COMPLEX STRUCTURES IN ELECTRODYNAMICS

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Abstract. In this paper we show that the basic external (i.e., not determined by the equations) object in Maxwell vacuum equations is a *complex structure*. In the three-dimensional standard form of Maxwell equations this complex structure $\mathcal{I}$ participates implicitly in the equations and its presence is responsible for the so called *duality invariance*. We give a new form of the equations showing explicitly the participation of $\mathcal{I}$. In the four-dimensional formulation the complex structure is extracted directly from the equations, it appears as a linear map Φ in the space of two-forms on $\mathbb{R}^4$. It is shown also that Φ may appear through the equivariance properties of the new formulation of the theory. Further we show how this complex structure Φ combines with the Poincaré isomorphism $\mathfrak{P}$ between the two-forms and two-tensors to generate all well known and used in the theory (pseudo)metric constructions on $\mathbb{R}^4$, and to define the conformal symmetry properties. The equations of Extended Electrodynamics (EED) do not also need these pseudometrics as beforehand necessary structures. A new formulation of the EED equations in terms of a generalized Lie derivative is given.

1. Introduction

We begin with two examples, showing that meeting with implicitly participating objects in some equations of mathematical physics is not an unknown phenomenon. Recall the wave D'Alembert equation (in standard form)

$$U_{tt} - c^2 \left(U_{xx} + U_{yy} + U_{zz} \right) = 0.$$

Except the constant c, no external objects participate (at first sight, explicitly) in this equation. During the first half of 20th century a new understanding of this equation was created, namely, that a new external object participates implicitly in it and it is the pseudoeuclidean metric tensor $g^{\mu\nu}$, $-g^{11} = -g^{22} = -g^{33} = g^{44} = 1$ on $\mathbb{R}^4$, so that the true form of this equation should read

$$g^{\mu\nu} \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} = 0.$$

This form of the equation, although written in canonical coordinates, may be considered as generally covariant, i.e., making use of corresponding connection coefficients we can pass to arbitrary coordinates. The symmetries of the equation, coming from transformations of the base manifold $\mathbb{R}^4$, as well as many other of its important properties, seem to be determined by the isometries of $g_{\mu\nu}$.

The posterior studies brought another view, saying that the Hodge $*$-operator, defined by $g_{\mu\nu}$, is the essential object, and the equation acquired any of the following two *coordinate free* forms

$$\mathrm{d} * \mathrm{d}U = 0, \qquad (\delta\mathrm{d} + \mathrm{d}\delta)U \equiv \Box U = 0$$

where d is the exterior derivative, and $\delta = (-1)^p *^{-1} \mathrm{d}*$ is the coderivative ($\delta U \equiv 0$).

If we continue this process of *revealing the structures* in defining the equation, we would come to the conclusion that, in fact, the pseudometric tensor $g_{\mu\nu}$ is *not* needed, and the *necessary and sufficient external* structure needed to give a coordinate free form of the equation is a linear map $f : \Lambda^1(\mathbb{R}^4) \to \Lambda^3(\mathbb{R}^4)$ (in fact, a linear isomorphism) from the space of one-forms on $\mathbb{R}^4$ to the space of three-forms on $\mathbb{R}^4$ defined (in canonical coordinates) by

$$f(\mathrm{d}x) = -\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\xi, \qquad f(\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}\xi$$
$$f(\mathrm{d}z) = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\xi, \qquad f(\mathrm{d}\xi) = -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

where $\xi = ct$. Then the above wave equation will read (in canonical coordinates)

$$\mathrm{d}f(\mathrm{d}U) = (-U_{xx} - U_{yy} - U_{zz} + U_{\xi\xi})\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\xi = 0.$$

That the linear map f can be defined by the pseudometric g through the Hodge $*$ is obvious, but it is also evident that f can be defined independently of g. Hence, since the exterior derivative d is defined only by the differential structure on $\mathbb{R}^4$, the symmetries and the other properties of the D'Alembert wave equation are determined entirely by f.

Another example comes from classical mechanics in its hamiltonian formulation. If q^i and p^i are the classical coordinates and momentum components and H is the hamiltonian, then we have the well known hamiltonian form of the basic equations of classical mechanics

$$\dot{q}^i = \frac{\partial H}{\partial p^i}, \qquad \dot{p}^i = -\frac{\partial H}{\partial q^i}.$$

These equations, have been studied quite a long time (nearly a century) before to become clear that there is an external object implicitly participating in these equations, namely the *symplectic* two-form $\omega = \mathrm{d}q^i \wedge \mathrm{d}p^i$ on the cotangent bundle

of $\mathbb{R}^3$. It became clear then that the so called *canonical transformations* coincide with the symmetries of the symplectic two-form ω. This fact turned out to be of great importance in mechanics and could hardly be overestimated. The tremendous mathematical developments that followed this simple observation (*symplectic geometry, contact geometry*) undoubtedly prove this and show the importance of having a clear and full knowledge of *every detail* when studying an equation. And the attention paid in this direction must correspond to the significance of the equations being studied. The right understanding and interpretation of every sign, of every participating constant, and even of the different "equivalent" forms of an equation, reveals its hidden nature, and gives sometimes powerful tools to get knowledge of its basic properties, and suggests various ways of using it and some appropriate generalizations.

Maxwell equations of classical electrodynamics are of exclusive importance in all theoretical and mathematical physics. Their deep study at the end of ninetieth and the beginning of twentieth centuries gave birth to the relativistic view on the physical world. Their duality properties generated the modern "dualities" in field (superstring) theories. Their gauge interpretation, introduced in the remarkable works of H. Weyl and A. Eddington [12], as well as the incorporation of E. Cartan's algebraic and geometric ideas and results in the frame of principle and jet bundle constructions, brought modern gauge theory, and the leading role of gauge theory in today's field theory is undeniable. Therefore, revealing the right external mathematical structures these equations use, we consider as a meaningful and important task.

In this paper, following the ideas previously stated in [3,4], we shall show that the standard complex structure $\mathcal{I}$ in $\mathbb{R}^2$ (here and further under *complex structure* we mean a linear map Φ having the property $\Phi_\circ\Phi = -\mathrm{id}$) implicitly participates in the three-dimensional formulation of Maxwell equations, making quite obvious their duality properties. The new mathematical representation of the field through the $\mathbb{R}^2$ valued one-form $\omega = \mathbf{E} \otimes e_1 + \mathbf{B} \otimes e_2$ on $\mathbb{R}^3$ and through the above mentioned complex structure $\mathcal{I}$ we consider as a more adequate one since it gives an explicit respect to the mathematical structures involved. Some analogy with the the symplectic mechanics development could be noticed: the symplectic two-form in hamilton equations interprets appropriately the minus sign in the second group equations and the canonical transformations; similarly, the complex structure $\mathcal{I}$ will interpret appropriately the minus sign in one of the "curl" Maxwell equations and the mentioned dual symmetry.

In the four-dimensional differential form formulation of Maxwell equations we get the possibility to transform the presence of the complex structure $\mathcal{I}$ in the third

form of the equations to a presence of a special complex structure $\Phi : \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4)$ in the space of differential two-forms on $\mathbb{R}^4$. And this linear map Φ is, in fact, the necessary and sufficient external object that is needed to build the whole picture.

May be it deserves to note that establishing the symmetry properties of the equations considered is not our main purpose in this paper. Our aim is to extract the *precise external structures* used by the equation(s) under consideration. All further reinterpretations of these initial structures in terms of other objects and operations, as well as all useful concequences (e.g. corresponding symmetry properties), we consider as a second step revealing the potential abilities of these structures. For example, we may define Maxwell equations on $\mathbb{R}^4$ through the Hodge $*$-operator defined by the pseudometric g: $\mathrm{d}F = 0$, $\mathrm{d} * F = 0$, but the important moment is that the second equation $\mathrm{d} * F = 0$ does *not* make use of the $*$-operator, it uses *only one* of its properties, namely the property that it defines a *concrete complex structure* in $\Lambda^2(\mathbb{R}^4)$ and *nothing more*, and this complex structure may be introduced *without making use of the pseudometric g*. So, if we start from the equations, we have to try to build all needed structures in terms of those already introduced by the very equations, and we should introduce new ones *only if it is impossible* to define them through the available ones. This is the philosophy we are going to follow in this paper.

