\big\|_{A_\infty} := \lim_{n\to\infty} \big\|f_n\big\|_{A_n} \tag{36}$$

with $f_n \in A_n$. We can use the embeddings (35) to identify A_n with a subalgebra of A_{n+1} as

$$A_n = \bigoplus_{j=1}^{l_n} \mathbb{M}_{d_j^{(n)}}(\mathbb{C}) \cong \bigoplus_{k=1}^{l_{n+1}} \bigoplus_{j=1}^{l_n} \mathbb{M}_{d_j^{(n)}}(\mathbb{C}) \otimes \mathbb{C}^{N_{kj}} \tag{37}$$

with $\bigoplus_{j=1}^{l_n} \mathbb{M}_{d_j^{(n)}}(\mathbb{C}) \otimes \mathbb{C}^{N_{kj}} \subset \mathbb{M}_{d_k^{(n+1)}}(\mathbb{C})$. The non-negative integers N_{kj} must obey the consistency conditions

$$\sum_{j=1}^{l_n} N_{kj}\, d_j^{(n)} = d_k^{(n+1)} \tag{38}$$

which define a collection of partial embeddings $d_j^{(n)} \searrow^{N_{kj}} d_k^{(n+1)}$.

The important point is that the algebra $\mathcal{A}_\theta$ is *not* an AF-algebra. This can be seen, for instance, at the level of K-theory. The degree 1 K-theory group of any finite-dimensional algebra is always trivial, and since K-theory is covariant under inductive limits one has $\mathrm{K}_1(A_\infty) = 0$. On the other hand, the noncommutative torus has non-trivial K-theory group $\mathrm{K}_1(\mathcal{A}_\theta) = \mathbb{Z} \oplus \mathbb{Z}$, with generators the unitary equivalence classes of U_1 and U_2. Thus, at the level of zero-dimensional matrix models, the large N limit can at best be defined by an embedding $\mathcal{A}_\theta \hookrightarrow A_\infty$. This has the consequence of making the large N limit rather complex, a property also witnessed of the complicated double-scaling continuum limit of the noncommutative lattice gauge theory of the previous section [1,2]. We will see a cleaner way to treat the large N limit later on via one-dimensional matrix models. An approximation of the noncommutative torus by fuzzy tori with respect to the quantum Gromov-Hausdorff metric has been recently constructed in [18].

4.2. Rational Approximation

To realize this construction explicitly [16, 20], we expand the irrational number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ into simple continued fractions as [12]

$$\theta = \lim_{n \to \infty} \theta_n \tag{39}$$

with the n-th convergent of the expansion given by

$$\theta_n = \frac{p_n}{q_n} := c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \; c_{n-1} + \cfrac{1}{c_n}}}} \tag{40}$$

where $c_k \in \mathbb{N}$ for $k \geq 1$, $c_0 \in \mathbb{Z}$, and the coprime integers p_n, q_n can be computed from the recursion relations

$$\begin{aligned} p_0 &= c_0, & p_1 &= c_0 c_1 + 1, & p_n &= c_n p_{n-1} + p_{n-2} \\ q_0 &= 1, & q_1 &= c_1, & q_n &= c_n q_{n-1} + q_{n-2} \end{aligned} \tag{41}$$

for $n \geq 2$. It follows from these relations that the positive sequences $\{q_n\}$ and $\{|p_n|\}$ are increasing with $q_n, |p_n| \to \infty$ as $n \to \infty$. The desired finite-dimensional algebra A_n at level $n \in \mathbb{N}_0$ is then given by

$$A_n := \mathbb{M}_{q_n}(\mathbb{C}) \oplus \mathbb{M}_{q_{n-1}}(\mathbb{C}) \tag{42}$$

with the embeddings $\rho_n : A_{n-1} \hookrightarrow A_n$ defined by

$$\boldsymbol{M} \oplus \boldsymbol{N} \stackrel{\rho_n}{\longmapsto} \big(\boldsymbol{M}^{\oplus c_n} \oplus \boldsymbol{N}\big) \oplus \boldsymbol{M} \tag{43}$$

for $\boldsymbol{M} \in \mathbb{M}_{q_{n-1}}(\mathbb{C})$ and $\boldsymbol{N} \in \mathbb{M}_{q_{n-2}}(\mathbb{C})$. At each finite level n, let $U_1^{(n)}, U_2^{(n)}$ be the generators of the noncommutative torus algebra $\mathcal{A}_{\theta_n}$ obeying

$$U_1^{(n)}\,U_2^{(n)} = \mathrm{e}^{\mathrm{i}2\pi\theta_n}\,U_2^{(n)}\,U_1^{(n)}\ . \tag{44}$$

