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ON MATRIX SOLVABLE CALOGERO MODELS OF B_2 TYPE

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Abstract. We use a method developed earlier to construct new matrix solvable models. We apply this method to the model of the B_2 type and obtain new solvable model of the system with matrix potential.

1. Introduction

Let us consider an eigenvalue problem

$$\mathbf{H}\psi = E\psi$$

where $\mathbf{H}$ is the differential operator of two variables

$$\mathbf{H} = \partial_{11} + \partial_{22} - \mathbf{U}(x_1, x_2) \tag{1}$$

and $\mathbf{U}(x_1, x_2)$ is, say, two by two matrix (we denote $\partial_k \equiv \dfrac{\partial}{\partial x_k}$). If we transform the operator (1) by using similarity transformation $\widehat{\mathbf{H}} = \mathbf{G}^{-1}\mathbf{H}\mathbf{G}$ and transformation of variables $y_1 = y_1(x_1, x_2)$, $y_2 = y_2(x_1, x_2)$ to the form

$$\widehat{\mathbf{H}} = g^{rs}(y)\partial_{rs} + 2\mathbf{b}^r(y)\partial_r + \mathbf{V}(y)$$

(here we of course differentiate with respect to y), for which we know infinite flag of finite dimensional invariant subspaces, it is possible to find the spectrum of $\widehat{\mathbf{H}}$ (and hence $\mathbf{H}$) by diagonalizing $\widehat{\mathbf{H}}$ on these subspaces using standard algebraic methods.

The basic idea used in this paper is to reverse such a process, i.e., to start from the operator $\widehat{\mathbf{H}}$ for which we know invariant subspaces and try to reconstruct the operator $\mathbf{H}$ so that the matrix potential $\mathbf{U}(x_1, x_2)$ is symmetric. There exist some necessary conditions on g^{rs}, $\mathbf{b}^r$ and $\mathbf{V}$, which ensure the existence of the operator $\mathbf{H}$. The detailed description of these conditions can be found in [1].

2. Construction of the New Model

Let us start with the case where

$$g^{11} = g^{22} = y_1, \qquad g^{12} = g^{21} = y_2$$

and the transformation of variables is determined by

$$y_1 = \tfrac{1}{4}\,(x_1^2 + x_2^2), \qquad y_2 = \tfrac{1}{4}\,(x_1^2 - x_2^2)$$

which is regular on the set where $x_1 > 0$ and $x_2 > 0$.

Because the part $g^{rs}(y)\partial_{rs}$ of the operator $\widehat{\mathbf{H}}$, which contains the second derivatives, preserves for each $N \in \mathbb{N}$ finite dimensional spaces V_N generated by the polynomials of the form $y_1^{n_1} y_2^{2n_2}$, where $n_1 + 2n_2 \leq N$, we can choose $\mathbf{V}$ to be a constant matrix and $\mathbf{b}^r$ to have the following form, so that they also preserve the spaces V_N

$$\mathbf{b}^1 = \mathbf{C}_0^1 + \mathbf{C}_1^1 y_1, \qquad \mathbf{b}^2 = \mathbf{C}_3^2 y_2 + \frac{\mathbf{C}_0^2 + \mathbf{C}_1^2 y_1 + \mathbf{C}_2^2 y_1^2}{y_2}$$

where $\mathbf{C}_j^i$ are constant matrices.

According to [1] the compatibility conditions which have to be fulfilled by $\mathbf{b}^r$ are

$$\partial_s\big(\mathbf{b}_r + \tfrac{1}{2}\,\Gamma_r\big) - \partial_r\big(\mathbf{b}_s + \tfrac{1}{2}\,\Gamma_s\big) = \big[\mathbf{b}_r, \mathbf{b}_s\big] \tag{2}$$

where $\mathbf{b}_r = \sum_s g_{rs}\mathbf{b}^s$, g_{rs} is inverse matrix of g^{rs} and $\Gamma_t = \sum_{r,s} g^{rs}\Gamma_{rs,t}$ where $\Gamma_{rs,t}$ is the connection corresponding to the metric tensor g^{rs} defined by

$$\Gamma_{st,k} = \frac{1}{2}\big(-\partial_k g_{st} + \partial_s g_{tk} + \partial_t g_{sk}\big).$$

From the condition (2) it follows

$$\begin{aligned}
\big[\mathbf{C}_1^1, \mathbf{C}_1^2\big] + \big[\mathbf{C}_0^1, \mathbf{C}_2^2\big] &= -\mathbf{C}_2^2 & \qquad \big[\mathbf{C}_1^1, \mathbf{C}_3^2\big] &= 0 \\
\big[\mathbf{C}_0^1, \mathbf{C}_3^2\big] &= \mathbf{C}_1^1 - \mathbf{C}_3^2 & \qquad \big[\mathbf{C}_0^1, \mathbf{C}_0^2\big] &= \mathbf{C}_0^2 \\
\big[\mathbf{C}_0^1, \mathbf{C}_1^2\big] + \big[\mathbf{C}_1^1, \mathbf{C}_0^2\big] &= 0 & \qquad \big[\mathbf{C}_1^1, \mathbf{C}_2^2\big] &= 0.
\end{aligned} \tag{3}$$