2. The Complex Structure in the Standard Third Formulation of Maxwell Equations

We consider the pure field Maxwell equations

$$\operatorname{curl} \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \operatorname{div} \mathbf{B} = 0 \tag{1}$$

$$\operatorname{curl} \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0, \qquad \operatorname{div} \mathbf{E} = 0. \tag{2}$$

First we note, that because of the linearity of these equations if $(\mathbf{E}_i, \mathbf{B}_i), i = 1, 2, ...$ are a collection of solutions, then every couple of linear combinations of the form

$$\mathbf{E} = a_i \mathbf{E}_i, \qquad \mathbf{B} = a_i \mathbf{B}_i \tag{3}$$

(sum over the repeated $i = 1, 2, ...$) with arbitrary constants a_i gives a new solution. This property will not be subject to our further considerations. Our attention will be focussed on possible linear combinations of the kind

$$\mathbf{E}' = a\mathbf{E} + b\mathbf{B}, \qquad \mathbf{B}' = m\mathbf{E} + n\mathbf{B}, \qquad a, b, m, n = \text{constant}.$$

The important observation made by Heaviside [6], and later considered by Larmor [7], is that the substitution

$$\mathbf{E} \to -\mathbf{B}, \qquad \mathbf{B} \to \mathbf{E} \tag{4}$$

transforms the first couple (1) of the pure field Maxwell equations into the second couple (2), and vice versa, the second couple (2) is transformed into the first one (1). This symmetry transformation (4) of the pure field Maxwell equations will be called *special duality transformation*, or SD-transformation. It clearly shows that the electric and magnetic components of the pure electromagnetic field are interchangeable and the interchange (4) transforms solution into solution. This feature of the pure electromagnetic field reveals its *dual* nature.

It is important to note that the SD-transformation (4) does not change the energy density $8\pi w = \mathbf{E}^2 + \mathbf{B}^2$, the Poynting vector $4\pi \mathbf{S} = c(\mathbf{E} \times \mathbf{B})$, and the (nonlinear) Poynting relation

$$\frac{\partial}{\partial t} \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = -\operatorname{div} \mathbf{S}.$$

Hence, from energy-momentum point of view two dual, in the sense of (4), solutions are indistinguishable.

Note that the transformation (4) does NOT come from transformation of the coordinates, it acts only in the space of solutions, and it may be written in a matrix form as follows:

$$(\mathbf{E}, \mathbf{B}) \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\| = (-\mathbf{B}, \mathbf{E}). \tag{5}$$

The following question now arises naturally: do there exist constants (a, b, m, n), such that the linear combinations

$$\mathbf{E}' = a\mathbf{E} + m\mathbf{B}, \qquad \mathbf{B}' = b\mathbf{E} + n\mathbf{B} \tag{6}$$

or in a matrix form

$$(\mathbf{E}', \mathbf{B}') = (\mathbf{E}, \mathbf{B}) \left\| \begin{matrix} a & b \\ m & n \end{matrix} \right\| = (a\mathbf{E} + m\mathbf{B}, b\mathbf{E} + n\mathbf{B}) \tag{7}$$

form again a vacuum solution? Substituting $\mathbf{E}'$ and $\mathbf{B}'$ into Maxwell's vacuum equations we see that the answer to this question is affirmative iff $m = -b, n = a$, i.e., iff the corresponding matrix S is of the form

$$S = \left\| \begin{matrix} a & b \\ -b & a \end{matrix} \right\|. \tag{8}$$

The new solution will have now energy density w' and momentum density $\mathbf{S}'$ as follows

$$w' = \frac{1}{8\pi}\left(\mathbf{E}'^2 + \mathbf{B}'^2\right) = \frac{1}{8\pi}(a^2 + b^2)\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$
$$\mathbf{S}' = (a^2 + b^2)\frac{c}{4\pi}\mathbf{E} \times \mathbf{B}.$$

Obviously, the new and the old solutions will have the same energy and momentum if $a^2 + b^2 = 1$, i.e., if the matrix S is *unimodular*. In this case we may put $a = \cos\alpha$ and $b = \sin\alpha$, where $\alpha = \text{constant}$, so transformation (8) becomes

$$\tilde{\mathbf{E}} = \mathbf{E}\cos\alpha - \mathbf{B}\sin\alpha, \qquad \tilde{\mathbf{B}} = \mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha. \qquad (9)$$

Transformation (9) is known as *electromagnetic duality transformation*, or D-transformation. It has been a subject of many detailed studies in various aspects and contexts [2,6,10,11]. It also has greatly influenced some modern developments in non-Abelian Gauge theories [1], as well as some recent general views on duality in field theory, especially in superstring and brane theories (classical and quantum).

From physical point of view a basic feature of the D-transformation (9) is, that the difference between the electric and magnetic fields becomes non-essential: we may superpose the electric and the magnetic vectors, i.e., vector-components, of a general electromagnetic field to obtain new solutions. From mathematical point of view we see that Maxwell's equations in vacuum, besides the usual linearity (3) mentioned above, admit also "cross"-linearity, i.e., linear combinations of $\mathbf{E}$ and $\mathbf{B}$ of a definite kind determine new solutions.

Any linear map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, having in the canonical basis of $\mathbb{R}^2$ a matrix S of the kind (8), is a symmetry of the canonical complex structure $\mathcal{I}$ of $\mathbb{R}^2$. We recall that if the canonical basis of $\mathbb{R}^2$ is denoted by $(\varepsilon^1, \varepsilon^2)$ then $\mathcal{I}$ is defined by $\mathcal{I}(\varepsilon^1) = \varepsilon^2$, $\mathcal{I}(\varepsilon^2) = -\varepsilon^1$, so if S is given by (8) we have: $S.\mathcal{I}.S^{-1} = \mathcal{I}$. Hence, *the electromagnetic* D-*transformations* (9) *coincide with the unimodular symmetries of the canonical complex structure $\mathcal{I}$ of $\mathbb{R}^2$*. This important in our view remark clearly points out that the canonical complex structure $\mathcal{I}$ in $\mathbb{R}^2$ should be an *essential element* of classical electromagnetic theory, so we should in no way neglect it. Moreover, in our opinion and recalling the above mentioned explicit introducing of the symplectic structure in hamiltonian mechanics, we *must find an appropriate way to introduce $\mathcal{I}$ explicitly* in the equations.

Finally we note that D-transformations change the two well known invariants:

$I_1 = (\mathbf{B}^2 - \mathbf{E}^2)$ and $I_2 = 2\mathbf{E}.\mathbf{B}$ in the following way

$$\tilde{I}_1 = \tilde{\mathbf{B}}^2 - \tilde{\mathbf{E}}^2 = (\mathbf{B}^2 - \mathbf{E}^2)\cos 2\alpha + 2\mathbf{E}.\mathbf{B}\sin 2\alpha = I_1 \cos 2\alpha + I_2 \sin 2\alpha$$

(10)

$$\tilde{I}_2 = 2\tilde{\mathbf{E}}.\tilde{\mathbf{B}} = (\mathbf{E}^2 - \mathbf{B}^2)\sin 2\alpha + 2\mathbf{E}.\mathbf{B}\cos 2\alpha = -I_1 \sin 2\alpha + I_2 \cos 2\alpha.$$

It is seen that even the SD-transformation, where $\alpha = \pi/2$, changes these two invariants: $I_1 \to -I_1$, $I_2 \to -I_2$. This shows that if these two invariants define which solutions should be called *different*, then by making an arbitrary dual transformation we will always produce different solutions, no matter if these solutions carry the same energy-momentum or not. In general we always have

$$\tilde{I}_1{}^2 + \tilde{I}_2{}^2 = I_1^2 + I_2^2$$

(11)

i.e., the sum of the squared invariants is a D-invariant.