4.3. Matrix Approximation

We can finally derive the matrix approximation to the algebra $\mathcal{A}_\theta$ which rigorously accomplishes the desired large N limit of the unitary matrix model. As mentioned in the previous section, for each n there is a surjective algebra homomorphism $\pi_n : \mathcal{A}_{\theta_n} \to \mathbb{M}_{q_n}(\mathbb{C})$ given by

$$\pi_n\big(U_i^{(n)}\big) = \Gamma_i^{(n)}, \qquad i = 1,2 \tag{45}$$

with $\Gamma_1^{(n)}, \Gamma_2^{(n)}$ the $q_n \times q_n$ clock and cyclic shift matrix generators of $\mathbb{M}_{q_n}(\mathbb{C})$ which obey the Weyl algebra

$$\Gamma_1^{(n)}\,\Gamma_2^{(n)} = \mathrm{e}^{\mathrm{i}2\pi p_n/q_n}\,\Gamma_2^{(n)}\,\Gamma_1^{(n)}\ . \tag{46}$$

Then the subalgebra $\pi_n(\mathcal{A}_{\theta_n}) \oplus \pi_{n-1}(\mathcal{A}_{\theta_{n-1}}) \subset A_n$ is a finite-dimensional approximation of $\mathcal{A}_\theta$ in the following sense [16]. Since [20]

$$\lim_{n\to\infty} \big\|\rho_n\big(\Gamma_i^{(n-1)} \oplus \Gamma_i^{(n-2)}\big) - \Gamma_i^{(n)} \oplus \Gamma_i^{(n-1)}\big\|_{A_n} = 0 \tag{47}$$

for $i = 1,2$, it follows that there exist unitary operators $U_i \in A_\infty$ which are limits of sequences of finite-rank operators in the inductive limit (34) with respect to the induced operator norm (36) on A_∞, and which obey the defining relation (5) of the noncommutative torus. Thus there exists a unital injective $*$-morphism $\rho : \mathcal{A}_\theta \to A_\infty$.

It is in this sense that the elements of the algebra $\mathcal{A}_\theta$ may be "approximated" by sufficiently large finite-dimensional matrices and the large N limit thus taken, since for n sufficiently large the generators $\Gamma_i^{(n)}$ are well approximated by the images under the injection ρ_n of the matrices $\Gamma_i^{(n-1)}$ generating $\mathcal{A}_{\theta_{n-1}}$. The embeddings $\rho_n : A_{n-1} \to A_n$ above are completely characterized by the sequence of partial embeddings $\{c_n\}_{n\in\mathbb{N}_0}$ associated with the positive maps $\varphi_n : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = \varphi_n \begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix} \qquad \varphi_n = \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{48}$$

It follows that the K-theory group $\mathrm{K}_0(A_\infty)$ can be obtained as the inductive limit of the inductive system of ordered groups $\{\varphi_n : \mathrm{K}_0(A_{n-1}) \to \mathrm{K}_0(A_n)\}_{n\in\mathbb{N}_0}$. Since $\mathrm{K}_0(A_n) = \mathbb{Z}\oplus\mathbb{Z}$ (with the canonical ordering $\mathrm{K}_0^+(A_n) = \mathbb{N}\oplus\mathbb{N}$) for all $n\geq 0$, there is an isomorphism of ordered groups $\mathrm{K}_0(A_\infty)\cong\mathbb{Z}+\mathbb{Z}\,\theta$ with positive cone $\mathrm{K}_0^+(A_\infty) = \{(p,q)\in\mathbb{Z}^2 ~;~ p-q\,\theta>0\}$. This coincides with the K-theory of the noncommutative torus algebra $\mathcal{A}_\theta$.