It is difficult to find a general solution to the set of equations in (3). We choose

$$\begin{aligned}
\mathbf{C}_0^1 &= \rho + \sigma + \tfrac{1}{2} + \tfrac{1}{2}\,\mathbf{e}_0 & \qquad \mathbf{C}_0^2 &= A\mathbf{e}_{12} \\
\mathbf{C}_1^1 &= -2\omega & \qquad \mathbf{C}_1^2 &= \rho + \tfrac{1}{2}\,\lambda\,\mathbf{e}_0 \\
\mathbf{C}_3^2 &= -2\omega + \mathbf{e}_{21} & \qquad \mathbf{C}_2^2 &= C\mathbf{e}_{21}.
\end{aligned}$$

where ω, ρ, σ, λ, A and C are arbitrary constants and

$$\mathbf{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

From the knowledge of matrices $\mathbf{b}^1$ a $\mathbf{b}^2$ it is possible to find the transformation matrix $\mathbf{G}$ as a solution of the following differential equation

$$\partial_r \mathbf{G} = \mathbf{G}\left(\mathbf{b}_r + \tfrac{1}{2}\,\Gamma_r\right). \tag{4}$$

This turns out to be useful for obtaining symmetric potential and we choose the constants A and C such that

$$A = \frac{\lambda+1}{2}, \qquad C = -\frac{\lambda-1}{2}\cdot$$

In this case one of the solutions of the equation (4) is

$$\mathbf{G} = y_2^{\rho-1/2}\left(y_1^2 - y_2^2\right)^{(\sigma-1)/2} \mathrm{e}^{-2\omega y_1} \begin{pmatrix} y_1 y_2 & y_2 \\ (1-\lambda)y_1^2 - 2y_2^2 & -(1+\lambda)y_1 \end{pmatrix}.$$

When we come back to the coordinates x_1, x_2, the matrix $\mathbf{G}$ takes the form

$$\mathbf{G} = (x_1x_2)^{\sigma-1}(x_1^2 - x_2^2)^{\rho-1/2}\mathrm{e}^{-\omega(x_1^2+x_2^2)/2} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

where

$$G_{11} = \frac{1}{4}\,(x_1^2+x_2^2)(x_1^2-x_2^2), \qquad G_{12} = x_1^2 - x_2^2$$
$$G_{21} = -\frac{1}{8}\,(\lambda+1)(x_1^2+x_2^2)^2 + x_1^2x_2^2, \qquad G_{22} = -\frac{1}{2}\,(\lambda+1)(x_1^2+x_2^2).$$

In [1] it is shown how to compute the matrix potential $\mathbf{U}$ in (1) when the matrices $\mathbf{V}$ and $\mathbf{G}$ are known. For simplicity we choose $\mathbf{V}$ to be

$$\mathbf{V} = -2\omega(2\rho + 2\sigma + 1 + \mathbf{e}_0) + d\mathbf{e}_{21} \quad \text{where} \quad d = \frac{2(\lambda(\rho+\sigma) + 3\rho - \sigma)}{\lambda+1}\cdot$$

Finally we get the potential

$$\mathbf{U} = U_s + U_{11}\mathbf{e}_0 + U_{12}\mathbf{e}_{12} + U_{21}\mathbf{e}_{21}$$

where

$$
\begin{aligned}
U_s &= \omega^2(x_1^2 + x_2^2) + 2\left(\rho - \tfrac{1}{2}\right)^2 \left(\frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 + x_2)^2}\right) \\
&\quad + \left(\sigma^2 - \sigma + 1\right)\left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right) \\
U_{11} &= (2\rho - 1)\left(\frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 + x_2)^2} - \frac{1}{x_1^2} - \frac{1}{x_2^2}\right) \\
U_{12} &= \frac{4(\rho - \sigma)}{\lambda + 1}\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right) \\
U_{21} &= -(\lambda + 1)(\sigma + \rho - 1)\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right).
\end{aligned}
$$

It is possible to choose the constant λ above so that the potential $\mathbf{U}$ becomes symmetric, e.g., when

$$
\lambda = -2\sqrt{\frac{\sigma - \rho}{\sigma + \rho - 1}} - 1
$$

we have

$$
U_{12} = U_{21} = 2\sqrt{(\sigma - \rho)(\sigma + \rho - 1)}\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right).
$$

3. Conclusion

Multi-component Calogero models of B_2 type was studied also by Yamamoto. When we compare our model to the one mentioned in his paper [3], we conclude that the models differ and can not be transformed one into another.

We now work on the generalization of this model to $N \times N$ matrix potential and generally to n variables. We suppose this generalization will appear in one of incoming papers. The interesting question is also to explore hidden Lie algebra symmetry for this model, as it is known in the scalar case, see [2].

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