The suggestion coming from the above notices is that the electromagnetic field, considered as *one physical object*, has *two physically distinguishable and dynamically interrelated vector components*, $(\mathbf{E}, \mathbf{B})$, so the adequate mathematical model-object must have two vector components and must admit two-dimensional linear transformations of its components, in particular, the two-dimensional rotations should be closely related to the invariance properties of the energy-momentum characteristics of the field. But every two-dimensional linear transformation requires a "room where to act", i.e., a two-dimensional real vector space has to be *explicitly pointed out* and properly incorporated in theory. This two-dimensional space has always been implicitly present inside the electromagnetic field theory, but has not been given a corresponding respect. We introduce it as follows:

> *The electromagnetic field is mathematically represented on $\mathbb{R}^3$ by an $\mathbb{R}^2$-valued differential one-form ω, such that in the canonical basis $(\varepsilon^1, \varepsilon^2)$ in $\mathbb{R}^2$ the one-form ω looks as follows*
>
> $$\omega = \mathbf{E} \otimes \varepsilon^1 + \mathbf{B} \otimes \varepsilon^2.$$

(12)

Remark 1. *In* (12)*, as well as later on, we identify the vector fields and one-forms on $\mathbb{R}^3$ through the euclidean metric and we write, e.g. $*(\mathbf{E} \wedge \mathbf{B}) = \mathbf{E} \times \mathbf{B}$, where $*$ is the Hodge star with respect to the euclidean metric. Also, we identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ through the euclidean metric.*

Now we have to present equations (1) and (2) correspondingly, i.e., in terms of $\mathbb{R}^2$-valued objects.

The above assumption (12) requires a general covariance with respect to transformations in $\mathbb{R}^2$, so, the complex structure $\mathcal{I}$ has to be introduced explicitly in the equations. In order to do this we recall that the linear map $\mathcal{I} : \mathbb{R}^2 \to \mathbb{R}^2$ induces a map

$$\mathcal{I}_* : \omega \to \mathcal{I}_*(\omega) = \mathbf{E} \otimes \mathcal{I}(\varepsilon^1) + \mathbf{B} \otimes \mathcal{I}(\varepsilon^2) = -\mathbf{B} \otimes \varepsilon^1 + \mathbf{E} \otimes \varepsilon^2.$$

We recall also that every operator $\mathcal{D}$ in the set of differential forms is naturally extended to vector-valued differential forms according to the rule $\mathcal{D} \to \mathcal{D} \times$ id, and id is usually omitted. Having in mind the identification of vector fields and one-forms through the euclidean metric we introduce now $\mathcal{I}$ in Maxwell's equations (1)-(2) through ω in the following way

$$*\mathrm{d}\omega - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = 0, \qquad \delta\omega = 0. \tag{13}$$

Two other equivalent forms of (13) are given as follows

$$\mathrm{d}\omega - *\frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = 0, \qquad \delta\omega = 0$$

$$*\mathrm{d}\mathcal{I}_*(\omega) + \frac{1}{c}\frac{\partial}{\partial t}\omega = 0, \qquad \delta\omega = 0.$$

In order to verify the equivalence of (13) to Maxwell equations (1) and (2) we compute the marked operations. We obtain

$$*\mathrm{d}\omega - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{I}_*(\omega) = \left(\mathrm{curl}\,\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) \otimes \varepsilon^1 + \left(\mathrm{curl}\,\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) \otimes \varepsilon^2.$$

The second equation $\delta\omega = 0$ is, obviously, equivalent to

$$\mathrm{div}\,\mathbf{E} \otimes \varepsilon^1 + \mathrm{div}\,\mathbf{B} \otimes \varepsilon^2 = 0$$

since $\delta = -\mathrm{div}$. Hence, (13) coincides with (1) and (2).

We shall emphasize once again that according to our general assumption (12) the field ω will have different representations in the different bases of $\mathbb{R}^2$. Changing the basis $(\varepsilon^1, \varepsilon^2)$ to any other basis $\varepsilon^{1'} = \varphi(\varepsilon^1), \varepsilon^{2'} = \varphi(\varepsilon^2)$, means, of course, that in equations (13) the field ω changes to $\varphi_*\omega$ and the complex structure $\mathcal{I}$ changes to $\varphi\mathcal{I}\varphi^{-1}$. In some sense this means that we have two fields now: ω and $\mathcal{I}$, but $\mathcal{I}$ is given beforehand and it is not determined by equations (13). So, in the new basis the $\mathcal{I}$-dependent equations of (13) will look like

$$*\mathrm{d}\varphi_*\omega - \frac{1}{c}\frac{\partial}{\partial t}(\varphi\mathcal{I}\varphi^{-1})_*(\varphi_*\omega) = 0.$$

If φ is a symmetry of $\mathcal{I}$: $\varphi \mathcal{I} \varphi^{-1} = \mathcal{I}$ (so, φ has a matrix representation as given by (8)), then we transform just ω to $\varphi_* \omega$.

In order to write down the Poynting energy-momentum balance relation we recall the product of vector-valued differential forms. Let $\Phi = \Phi^a \otimes e_a$ and $\Psi = \Psi^b \otimes k_b$ be two differential forms on some manifold with values in the vector spaces V_1 and V_2 with bases $\{e_a\}, a = 1, ..., n$ and $\{k_b\}, b = 1, ..., m$, respectively. Let $f : V_1 \times V_2 \to W$ be a bilinear map valued in a third vector space W. Then a new differential form, denoted by $f(\Phi, \Psi)$, on the same manifold and valued in W is defined by

$$f(\Phi, \Psi) = \Phi^a \wedge \Psi^b \otimes f(e_a, k_b).$$

Clearly, if Φ and Ψ are of degree p and q respectively, then the product is a $(p + q)$-form.

Assume now that $V_1 = V_2 = \mathbb{R}^2$ and the bilinear map is the exterior product: $\wedge : \mathbb{R}^2 \times \mathbb{R}^2 \to \Lambda^2(\mathbb{R}^2)$.

Let us compute the expression $\wedge(\omega, d\omega)$.

$$
\begin{aligned}
\wedge(\omega, d\omega) &= \wedge(\mathbf{E} \otimes \varepsilon^1 + \mathbf{B} \otimes \varepsilon^2, d\mathbf{E} \otimes \varepsilon^1 + d\mathbf{B} \otimes \varepsilon^2) \\
&= (\mathbf{E} \wedge d\mathbf{B} - \mathbf{B} \wedge d\mathbf{E}) \otimes \varepsilon^1 \wedge \varepsilon^2 \\
&= -d(\mathbf{E} \wedge \mathbf{B}) \otimes \varepsilon^1 \wedge \varepsilon^2 = -d(* * (\mathbf{E} \wedge \mathbf{B})) \otimes \varepsilon^1 \wedge \varepsilon^2 \\
&= *\delta(\mathbf{E} \times \mathbf{B}) \otimes \varepsilon^1 \wedge \varepsilon^2 = - * \mathrm{div}(\mathbf{E} \times \mathbf{B}) \otimes \varepsilon^1 \wedge \varepsilon^2 \\
&= -\mathrm{div}(\mathbf{E} \times \mathbf{B}) dx \wedge dy \wedge dz \otimes \varepsilon^1 \wedge \varepsilon^2.
\end{aligned}
$$

Following the same rules we obtain

$$\wedge \left(\omega, * \frac{1}{c} \frac{\partial}{\partial t} \mathcal{I}_* \omega \right) = \frac{1}{c} \frac{\partial}{\partial t} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} dx \wedge dy \wedge dz \otimes \varepsilon^1 \wedge \varepsilon^2.$$

So, the Poynting energy-momentum balance relation is given by

$$\wedge \left(\omega, d\omega - * \frac{1}{c} \frac{\partial}{\partial t} \mathcal{I}_* \omega \right) = 0. \tag{14}$$

Since the two-form $\varepsilon^1 \wedge \varepsilon^2$ is invariant with respect to rotations in $\mathbb{R}^2$ we have the D-invariance of the above energy-momentum quantities and relations.