The sets $\mathbb{Z}+\mathbb{Z}\,\theta$ and $\mathbb{Z}+\mathbb{Z}\,\theta'$ are isomorphic as ordered groups if and only if the irrational numbers θ,θ' lie in the same $\mathrm{SL}(2,\mathbb{Z})$ orbit as in (18) [12], i.e., the algebras $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are Morita equivalent. Equivalently, the continued fraction expansions of θ and θ' have the same "tails". This has two important consequences [16]. Firstly, Morita equivalent tori have the same K-theory group. Secondly, Morita equivalent noncommutative tori can be embedded in the *same* AF-algebra A_∞ (up to isomorphism), because their sequences of embeddings are the same up to a finite number of terms.

5. Noncommutative Solitons

Given the complexity of the large N limit required of the zero-dimensional matrix models of noncommutative gauge theory, it is desirable both physically and mathematically to seek alternative matrix regularizations for which a simpler continuum limit exists. We will now describe precisely how to do this using one-dimensional matrix models, i.e., matrix quantum mechanics. The large N limit of these matrix models does not require a complicated double-scaling, nor is it the conventional 't Hooft planar limit. The matrix approximation is intimately related to the regularization of generic noncommutative field theories by means of solitons on the noncommutative torus, to which the present section is devoted. This formalism also has the virtue of making contact with the relationship between noncommutative field theory and the dynamics of D-branes in string theory.

5.1. D-Branes and Solitons

Let us begin by briefly describing the somewhat simpler situation of solitons on the noncommutative plane, viewed in terms of operators on the Fock module (Schrödinger representation) over the Heisenberg algebra. Open string field theory on this module is described by a potential energy functional V. Restrict to static solutions of the equations of motion. Suppose that $V = V(\mathsf{T}^2)$ is an even functional of hermitian elements T corresponding to tachyon fields. Then the equations of motion are $\mathsf{T}\,V'(\mathsf{T}^2)=0$, which for polynomial functions V can be solved in terms of projections $\mathsf{T} = \mathsf{T}^2$. Let $\{|n\rangle\}_{n\in\mathbb{N}_0}$ be the standard orthonormal number basis for the Fock space. Then the basic rank N projector T_N, having $\mathrm{Tr}(\mathsf{T}_N) = N$, is given by

$$\mathsf{T}_N = \sum_{n=0}^{N-1} |n\rangle\langle n| \tag{49}$$

and it describes N D0-branes sitting inside a D2-brane. In deformation quantization [22], the Wigner function corresponding to the operator T_1 is a gaussian field on $\mathbb{R}^2$ centered about the origin with width proportional to $\theta^{-1/2}$, and hence corresponds to a solitonic "lump". This is the basic GMS soliton [11]. More generally, if $V = V(\mathsf{T}\,\mathsf{T}^\dagger - \mathbb{1}) + V(\mathsf{T}^\dagger\,\mathsf{T} - \mathbb{1})$ is a functional of generically complex operators T, then the static equations of motion are solved by partial isometries $\mathsf{T} = \mathsf{T}\,\mathsf{T}^\dagger\,\mathsf{T}$. Equivalently, the operators $\mathsf{T}^\dagger\,\mathsf{T}$ and $\mathsf{T}\,\mathsf{T}^\dagger$ are projections. Such tachyon fields describe brane-antibrane systems and the basic partial isometry $\mathsf{T} = \mathsf{S}$ of the Fock module is provided by the standard shift operator

$$\mathsf{S} = \sum_{n\in\mathbb{N}_0} |n+1\rangle\langle n| \ . \tag{50}$$

The analogous quantities on the noncommutative torus are much richer and intricate, and it is the purpose of the remainder of this section to demonstrate how they are constructed. In the next section we will then show that fields on the noncommutative torus $\mathcal{A}_\theta$ are expandable in a basis of projection and partial isometry solitons. The solitons generate *subalgebras* $\mathcal{A}_n \subset \mathcal{A}_\theta$ which are isomorphic to two copies of the algebra of matrix-valued functions on a circle of the form $\mathcal{A}_n \cong \mathbb{M}_{q_{2n}}(\mathbb{S}^1) \oplus \mathbb{M}_{q_{2n-1}}(\mathbb{S}^1)$, and for which the convergence to $\mathcal{A}_\theta$ as $n \to \infty$ is "exact" in the sense that $\mathcal{A}_\theta$ is the inductive limit [9, 17]

$$\mathcal{A}_\theta = \lim_{\substack{\longrightarrow \\ n\in\mathbb{N}_0}} \ \mathcal{A}_n \ . \tag{51}$$