Note the following simple forms of the energy density

$$\frac{1}{8\pi} * \wedge (\omega, * \mathcal{I}_* \omega) = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} \varepsilon^1 \wedge \varepsilon^2$$

and of the Poynting vector

$$\frac{c}{8\pi} * \wedge(\omega, \omega) = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \otimes \varepsilon^1 \wedge \varepsilon^2$$

the D-invariance is obvious. As for the general $\mathbb{R}^2$ covariance of the second equation of (13) it is obvious.

Resuming, we may say that pursuing the mathematical adequacy of the correspondence: *one physical object - one mathematical model-object*, we came to the idea to introduce the $\mathbb{R}^2$-valued one-form ω as the mathematical model-field. This, in turn, set the problem for general $\mathbb{R}^2$ covariance of the equations and this problem was solved through introducing explicitly the canonical complex structure $\mathcal{I}$ in the dynamical equations (13) of the theory. This means that a linear transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{E}' = a\mathbf{E} + m\mathbf{B}, \qquad \mathbf{B}' = b\mathbf{E} + n\mathbf{B}$$

will give again a solution if $\mathcal{I}$ participates in the equations appropriately and the transformation φ is a symmetry of $\mathcal{I}$.

3. Four-Dimensional Consideration

3.1. Classical Electrodynamics

We are going to reveal the complex structure in the four-dimensional formulation of Maxwell equations in two ways. The first way is quite direct and consists in the following.

In the four-dimensional formulation of Maxwell equations we consider the time variable $x^4 = \xi = ct$, where c is the velocity of light, as a coordinate and treat it in the same way as the other three spatial coordinates $(x^1, x^2, x^3 = x, y, z)$. So, the base manifold becomes four-dimensional, in fact, $\mathbb{R}^4$. Since the vector fields on $\mathbb{R}^4$ form a four-dimensional module, $\mathbf{E}$ and $\mathbf{B}$ can not be considered as vector fields on $\mathbb{R}^4$. But the couple $(\mathbf{E}, \mathbf{B})$ has six components, therefore we consider the space $\Lambda^2(\mathbb{R}^4)$ of two-forms, which is a six-dimensional module, as a natural solution space. Moreover, as it is well known, in the basis

$$\mathrm{d}x \wedge \mathrm{d}y, \qquad \mathrm{d}x \wedge \mathrm{d}z, \qquad \mathrm{d}y \wedge \mathrm{d}z, \qquad \mathrm{d}x \wedge \mathrm{d}\xi, \qquad \mathrm{d}y \wedge \mathrm{d}\xi, \qquad \mathrm{d}z \wedge \mathrm{d}\xi$$

of $\Lambda^2(\mathbb{R}^4)$ if we put for $F \in \Lambda^2(\mathbb{R}^4)$

$$F_{12} = B^3, \quad F_{13} = -B^2, \quad F_{23} = B^1, \quad F_{14} = E^1, \quad F_{24} = E^2, \quad F_{34} = E^3$$

then the equation $\mathrm{d}F = 0$ gives the first couple (1) of Maxwell equations. Similarly, if we consider the two-form $\Phi(F)$, given in this basis by

$$(\Phi F)_{12} = E^3, \qquad (\Phi F)_{13} = -E^2, \qquad (\Phi F)_{23} = E^1$$
$$(\Phi F)_{14} = -B^1, \qquad (\Phi F)_{24} = -B^2, \qquad (\Phi F)_{34} = -B^3$$

then the equation $\mathrm{d}(\Phi F) = 0$ gives the second couple (2) of Maxwell equations. Hence, Maxwell equations in vacuum become

$$\mathrm{d}F = 0, \qquad \mathrm{d}(\Phi F) = 0. \tag{15}$$

We especially note that *no pseudometric is needed* to write down these equations. We also note that if $(F, \Phi(F))$ is a solution then the two linear combinations

$$F' = aF + b(\Phi F), \qquad G = mF + n(\Phi F) \tag{16}$$

with arbitrary (a, b, m, n) define again two closed two-forms, but these two closed two-forms will define a solution only if $G = \Phi(F')$.

The linear map $\Phi : \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4)$, as defined above, has the property

$$\Phi_{\circ} \Phi = -\mathrm{id}_{\Lambda^2(\mathbb{R}^4)}$$

hence *it introduces complex structure* in the space $\Lambda^2(\mathbb{R}^4)$.

The second way follows the considerations in the three-dimensional case and makes use of the complex structure $\mathcal{I}$ of $\mathbb{R}^2$, and of the space $\Lambda^2(\mathbb{R}^4, \mathbb{R}^2)$ of $\mathbb{R}^2$-valued differential two-forms on $\mathbb{R}^4$. In general an $\mathbb{R}^2$ valued two-form Ω on $\mathbb{R}^4$ looks as follows

$$\Omega = F_1 \otimes \varepsilon^1 + F_2 \otimes \varepsilon^2.$$

Consider now the two linear maps

$$\mathcal{F} : \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4), \qquad \varphi : \mathbb{R}^2 \to \mathbb{R}^2.$$

These maps induce a map $(\mathcal{F}, \varphi) : \Lambda^2(\mathbb{R}^4, \mathbb{R}^2) \to \Lambda^2(\mathbb{R}^4, \mathbb{R}^2)$ by the rule:

$$(\mathcal{F}, \varphi)(\Omega) = (\mathcal{F}, \varphi)(\mathbf{F}_a \otimes \varepsilon^a) = \mathcal{F}(\mathbf{F}_a) \otimes \varphi(\varepsilon^a), \qquad \text{summation over } a=1,2.$$

It is natural to ask now is it possible the joint action of these two maps to keep Ω unchanged, i.e., to have

$$(\mathcal{F}, \varphi)(\Omega) = \Omega.$$

In such a case the form Ω is called $(\mathcal{F}, \varphi)$-*equivariant*. If φ is a linear isomorphism and we identify $\mathcal{F}$ with $(\mathcal{F}, \mathrm{id}_{\mathbb{R}^2})$ and φ with $(\mathrm{id}_{\Lambda^2(\mathbb{R}^4)}, \varphi)$, we can equivalently write

$$\mathcal{F}(\Omega) = \varphi^{-1}(\Omega).$$

If we specialize now: $\varphi = \mathcal{I}$ we readily find that the $(\mathcal{F}, \mathcal{I})$-equivariant forms Ω must satisfy

$$(\mathcal{F}, \mathcal{I})(\Omega) = -\mathcal{F}(F_2) \otimes \varepsilon^1 + \mathcal{F}(F_1) \otimes \varepsilon^2 = F_1 \otimes \varepsilon^1 + F_2 \otimes \varepsilon^2 = \Omega.$$

Hence, we must have $\mathcal{F}(F_1) = F_2$ and $\mathcal{F}(F_2) = -F_1$, i.e., $\mathcal{F}_\circ \mathcal{F} = -\mathrm{id}$. In other words, the property $\mathcal{I}_\circ \mathcal{I} = -\mathrm{id}_{\mathbb{R}^2}$ is carried over to $\mathcal{F}$: $\mathcal{F}_\circ \mathcal{F} = -\mathrm{id}_{\Lambda^2(\mathbb{R}^4)}$.

Hence, recalling the linear map Φ, introduced above, and working with $(\Phi, \mathcal{I})$-equivariant two-forms on $\mathbb{R}^4$, we can replace the action of $\mathcal{I}$ with the action of Φ. And that's why in the four-dimensional formulation of electrodynamics we have general $\mathbb{R}^2$ covariance if we work with forms Ω of the kind $\Omega = F \otimes \varepsilon^1 + \Phi F \otimes \varepsilon^2$. In the three-dimensional formulation this is not possible to be done since we work there on $\mathbb{R}^3$ and no map $\Phi : \Lambda^1(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$ with the property $\Phi_\circ \Phi = -\mathrm{id}_{\Lambda^1(\mathbb{R}^3)}$ exists since $\Lambda^1(\mathbb{R}^3)$ is a three-dimensional space and the relation $\Phi_\circ \Phi = -\mathrm{id}$ requires *even*-dimensional space, so we have to introduce the complex structure through $\mathbb{R}^2$ only.

Having in view these considerations our basic assumption for the algebraic nature of the mathematical-model object must read:

> *The electromagnetic field is mathematically represented on $\mathbb{R}^4$ by a $(\Phi, \mathcal{I})$-equivariant $\mathbb{R}^2$ valued two-form Ω such that in the canonical basis $(\varepsilon^1, \varepsilon^2)$ in $\mathbb{R}^2$ the one-form Ω looks as follows*
>
> $$\Omega = F \otimes \varepsilon^1 + (\Phi F) \otimes \varepsilon^2.$$

(17)

The pure field Maxwell equations, expressed through the $(\Phi, \mathcal{I})$-equivariant two-form Ω have, obviously, general $\mathbb{R}^2$ covariance and are equivalent to

$$\mathrm{d}\Omega = 0. \tag{18}$$

Now we are going to show how the well known from standard relativistic electrodynamics on Minkowski space-time pseudometric structures can be introduced by means of the complex structure Φ, which is the *only external mathematical structure* on $\mathbb{R}^4$ introduced by Maxwell equations.

We begin with giving the matrix of Φ in the above given coordinate basis $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, $\mu < \nu$, where $x^4 = \xi = ct$, of the space $\Lambda^2(\mathbb{R}^4)$ considered as a six-dimensional module over the algebra $C^\infty(\mathbb{R}^4)$ of all smooth real valued functions

on $\mathbb{R}^4$

$$\Phi^{(\alpha\beta)}_{(\mu\nu)} = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix} \tag{19}$$

where $\alpha < \beta$, $\mu < \nu$, $(\alpha\beta)$ numbers the rows and $(\mu\nu)$ numbers the columns. Hence,

$$\Phi(\mathrm{d}x \wedge \mathrm{d}y) = -\mathrm{d}z \wedge \mathrm{d}\xi, \quad \Phi(\mathrm{d}x \wedge \mathrm{d}z) = \mathrm{d}y \wedge \mathrm{d}\xi, \quad \Phi(\mathrm{d}y \wedge \mathrm{d}z) = -\mathrm{d}x \wedge \mathrm{d}\xi$$

$$\Phi(\mathrm{d}x \wedge \mathrm{d}\xi) = \mathrm{d}y \wedge \mathrm{d}z, \quad \Phi(\mathrm{d}y \wedge \mathrm{d}\xi) = -\mathrm{d}x \wedge \mathrm{d}z, \quad \Phi(\mathrm{d}z \wedge \mathrm{d}\xi) = \mathrm{d}x \wedge \mathrm{d}y.$$

Remark 2. *If we introduce the notation* $\mathrm{d}x \wedge \mathrm{d}y = e^1, \ldots, \mathrm{d}z \wedge \mathrm{d}\xi = e^6$*, and* $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = e_1, \ldots, \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi} = e_6$*, so that* $\{e^i\}$ *and* $\{e_j\}$ *are dual bases, then* Φ *may be considered as* $(2,2)$ *tensor and represented by*

$$\Phi = \sum_{1 \leq i \leq 6} (-1)^i e_i \otimes e^{7-i}.$$

In order to build all additional structures needed and used in Electrodynamics we are going to make use of the *Poincaré* isomorphism $\mathfrak{P}$, which is algebraically built as follows [5]. Let $\Lambda^p(V)$ be the space of antisymmetric contravariant p-tensors, or p-vectors for short, and $\Lambda^{(n-p)}(V^*)$ be the space of $(n-p)$-forms over the pair of dual n-dimensional real vector spaces (V, V^*), and $p = 1, \ldots, n$. If $\{e_i\}$ and $\{\varepsilon^i\}$ are two dual bases we have the n-vector $\omega = e_1 \wedge \cdots \wedge e_n$ and the n-form $\omega^* = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$. The duality requires $\langle \varepsilon^i, e_j \rangle = \delta^i_j$ and $\langle \omega^*, \omega \rangle = 1$. If $x \in V$ and $\alpha \in \Lambda^p(V^*)$ then by means of the insertion operator $i(x)$ we obtain the $(p-1)$-form $i(x)\alpha$. If α is decomposable: $\alpha = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p$, where $\alpha^1, \ldots, \alpha^p$ are one-forms, then

$$i(x)\alpha = \langle \alpha^1, x \rangle \alpha^2 \wedge \cdots \wedge \alpha^p - \langle \alpha^2, x \rangle \alpha^1 \wedge \alpha^3 \wedge \cdots \wedge \alpha^p$$
$$+ \ldots (-1)^{p-1} \langle \alpha^p, x \rangle \alpha^1 \wedge \cdots \wedge \alpha^{p-1}.$$

Now let $x_1 \wedge x_2 \wedge \cdots \wedge x_p \in \Lambda^p(V)$. The Poincaré isomorphism $\mathfrak{P} : \Lambda^p(V) \rightarrow \Lambda^{n-p}(V^*)$ acts as follows

$$\mathfrak{P}_p(x_1 \wedge x_2 \wedge \cdots \wedge x_p) = i(x_p) \circ i(x_{p-1}) \circ \ldots \circ i(x_1)\omega^*. \tag{20}$$

In the same way, for the opposite isomorphism $\mathfrak{P}^p$ we have

$$\mathfrak{P}^p(\alpha^1 \wedge \cdots \wedge \alpha^p) = i(\alpha^p) \circ \ldots \circ i(\alpha^1)\omega. \tag{21}$$

For nondecomposable tensors $\mathfrak{P}$ is extended by linearity. We note also the relations:

$$\mathfrak{P}_{n-p}\mathfrak{P}^p = (-1)^{p(n-p)}\mathrm{id}\,, \qquad \mathfrak{P}^{n-p}\mathfrak{P}_p = (-1)^{p(n-p)}\mathrm{id}$$

$$\langle \mathfrak{P}^p(\alpha^1 \wedge \cdots \wedge \alpha^p), \mathfrak{P}_p(x_1 \wedge \cdots \wedge x_p)\rangle = \langle \alpha^1 \wedge \cdots \wedge \alpha^p, x_1 \wedge \cdots \wedge x_p\rangle.$$

On an arbitrary basis element $\mathfrak{P}$ acts in the following way:

$$\mathfrak{P}^p(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_p}) = (-1)^{\sum\limits_{k=1}^{p}(i_k-k)} e_{i_{p-1}} \wedge \cdots \wedge e_{i_n} \tag{22}$$

where $i_1 < \cdots < i_p$ and $i_{p+1} < \cdots < i_n$ are complementary $p-$ and $(n-p)-$ tuples.

On the manifold $\mathbb{R}^4$ we have the dual bases $\{\mathrm{d}x^i\}$ and $\left\{\frac{\partial}{\partial x^i}\right\}$ of the two modules of one-forms and vector fields. The corresponding ω and ω^* are

$$\omega = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}, \qquad \omega^* = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\xi.$$

Now we define the nondegenerate operator $\mathfrak{D} : \Lambda^2(T^*\mathbb{R}^4) \to \Lambda^2(T\mathbb{R}^4)$ as follows

$$\mathfrak{D} = -\mathfrak{P}_\circ\Phi \tag{23}$$

where Φ is defined in (19). On the basis elements $\mathfrak{D}$ acts in the following way:

$$\mathfrak{D}(\mathrm{d}x \wedge \mathrm{d}y) = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \quad \mathfrak{D}(\mathrm{d}x \wedge \mathrm{d}z) = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}, \quad \mathfrak{D}(\mathrm{d}y \wedge \mathrm{d}z) = \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

$$\mathfrak{D}(\mathrm{d}x \wedge \mathrm{d}\xi) = -\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \xi}, \quad \mathfrak{D}(\mathrm{d}y \wedge \mathrm{d}\xi) = -\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \xi}, \quad \mathfrak{D}(\mathrm{d}z \wedge \mathrm{d}\xi) = -\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}.$$

Note that

$$\mathfrak{D}(\mathrm{d}x \wedge \mathrm{d}y) \wedge \mathfrak{D}(\mathrm{d}z \wedge \mathrm{d}\xi) = -\omega.$$

The isomorphism $\mathfrak{D}$ defines a bilinear form $h^2 : \Lambda^2(T^*\mathbb{R}^4) \times \Lambda^2(T^*\mathbb{R}^4) \to \mathbb{R}$ according to the rule: $h^2(\alpha, \beta) = \langle \mathfrak{D}(\alpha), \beta\rangle$. In our coordinate basis h^2 has components as follows

$$h^{2(12,12)} = h^{2(13,13)} = h^{2(23,23)} = -h^{2(14,14)} = -h^{2(24,24)} = -h^{2(34,34)} = 1$$

and all other components are equal to zero. On the other hand the complex structure Φ defines a bilinear form $\tilde{h}^2 : \Lambda^2(T^*\mathbb{R}^4) \times \Lambda^2(T^*\mathbb{R}^4) \to \mathbb{R}$ according to the rule

$$\alpha \wedge (\Phi\beta) = -\tilde{h}^2(\alpha, \beta)\omega^*.$$

It is easy to show that *these two bilinear forms coincide*: $h^2 = \tilde{h}^2$.