It follows that any field theory on $\mathcal{A}_\theta$ is a **matrix quantum mechanics** with a much simpler large N limit than before.

5.2. Powers-Rieffel Projections

Given the generators U_1, U_2 of $\mathcal{A}_\theta$ obeying (5) and the continued fraction expansion (39,40), we define two *towers* of projections

$$\begin{aligned} \mathsf{P}_n &= U_1^{-q_{2n-1}}\,\hat{g}_n + \hat{f}_n + \hat{g}_n U_1^{q_{2n-1}} \\ \mathsf{P}'_n &= U_2^{q_{2n}}\,\hat{g}'_n + \hat{f}'_n + \hat{g}'_n U_2^{-q_{2n}} . \end{aligned} \tag{52}$$

To ease notation, we will drop the sequence labels $n \in \mathbb{N}_0$ until we look at the $n \to \infty$ limits explicitly, and denote $q := q_{2n}$ and $q' := q_{2n-1}$. Then $q, q' \to \infty$ in the limit we are interested in. The algebra element $\hat{f} = \rho(f)$ is in correspondence with a function $f \in \mathrm{C}^\infty(\mathbb{S}^1)$ through the map $\rho : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathcal{A}_\theta$ defined on generators by $\rho(z) := U_1$, where z is the coordinate of the circle $\mathbb{S}^1$. Similarly, $\hat{f}' = \rho'(f')$ corresponds to a function f' on $\mathbb{S}^1$ through the dual map $\rho' : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathcal{A}_\theta$ given by $\rho'(z) := U_2$, and analogous statements are true of $\hat{g}, \hat{g}'$. The traces of these elements are given by $⨍ \hat{f} = f(1)$, and so on.

The functions $f, g, f', g' \in \mathrm{C}^\infty(\mathbb{S}^1)$ take values in the interval $[0,1]$ and are called **bump functions** because they are zero almost everywhere on $\mathbb{S}^1$ [17]. They are chosen so that the elements (52) satisfy three basic requirements: (a) They are projectors, $\mathsf{P}^2 = \mathsf{P}$ and $\mathsf{P}'^2 = \mathsf{P}'$; (b) They have ranks $⨍ \mathsf{P} = p' - q'\,\theta =: \beta$ and $⨍ \mathsf{P}' = -(p - q\,\theta) =: \beta'$; and (c) They have Chern numbers $c_1(\mathsf{P}) = -q'$ and $c_1(\mathsf{P}') = q$. This fixes the K-theory classes of the projectors (52) which are interpreted as $(\mathrm{D}2, \mathrm{D}0)$-brane charges $(p', -q')$ and $(-p, q)$ respectively in open string field theory [4]. The non-trivial generator of $\mathrm{K}_0(\mathcal{A}_\theta)$ has charge $(0,1)$ and is called the *Powers-Rieffel projector* [21].

5.3. Orthogonal Projections

Let us focus for the moment on the first tower of projectors P in (52). In deformation quantization [22], the Wigner function corresponding to P is not a lump as in the case of the noncommutative plane, but rather exhibits stripe patterns on the torus $\mathbb{T}^2$ with area proportional to β [17]. One set of stripes displays periodic lumps with period q', which is a manifestation of the UV/IR mixing phenomenon in noncommutative field theory since the size of the soliton grows with its oscillation period. We can now "translate" the projector P along the first cycle of $\mathbb{T}^2$ through the outer automorphism $\alpha : \mathcal{A}_\theta \to \mathcal{A}_\theta$ defined by

$$\alpha(U_2) = \mathrm{e}^{\mathrm{i}2\pi p/q}\, U_2, \qquad \alpha(U_1) = U_1. \tag{53}$$