We are going to extend the complex structure Φ to a map $\circledast$: $\Lambda^p(T^*\mathbb{R}^4) \to \Lambda^{4-p}(T^*\mathbb{R}^4), p = 1, \ldots, 4$. To this end we first find that there exists unique (up to a sign) linear isomorphism $\varphi : T^*(\mathbb{R}^4) \to T(\mathbb{R}^4)$ satisfying the two conditions:

1. φ is represented by a diagonal matrix in the bases $\{\mathrm{d}x^i\}$, $\{\frac{\partial}{\partial x^i}\}$
2. $\wedge^2\varphi = \mathfrak{D}$.

Actually, these two conditions imply

$$\varphi(\mathrm{d}x^\mu) = \lambda^\mu \frac{\partial}{\partial x^\mu}, \quad \text{no summation over } \mu$$

$$(\wedge^2\varphi)^{\mu\nu,\alpha\beta} = \varphi^{\mu\alpha}\varphi^{\nu\beta} - \varphi^{\mu\beta}\varphi^{\nu\alpha}, \quad \mu < \nu, \ \alpha < \beta$$

and lead to

$$-\lambda^1 = -\lambda^2 = -\lambda^3 = \lambda^4 = 1, \quad \text{or} \quad \lambda^1 = \lambda^2 = \lambda^3 = -\lambda^4 = 1.$$

We are going to work further with the first of these two solutions.

Thus we have φ and $\wedge^2\varphi$. Computing $\Lambda^3\varphi$ in the bases

$$\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\mu, \qquad \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta} \wedge \frac{\partial}{\partial x^\mu}, \qquad \alpha < \beta < \mu$$

we find that $\wedge^3\varphi$ has only diagonal components equal to $(-1, 1, 1, 1)$. Finally, $\wedge^4\varphi$ has only one component $(\wedge^4\varphi)^{1234} = -1$, so that $\wedge^4\varphi(\omega^*) = -\omega$.

In the well known way the linear isomorphisms $\wedge^p\varphi$, $p = 1, \ldots, 4$, define bilinear forms h^p in $\Lambda^p(T^*\mathbb{R}^4)$ by the rule $h^p(\alpha, \beta) = \langle \wedge^p\varphi(\alpha), \beta \rangle$, and all these four bilinear forms are *nondegenerate*.

Now we extend Φ to $\circledast$ by the rule

$$\alpha^p \wedge \circledast\beta^p = -h^p(\alpha^p, \beta^p)\omega^*. \tag{24}$$

The above algebraic considerations show that:

1. The bilinear forms h^p define *pseudoeuclidean metric structures* in the spaces $\Lambda^p(T^*\mathbb{R}^4)$, $p = 1, \ldots, 4$, and in fact, in all tensor bundles over the manifold $\mathbb{R}^4$.
2. The following relation holds:

$$\circledast_p = -\mathfrak{P}_\circ \wedge^p \varphi. \tag{25}$$

3.2. Symmetries of Maxwell Equations Generated by Φ

In order to find the symmetries of Maxwell equations, generated by vector fields X on the base manifold $\mathbb{R}^4$ we have to solve the equation $L_X\Phi = 0$, where L_X is the Lie derivative along X. We shall solve this problem as a particular case of the more general problem to find the symmetries of the operator $\circledast$, i.e., to find those X along which we have

$$[L_X, \circledast] = (L_X)_\circ \circledast - \circledast_\circ (L_X) = 0.$$

First we note

$$\begin{aligned}
L_X (\alpha \wedge \circledast\beta) &= (L_X\alpha) \wedge \circledast\beta + \alpha \wedge L_X (\circledast\beta) \\
&= (L_X\alpha) \wedge \circledast\beta + \alpha \wedge [L_X, \circledast]\,\beta + \alpha \wedge \circledast L_X\beta.
\end{aligned}$$

On the other hand, making use of relation (24), we obtain $(\alpha, \beta \in \Lambda^p(T^*\mathbb{R}^4))$

$$\begin{aligned}
&L_X(\alpha \wedge \circledast\beta) \\
&= -(L_X h^p)(\alpha, \beta)\omega^* - h^p(L_X\alpha, \beta)\omega^* - h^p(\alpha, L_X\beta)\omega^* - h^p(\alpha, \beta)L_X\omega^* \\
&= -(L_X h^p)(\alpha, \beta)\omega^* + (L_X\alpha) \wedge \circledast\beta + \alpha \wedge \circledast L_X\beta - h^p(\alpha, \beta)\mathrm{div}X.\omega^* \\
&= -(L_X h^p)(\alpha, \beta)\omega^* + L_X (\alpha \wedge \circledast\beta) - \alpha \wedge [L_X, \circledast]\,\beta - h^p(\alpha, \beta)\mathrm{div}X.\omega^*.
\end{aligned}$$

Since α and β are arbitrary p-forms from this relation it follows that $[L_X, \circledast] = 0$ if

$$L_X h^p = -\mathrm{div}X.h^p, \qquad p = 1, 2, 3, 4. \tag{26}$$

Therefore, for the case we are interesting in we obtain

$$L_X\Phi = 0 \Longleftrightarrow L_X h^2 = -\mathrm{div}X.h^2. \tag{27}$$

From this relation we obtain the following (independent) equations for the components of any local symmetry X of Φ.

$$2\left(\frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y}\right) = \mathrm{div}X, \quad 2\left(\frac{\partial X^1}{\partial x} + \frac{\partial X^1}{\partial \xi}\right) = \mathrm{div}X$$

$$2\left(\frac{\partial X^1}{\partial x} + \frac{\partial X^3}{\partial z}\right) = \mathrm{div}X, \quad 2\left(\frac{\partial X^2}{\partial y} + \frac{\partial X^4}{\partial \xi}\right) = \mathrm{div}X$$

$$2\left(\frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z}\right) = \mathrm{div}X, \quad 2\left(\frac{\partial X^3}{\partial z} + \frac{\partial X^4}{\partial \xi}\right) = \mathrm{div}X$$

$$\left(\frac{\partial X^2}{\partial x} + \frac{\partial X^1}{\partial y}\right) = 0, \qquad \left(\frac{\partial X^4}{\partial x} - \frac{\partial X^1}{\partial \xi}\right) = 0$$

$$\left(\frac{\partial X^3}{\partial x} + \frac{\partial X^1}{\partial z}\right) = 0, \qquad \left(\frac{\partial X^4}{\partial y} - \frac{\partial X^2}{\partial \xi}\right) = 0$$

$$\left(\frac{\partial X^3}{\partial y} + \frac{\partial X^2}{\partial z}\right) = 0, \qquad \left(\frac{\partial X^4}{\partial z} - \frac{\partial X^3}{\partial \xi}\right) = 0.$$

These equations have the following solutions:

1. Translations:

$$X = \frac{\partial}{\partial x}, \qquad X = \frac{\partial}{\partial y}, \qquad X = \frac{\partial}{\partial z}, \qquad X = \frac{\partial}{\partial \xi}$$

as well as any linear combination with *constant* coefficients of these four vector fields;

2. Spatial rotations:

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad X = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

3. Space-time rotations:

$$X = x\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial x}, \qquad X = y\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial y}, \qquad X = z\frac{\partial}{\partial \xi} + \xi\frac{\partial}{\partial z}$$

4. Dilatations:

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \xi\frac{\partial}{\partial \xi}, \qquad \text{or} \quad X = x^\mu \frac{\partial}{\partial x^\mu}$$

5. Conformal (with respect to the bilinear form h^1) vector fields:

$$X_\mu = \left(h^1_{\alpha\beta}x^\alpha x^\beta\right)\frac{\partial}{\partial x^\mu} - 2h^1_{\mu\nu}x^\nu\left(x^\sigma\frac{\partial}{\partial x^\sigma}\right), \qquad \mu = 1,\ldots,4.$$

Let us consider the flows generated by the above vector fields.

1. The translation vector fields generate flows as follows:

$$x^{\mu'} = x^\mu + a^\mu, \qquad a^\mu \text{ are four constants.}$$

2. The spatial rotations generate "rotational" flows inside the three planes (x, y), (x, z) and (y, z) as follows

$$x' = x\cos s + y\sin s, \qquad x' = x\cos s + z\sin s, \qquad y' = y\cos s + z\sin s$$
$$y' = -x\sin s + y\cos s, \qquad z' = -x\sin s + z\cos s, \qquad z' = -y\sin s + z\cos s.$$

3. The space-time rotations generate the following flows

$$x' = x\operatorname{ch}s + \xi\operatorname{sh}s, \quad y' = y\operatorname{ch}s + \xi\operatorname{sh}s, \quad z' = z\operatorname{ch}s + \xi\operatorname{sh}s$$
$$\xi' = x\operatorname{sh}s + \xi\operatorname{ch}s, \quad \xi' = y\operatorname{sh}s + \xi\operatorname{ch}s, \quad \xi' = z\operatorname{sh}s + \xi\operatorname{ch}s.$$

We will concentrate for a while on the flow in the plane (x, ξ). It is obtained by solving the equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \xi, \qquad \frac{\mathrm{d}\xi}{\mathrm{d}s} = x.$$

Let $x_{s=0} = x_\circ, \ \xi_{s=0} = \xi_\circ$. Then the solution is

$$x = x_\circ \operatorname{ch}s + \xi_\circ \operatorname{sh}s = \frac{x_\circ + \operatorname{th}s\xi_\circ}{\sqrt{1 - \operatorname{th}^2 s}} = \frac{x_\circ + \beta c t_\circ}{\sqrt{1 - \beta^2}}$$

$$\xi = x_\circ \operatorname{sh}s + \xi_\circ \operatorname{ch}s = \frac{x_\circ \operatorname{th}s + c t_\circ}{\sqrt{1 - \operatorname{th}^2 s}} = \frac{x_\circ \beta + c t_\circ}{\sqrt{1 - \beta^2}},$$

where $\beta^2 = \operatorname{th}^2 s \leq 1$, and s is fixed. The standard physical interpretation of these relations is that the frame $(x_\circ, \xi_\circ)$ moves with respect to the frame (x, ξ) along the common axis $x \equiv x_\circ$ with the velocity $v = \beta c$, and since $|\beta| \leq 1$ then $|v| \leq c$. It is important to have in mind that this interpretation requires that c has the same value in all such frames. This special invariance property of Maxwell equations, i.e., of the complex structure Φ, brought to life the Poincaré-Einstein relativity principle.

4. The dilatation vector field generates the flow

$$x^{\mu'} = a x^\mu, \qquad a = \exp(s) = \text{constant}.$$

5. The conformal (with respect to the bilinear form h^1) vector fields generate the nonlinear flows

$$x' = \frac{x + \langle \boldsymbol{x}, \boldsymbol{x} \rangle d^1}{1 + 2\langle \boldsymbol{x}, \boldsymbol{d} \rangle + \langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{d}, \boldsymbol{d} \rangle} \tag{28}$$

$$y' = \frac{y + \langle \boldsymbol{x}, \boldsymbol{x} \rangle d^2}{1 + 2\langle \boldsymbol{x}, \boldsymbol{d} \rangle + \langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{d}, \boldsymbol{d} \rangle} \tag{29}$$

$$z' = \frac{z + \langle \boldsymbol{x}, \boldsymbol{x} \rangle d^3}{1 + 2\langle \boldsymbol{x}, \boldsymbol{d} \rangle + \langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{d}, \boldsymbol{d} \rangle} \tag{30}$$

$$\xi' = \frac{\xi + \langle \boldsymbol{x}, \boldsymbol{x} \rangle d^4}{1 + 2\langle \boldsymbol{x}, \boldsymbol{d} \rangle + \langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{d}, \boldsymbol{d} \rangle} \tag{31}$$

where $\boldsymbol{d} = (d^1, \dots, d^4)$ are the four constants-parameters of the special conformal transformations and $\langle \boldsymbol{x}, \boldsymbol{d} \rangle = -xd^1 - yd^2 - zd^3 + \xi d^4$. Note that these

transformations may be considered as coordinate transformations only if the corresponding denominators are different from zero.

The above four relations may be written together in the following way

$$x^{\mu'} = \frac{x^\mu + (h^1_{\alpha\beta}x^\alpha x^\beta)d^\mu}{1 + 2h^1_{\alpha\beta}d^\alpha x^\beta + (h^1_{\alpha\beta}x^\alpha x^\beta)(h^1_{\mu\nu}d^\mu d^\nu)}.$$

These symmetry considerations show undoubtedly the above mentioned analogy with the symplectic mechanics: the canonical (q, p)-transformations defined as symmetries of the symplectic two-form on $T^*(\mathbb{R}^3)$ determine symmetries of the hamilton equations; in the same way, the transformations of $\mathbb{R}^4$, defined as (or generated by) symmetries of the complex structure Φ, determine symmetries of Maxwell equations.

3.3. Extended Electrodynamics

Extended Electrodynamics (EED) [5] was brought to life in pursue of a theoretical ability to describe mathematically *time-stable finite field configurations* having the basic features of finite photon-like field objects, i.e., to have a consistent translational-rotational propagation, to carry finite energy-momentum and spin-momentum, the translational component of propagation to follow straight line and to have velocity of light, to have polarization properties, etc. From formal point of view EED extends Maxwell vacuum equations $\mathrm{d}F = 0$, $\mathrm{d} * F = 0$ to nonlinear vacuum equations. In standard Minkowski space terms this extension looks as follows

$$F \wedge *\mathrm{d}F = 0, \quad (*F) \wedge *\mathrm{d} * F = 0, \quad F \wedge *\mathrm{d} * F + (*F) \wedge *\mathrm{d}F = 0 \quad (32)$$

or in terms of the coderivative δ

$$\delta F \wedge *F = 0, \qquad (\delta * F) \wedge F = 0, \qquad \delta F \wedge F - (\delta * F) \wedge (*F) = 0.$$

In order to make use of the introduced through Φ in the previous subsection operators we first recall [10] that the insertion operator $i(x)$ may be extended to an insertion operator $i(t) : \alpha \rightarrow i(t)\alpha$, where t is a q-vector, α is a p-form, $i(t)\alpha$ is a $(p - q)$-form, and $q \leq p$. For the decomposable case $t = x_1 \wedge x_2 \wedge \cdots \wedge x_q$ this extension is defined by

$$i(t)\alpha = i(x_p)_\circ \ldots_\circ i(x_1)\alpha$$

and in the nondecomposable case it is extended by linearity. In components we obtain

$$(i(t)\alpha)_{i_1...i_{p-q}} = t^{k_1...k_q}\alpha_{k_1...k_q i_1...i_{p-q}}, \quad k_1 < k_2 < \cdots < k_p, \quad i_1 < \cdots < i_{p-q}$$

where summation over $k_1, \ldots, k_q$ is understood.