Iterating, we may then define a new set of projections for $i = 1, \dots, q$ by

$$\mathsf{P}^{ii} := \alpha^{i-1}(\mathsf{P})\ . \tag{54}$$

They form a system of mutually orthogonal projection operators with

$$\mathsf{P}^{ii}\, \mathsf{P}^{jj} = \delta_{ij}\, \mathsf{P}^{ii}\ . \tag{55}$$

It is convenient in this construction to represent the algebra on the GNS representation space $\mathcal{H} := \mathrm{L}^2(\mathcal{A}_\theta\,, ⨍\,)$. The images of the projectors (54) then define the *Chan-Paton subspaces* $\mathcal{H}_i := \mathrm{im}(\mathsf{P}^{ii}) \subset \mathcal{H}$ with $\mathrm{End}(\mathcal{H}_i) = \mathsf{P}^{ii}\,\mathcal{A}_\theta\,\mathsf{P}^{ii}$ the algebra of open string modes ending on the D-brane state determined by P^{ii}. One has $\mathsf{P}^{ii}|_{\mathcal{H}_i} = 1\!\!1$ and $\mathcal{H}_i \subset \ker(\mathsf{P}^{jj})$ for $j \neq i$.

5.4. Partial Isometries

Let us define the operator

$$\Pi^{21} := \mathsf{P}^{22}\,U_1\,\mathsf{P}^{11}\ . \tag{56}$$

It can be regarded as a closed bounded operator $\Pi^{21} : \mathcal{H}_1 \to \mathcal{H}_2$, but it is not an isometry since $(\Pi^{21})^\dagger\,\Pi^{21} \neq 1\!\!1$. On $\mathcal{H}$ it admits a polar decomposition given by [23]

$$\Pi^{21} := \mathsf{P}^{21}\,\left|\Pi^{21}\right| \tag{57}$$

where $|\Pi^{21}|$ is a hermitian operator and $\mathsf{P}^{21} \in \mathcal{A}_\theta$ is a partial isometry with [9]

$$\lim_{n\to\infty}\left\|\Pi^{21}_n - \mathsf{P}^{21}_n\right\|_{\mathcal{H}} = 0\ . \tag{58}$$

For $i = 1,\dots,q-2$ the translated partial isometries

$$\mathsf{P}^{i+2,i+1} := \alpha^i\big(\mathsf{P}^{21}\big)\ , \qquad \mathsf{P}^{ji} := \big(\mathsf{P}^{ij}\big)^\dagger \tag{59}$$

satisfy the *matrix unit relations*

$$\mathsf{P}^{ij}\,\mathsf{P}^{kl} = \delta_{jk}\,\mathsf{P}^{il}\ . \tag{60}$$

These relations can be used to generate a set of q^2 operators $\mathsf{P}^{ij} : \mathcal{H}_j \to \mathcal{H}_i$, i.e., $\mathsf{P}^{ij} \in \mathsf{P}^{ii}\,\mathcal{A}_\theta\,\mathsf{P}^{jj}$. In a similar fashion, from the second tower we can construct $q'^{\,2}$ matrix units $\mathsf{P}'^{\,i'j'}$ which are orthogonal to P^{ij}.

6. Noncommutative Field Theory as Matrix Quantum Mechanics

We will now use the systems of projections and partial isometries above to construct the desired one-dimensional matrix model [17]. We will first build the subalgebras $\mathcal{A}_n$ and state the basic matrix approximation theorem at the algebraic level. Then we demonstrate how to transcribe generic field theory actions on $\mathcal{A}_\theta$ into matrix quantum mechanics which can serve as precise non-perturbative regularizations of the continuum field theories with tractable large N limits.