In components equations (28) are given by

$$F^{\mu\nu}(\mathrm{d}F)_{\mu\nu\sigma} = 0, \qquad (*F)^{\mu\nu}(\mathrm{d}*F)_{\mu\nu\sigma} = 0$$
$$F^{\mu\nu}(\mathrm{d}*F)_{\mu\nu\sigma} + (*F)^{\mu\nu}(\mathrm{d}F)_{\mu\nu\sigma} = 0, \qquad \mu < \nu$$

or in terms of the coderivative

$$(\delta * F)^{\mu}(*F)_{\mu\nu} = 0, \quad (\delta F)^{\mu}F_{\mu\nu} = 0, \quad (\delta * F)^{\mu}F_{\mu\nu} + (\delta F)^{\mu}(*F)_{\mu\nu} = 0.$$

Hence, we can make use of the operator $\mathfrak{D}$ to get $\mathfrak{D}F$ and $\mathfrak{D}\Phi F = -\mathfrak{P}_{\circ}\Phi_{\circ}\Phi F = \mathfrak{P}F$, and then we form the corresponding products. So, the EED vacuum equations (28) look as follows in these terms

$$i(\mathfrak{D}F)\mathrm{d}F = 0, \quad i(\mathfrak{P}F)\mathrm{d}(\Phi F) = 0, \quad i(\mathfrak{D}F)\mathrm{d}(\Phi F) + i(\mathfrak{P}F)\mathrm{d}F = 0. \quad (33)$$

Hence, EED vacuum equations also do *not* need and do *not* use pseudoeuclidean metric.

We are going to give one more formulation of the EED vacuum equations, and to this end we first extend the Lie derivative operator. Usually the Lie derivative is defined with respect to a given (but arbitrary) vector field and, considered in the frame of differential forms, it is given by $L_X\alpha = i(X)\mathrm{d}\alpha + \mathrm{d}i(X)\alpha$, where X is a vector field and α is a p-form. Thus, $L_X\alpha$ describes how α changes along the trajectories of X. A natural question arises: is it possible to define a "generalized Lie derivative", which more or less would describe the *simultaneous* change of α along several vector fields? In order to answer (although in some extent) positively to this question we make the following construction.

Let $X_1, X_2, \ldots, X_q$ be vector fields on an n-manifold M, Then we have the q-vector $T = X_1 \wedge X_2 \wedge \cdots \wedge X_q$. Let α be a p-form on M and $q \leq p$. Then the generalized Lie derivative $\mathfrak{L}_T\alpha$ of α along the q-vector T is defined by

$$\mathfrak{L}_T\alpha = i(T)\mathrm{d}\alpha + \mathrm{d}i(T)\alpha. \quad (34)$$

As in the usual case we always have commutation with d

$$\mathfrak{L}_T\, \mathrm{d} = \mathrm{d}\,\mathfrak{L}_T.$$

Of course, the above definition may be immediately extended by linearity and used for any two q-vector and p-form if $q \leq p$, and, if $q > p$, it seems naturally to put $\mathfrak{L}_T \alpha \equiv 0$. If the vector fields $X_1, \ldots, X_q$ define an integrable distribution, and the Pfaff one-forms $\alpha^1, \alpha^2, \ldots, \alpha^p$ define the corresponding integrable co-distribution, $\langle \alpha^i, X_j \rangle = 0$, $q + p = n$, then

$$\mathfrak{L}_T(\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p) = 0.$$

Now the EED vacuum equations (28) are given by

$$\mathfrak{L}_{\mathfrak{D}F}F = 0, \qquad \mathfrak{L}_{\mathfrak{P}F}\Phi F = 0, \qquad \mathfrak{L}_{\mathfrak{D}F}\Phi F + \mathfrak{L}_{\mathfrak{P}F}F = 0$$
$$i(\mathfrak{D}F)F = 0, \qquad i(\mathfrak{D}F)\Phi F = 0$$

where the last two equations require zero values of the two well known invariants I_1 and I_2.

4. Conclusion

Here we are going to mention those points of the paper which from our point of view seem most important.

The duality properties of the solutions to Maxwell equations reveal the internal structure of the field as having *two vector components*, which are *differentially interrelated* (through the equations), but *algebraically distinguished*.

From the point of view advocated in this paper, an adequate understanding of these duality properties of the solutions to Maxwell equations *requires explicitly introduced complex structure* in the equations. This brought us to make use of $\mathbb{R}^2$-valued differential forms, ω and Ω, as mathematical model objects of the electromagnetic field. This is the *first important step* towards further analysis.

In the traditional three-dimensional formulation of the theory it is important to introduce the canonical complex structure $\mathcal{I}$ of $\mathbb{R}^2$ in the equations in order to be able to represent *any* solution in *any* basis of $\mathbb{R}^2$, the presence of $\mathcal{I}$ makes it possible. Having this at hand we showed that the unimodular symmetries of $\mathcal{I}$, i.e., the two-dimensional rotations, are in a close connection with the conservative properties of the energy-momentum.

Much more interesting and fruitful turned out to be the four-dimensional formulation. Combining the first couple (1) of Maxwell equations into one relation through the appropriately constructed two-form F, $\mathrm{d}F = 0$, made somewhere in the beginning of twentieth century, we consider as a great achievement in

field theory, because no new or extra objects are needed to formulate this couple of equations. The exterior derivative d commutes with every transformation $f : \mathbb{R}^4 \to \mathbb{R}^4$, so, any such f-symmetry of the whole Maxwell system of equations shall be determined by the second couple (2) of equations. The particular symmetry $(\mathbf{E}, \mathbf{B}) \to (-\mathbf{B}, \mathbf{E})$ interchanges the two couples of equations, so, the second couple (2) of equations should also be possible to be cast into the form of $\mathrm{d}F' = 0$. The explicit form of F' is determined entirely by the equations: they require and make use of the linear map Φ, as given in the coordinate basis of $\Lambda^2(T^*\mathbb{R}^4)$ by the matrix (19), and satisfying the condition $\Phi_\circ \Phi = -\mathrm{id}_{\Lambda^2(T^*\mathbb{R}^4)}$, which is the *defining condition* for a linear map to be a *complex structure*. Hence, the *only* external structure used by Maxwell equations in their four-dimensional formulation, is the complex structure Φ, *no pseudoeuclidean metric is needed*. Therefore, all symmetries of these equations coming from transformations f of the base manifold $\mathbb{R}^4$ have to be symmetries of Φ. These symmetries f of Φ include translations, spatial rotations, space-time (Lorentz) rotations, dilatations, and (nonlinear space-time) special conformal transformations. The physical interpretation of a Lorentz rotation as a "change of frame" requires the *same* value of the constant c in the time coordinate $\xi = ct$ with respect to the various inertial frames.

We showed further that, combining appropriately Φ with the naturally existing Poincaré isomorphism $\mathfrak{P}$, makes possible to produce all pseudometric structures h^p needed in the theory. Hence, all these pseudometric structures are *secondary* objects. Therefore, in our view, they should *not* be preliminary introduced and considered as necessary (as it is usually done in many textbooks). The Poincaré-Einstein relativity principle privileges only a *part* of the known symmetries of Φ and says *nothing* about the rest part of the symmetries of Φ.

It was further shown that the equations of Extended Electrodynamics *also* do *not* need pseudoeuclidean metric structures, although their nonlinear character. Moreover, EED suggested a natural generalization of the Lie derivative, describing explicitly how a p-form changes along a given q-tensor , $q \leq p$. This generalization was used to give a new form if the EED vacuum equations, and this last form recalls very much "symmetry conditions".

Finally we note that, having at hand the bilinear forms h^p, the four-dimensional formulation of CED and EED in presence of external fields (charges, currents, media) is straightforward, no special efforts are needed.

In conclusion we shall state once again our view, based on the above considerations:

The only external structure required by CED and EED in their four-dimensional formulations on $\mathbb{R}^4$ is the complex structure Φ. All pseudometric structures arise as secondary objects and may be used when needed, but they should not be considered as beforehand necessary structures in electrodynamics.

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