6.1. The Matrix Approximation

The collection of operators $\{\mathsf{P}^{ij}\}$ do not quite close a $q\times q$ matrix algebra, because

$$\mathsf{P}^{1q} := \mathsf{P}^{12}\,\mathsf{P}^{23}\cdots\mathsf{P}^{q-1,q} \neq \alpha^{q-1}\big(\mathsf{P}^{21}\big)\ . \tag{61}$$

However, both operators in (61) are isometries on $\mathcal{H}_q \to \mathcal{H}_1$ and consequently are related as

$$\alpha^{q-1}\big(\mathsf{P}^{21}\big) = z\,\mathsf{P}^{1q} \tag{62}$$

where z is a unitary operator on $\mathcal{H}_1$, i.e., a partial isometry on the whole of the Hilbert space $\mathcal{H}$. We may regard z as the generator of the algebra $\mathrm{C}^\infty(\mathbb{S}^1)$. Then the operators $\{\mathsf{P}^{ij},z\}$ close a subalgebra of $\mathcal{A}_\theta$ which is naturally isomorphic to the algebra $\mathbb{M}_q(\mathbb{S}^1)$ of $q\times q$ matrix-valued functions on a circle. An analogous construction in the second tower gives a collection $\{\mathsf{P}'^{\,i'j'},z'\}$, and combining the two towers thus gives the matrix subalgebras

$$\mathcal{A}_n \cong \mathbb{M}_q\big(\mathbb{S}^1\big)\oplus\mathbb{M}_{q'}\big(\mathbb{S}^1\big)\subset\mathcal{A}_\theta\ . \tag{63}$$

The important point here is that $\mathcal{A}_n$ is a *subalgebra* of $\mathcal{A}_\theta$.

By using the continued fraction expansion (39,40) it is possible to define a system of embeddings on the sequence of subalgebras $\{\mathcal{A}_n\}_{n\in\mathbb{N}_0}$ and realize the noncommutative torus algebra $\mathcal{A}_\theta$ as an inductive limit (51) [17]. To describe the convergence theorem explicitly, we define the operators

$$\boldsymbol{U}_1 = \Gamma_1^{(q)}\oplus\Gamma_2^{(q')}(z')\ ,\qquad \boldsymbol{U}_2 = \Gamma_2^{(q)}(z)\oplus\Gamma_1^{(q')}\ . \tag{64}$$

The operator $\Gamma_1^{(q)}$ is, with respect to the system of matrix units $\mathsf{P}^{ij}\in\mathcal{A}_\theta$, the standard $q\times q$ clock matrix in the q-th root of unity $\mathrm{e}^{\mathrm{i}2\pi\theta_{2n}}$. The operator $\Gamma_2^{(q)}(z)$ is the same as the standard shift matrix except that its component multiplying the matrix unit P^{q1} is z. It coincides with the standard cyclic shift matrix $\Gamma_2^{(q)} = \Gamma_2^{(q)}(1)$ at $z=1$. One still has the usual Weyl algebra (46), and also $\big(\Gamma_1^{(q)}\big)^q = \mathbb{1}_q$, $\big(\Gamma_2^{(q)}(z)\big)^q = z\,\mathbb{1}_q$. Completely analogous relations hold for the second tower, and it follows that the elements $\boldsymbol{U}_1,\boldsymbol{U}_2$ generate the matrix subalgebra (63). We may define a restriction map $\gamma_n:\mathcal{A}_\theta\to\mathcal{A}_n$ on generators by

$$\gamma_n(U_i) = \boldsymbol{U}_i\ ,\qquad i=1,2\ . \tag{65}$$

Then for any element $f\in\mathcal{A}_\theta$ the image $\gamma_n(f) := \boldsymbol{f}\oplus\boldsymbol{f}'\in\mathcal{A}_n$ converges to f in the sense that [17]

$$\lim_{n\to\infty}\big\|f-\gamma_n(f)\big\|_{\mathcal{H}} = 0\ . \tag{66}$$

The mapping γ_n, which is *not* an algebra homomorphism, defines the *soliton expansion* of noncommutative fields.

6.2. The One-Dimensional Matrix Model

We will now describe how to construct matrix model actions which approximate generic field theories on the noncommutative torus [17]. Let $\partial_i : \mathcal{A}_\theta \to \mathcal{A}_\theta$ be the outer derivations representing the action of the translation group of $\mathbb{T}^2$ which are defined on generators by

$$\partial_i(U_j) = \mathrm{i}2\pi\delta_{ij}\, U_j \ , \qquad i,j = 1,2 \ . \tag{67}$$

Let Φ be a collection of fields in $\mathcal{A}_\theta$ with lagrangian $\mathcal{L}(\Phi, \partial_i(\Phi)) \in \mathcal{A}_\theta$. Field theory on $\mathcal{A}_\theta$ is then defined by the action

$$S(\Phi) = ⨍ \mathcal{L}\big(\Phi\, ,\, \partial_i(\Phi)\big) \ . \tag{68}$$

In the setting of Sections 3 and 4, we use the map $\pi_n : \mathcal{A}_\theta \to \mathbb{M}_{q_n}(\mathbb{C})$ defined by $\pi_n(U_i) = \Gamma_i^{(q_n)}$ (c.f. equation (45)) and the discrete derivatives (28) to approximate this field theory by a zero-dimensional matrix model with action [2]

$$S_n^{(0)}(\Phi) = \mathrm{Tr}\, \mathcal{L}\big(\pi_n(\Phi)\, ,\, D_i\, \pi_n(\Phi)\, D_i^\dagger - \pi_n(\Phi)\big) \ . \tag{69}$$

We will now find a matrix quantum mechanics action $S_n^{(1)}(\Phi)$ corresponding to the approximation $\gamma_n : \mathcal{A}_\theta \to \mathcal{A}_n$ defined by (65).

Let us begin by describing how to transcribe the trace in (68). On $\mathcal{A}_\theta$ it is defined on generic elements (6) by $⨍ U_1^{n_1}\, U_2^{n_2} = \delta_{n_1,0}\, \delta_{n_2,0}$. On the matrix subalgebra $\mathcal{A}_n$, it is possible to work out $⨍ \boldsymbol{U}_1^{n_1}\, \boldsymbol{U}_2^{n_2}$ using the trace properties $⨍ \mathsf{P}^{ij} = \beta\, \delta_{ij}$ of the matrix units. Using the rapid decay property of Schwartz sequences, in the limit $n \to \infty$ one finds that a good approximation is given by the expected definition

$$⨍ \gamma_n(f) = \beta \int_0^1 \mathrm{Tr}\, \boldsymbol{f}(\tau)\mathrm{d}\tau + \beta' \int_0^1 \mathrm{Tr}\, \boldsymbol{f}'(\tau')\mathrm{d}\tau' \tag{70}$$

where we have parametrized the circles in the two towers by $z = \mathrm{e}^{\mathrm{i}2\pi\tau}$ and $z' = \mathrm{e}^{\mathrm{i}2\pi\tau'}$ with $\tau, \tau' \in [0,1)$.

The approximation of the derivations (67) is much more involved. Let us focus on the first tower in (63) and expand the component of $\gamma_n(f)$ in this tower as

$$\boldsymbol{f} = \sum_{l,m=1}^{q} \sum_{k\in\mathbb{Z}} \phi_{lm;k}\, z^k\, \Gamma_1^l\, \Gamma_2(z)^m \tag{71}$$

with $\phi_{lm;k} := \sum_{r\in\mathbb{Z}} f_{(l+r\,q,m+k\,q)}$. By considering the projection $\gamma_n(\partial_i(f))$ on $\mathcal{A}_\theta \to \mathcal{A}_n$ and using the Schwartz property in the limit $q \to \infty$, we define the operators

$$\begin{aligned}\Delta_1 \boldsymbol{f}(z) &:= \mathrm{i}2\pi \sum_{l,m=1}^{q} \sum_{k\in\mathbb{Z}} l\,\phi_{lm;k}\, z^k\, \Gamma_1^l\, \Gamma_2(z)^m \\ \Delta_2 \boldsymbol{f}(z) &:= \mathrm{i}2\pi \sum_{l,m=1}^{q} \sum_{k\in\mathbb{Z}} (m + k\,q)\,\phi_{lm;k}\, z^k\, \Gamma_1^l\, \Gamma_2(z)^m\ .\end{aligned} \tag{72}$$

These operators converge to the derivations ∂_i in the limit $n \to \infty$. At finite n they satisfy an "approximate" Leibnitz rule, in the sense that they only become derivations at $n \to \infty$. To write these as operators acting on the expansion of $\boldsymbol{f}$ in the system of matrix units P^{lm}, one now needs to compute the change of orthogonal bases between P^{lm} and $\Gamma_1^l\,\Gamma_2(z)^m$ [17].

By performing an identical analysis in the second tower, in this way one finds that the action functional (68) is well approximated by the one-dimensional matrix model with action

$$S_n^{(1)}(\Phi) = \beta \int_0^1 \mathrm{Tr}\,\mathcal{L}\big(\boldsymbol{\Phi}(\tau),\, \Delta_i \boldsymbol{\Phi}(\tau)\big)\mathrm{d}\tau + \beta' \int_0^1 \mathrm{Tr}\,\mathcal{L}\big(\boldsymbol{\Phi}'(\tau'),\, \Delta_i \boldsymbol{\Phi}'(\tau'\,)\big)\mathrm{d}\tau' \tag{73}$$

where $\gamma_n(\Phi) := \boldsymbol{\Phi}(\tau) \oplus \boldsymbol{\Phi}'(\tau'\,)$ and

$$\begin{aligned}\Delta_1 \boldsymbol{\Phi}(\tau) &= \Sigma \boldsymbol{\Phi}(\tau), &\qquad \Delta_1 \boldsymbol{\Phi}'(\tau'\,) &= q' \dot{\boldsymbol{\Phi}}'(\tau') + \big[\Xi', \boldsymbol{\Phi}'(\tau')\big] \\ \Delta_2 \boldsymbol{\Phi}(\tau) &= q \dot{\boldsymbol{\Phi}}(\tau) + \big[\Xi, \boldsymbol{\Phi}(\tau)\big], &\qquad \Delta_2 \boldsymbol{\Phi}'(\tau') &= \Sigma' \boldsymbol{\Phi}'(\tau')\end{aligned} \tag{74}$$

with the dots denoting a derivative with respect to τ or τ'. The $q \times q$ matrix

$$\Xi_{lm} = \mathrm{i}2\pi m\, \delta_{lm}, \qquad 1 \leq l, m \leq q \tag{75}$$

is an "infinitesimal" version of the clock matrix $\big(\Gamma_1^{(q)}\big)^\dagger$. The antihermitian operator Σ is defined by $(\Sigma \boldsymbol{f})(\tau)_{lm} = \sum_{s,t} \Sigma(\tau)_{lm,st}\, \boldsymbol{f}(\tau)_{st}$ with

$$\Sigma(\tau)_{lm,st} = -\frac{\mathrm{i}2\pi}{q} \sum_{s'=1}^{q} s' \mathrm{e}^{\mathrm{i}2\pi s'(l-s)\theta_{2n}} \begin{cases} \delta_{t,s+l-m},\ m \leq l,\ 1 \leq s \leq q+m-l \\ \delta_{t,s+l-m-q} \mathrm{e}^{-\mathrm{i}2\pi\tau},\ j<i,\ q+m-l+1 \leq s \leq q \end{cases} \tag{76}$$

and it can be regarded as an "infinitesimal" version of the shift matrix $\Gamma_2^{(q)}$.

The forms (74) (along with (64)) of the derivation Δ illustrate the role of the two towers in this matrix approximation. The roles of the components of Δ are interchanged between the two towers, and acting on Schwartz sequences they "compensate" each other in the $n \to \infty$ limit. Because of the inductive limit property (51), the convergence of (73) to the original continuum action (68) is "exact" and the large n limit may be taken directly. This is in marked contrast to the large n limit required of (69) which involves a complex double scaling limit via an embedding of the noncommutative torus into a homotopically equivalent AF-algebra A_∞. The zero-dimensional matrix model has a clear geometric origin through toroidal compactification, while the one-dimensional matrix model has a nice physical interpretation. The mapping $f \mapsto \gamma_n(f)$ truncates the infinite number of image D-branes living on the covering space of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to a finite number $q\,q'$, corresponding to the physical open string modes which are invariant under the action of the truncated momentum lattice $(\mathbb{Z}/q\,q'\,\mathbb{Z})^2$.

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