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ACCARDI COMPLEMENTARITY IN μ-DEFORMED QUANTUM MECHANICS

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Abstract. In this note we show that the momentum and position operators of μ-deformed quantum mechanics for $\mu > 0$ are not Accardi complementary in a sense that we will define. We conjecture that this is also true if $-1/2 < \mu < 0$.

1. Introduction

We begin by reviewing some basics of μ-deformed quantum mechanics. This comes from Rosenblum [8]. For recent related work see references [2], [6] and [7]. We consider a deformation of quantum mechanics depending on a parameter $\mu > -1/2$, which will be fixed throughout this discussion. We work in the complex Hilbert space $L^2(\mathbb{R}, m_\mu)$, where the measure m_μ for $x \in \mathbb{R}$ (the real line) is given by $\mathrm{d}m_\mu(x) := [2^{\mu+1/2}\Gamma(\mu+1/2)]^{-1}|x|^{2\mu}\mathrm{d}x$. Here $\mathrm{d}x$ is Lebesgue measure on $\mathbb{R}$ and Γ is the Euler gamma function. (The normalization constant will be explained later.) In this Hilbert space we have two unbounded self-adjoint operators: Q_μ, the μ-deformed position operator, and P_μ, the μ-deformed momentum operator. They are defined for $x \in \mathbb{R}$ and certain elements $\psi \in L^2(\mathbb{R}, m_\mu)$ by

$$Q_\mu\psi(x) := x\psi(x)$$
$$P_\mu\psi(x) := \frac{1}{\mathrm{i}}\left(\psi'(x) + \frac{2\mu}{x}(\psi(x) - \psi(-x))\right).$$

We omit details about exact domains of definition. Interest in these operators originates in Wigner [9] where equivalent forms of them are used as examples of operators that do not satisfy the usual canonical commutation relation in spite of the fact that they do satisfy the equations of motion $[H_\mu, Q_\mu] = P_\mu$ and $[H_\mu, P_\mu] = -Q_\mu$ for the Hamiltonian $H_\mu := \frac{1}{2}(Q_\mu^2 + P_\mu^2)$. What does hold is the μ-deformed canonical commutation relation: $\mathrm{i}[P_\mu, Q_\mu] = I + 2\mu J$, where I is the identity operator and J is the parity operator $J\psi(x) := \psi(-x)$.

Many concepts from ordinary analysis also have μ-deformations. This material also comes form Rosenblum [8]. We start with a μ-deformed factorial function $\gamma_\mu(n)$ defined recursively for integers $n \geq 0$ by $\gamma_\mu(0) := 1$ and

$$\gamma_\mu(n) := (n + 2\mu\theta(n))\gamma_\mu(n-1)$$

for $n \geq 1$. Here θ is the characteristic function of the odd integers. Using this, we define a μ-deformed exponential function $\exp_\mu(z)$ for $z \in \mathbb{C}$ by

$$\exp_\mu(z) := \sum_{n=0}^{\infty} \frac{1}{\gamma_\mu(n)} z^n.$$

This can be shown to be a holomorphic (entire) function of z. Next, we define a μ-deformed Fourier transform $\mathcal{F}_\mu$ by

$$\mathcal{F}_\mu \psi(k) := \int_{\mathbb{R}} \mathrm{d}m_\mu(x) \exp_\mu(-\mathrm{i}kx)\psi(x)$$

for $k \in \mathbb{R}$ and $\psi \in L^1(\mathbb{R}, m_\mu)$. In analogy with the well-known case when $\mu = 0$, this can be shown to define uniquely a unitary onto transform at the level of L^2 spaces, that is $\mathcal{F}_\mu : L^2(\mathbb{R}, m_\mu) \to L^2(\mathbb{R}, m_\mu)$ is an isomorphism of Hilbert spaces. Given the formula for $\mathcal{F}_\mu$, this unitarity condition fixes the normalization constant in the definition of m_μ.

In [1] Accardi introduced a definition of complementary observables in quantum mechanics. We now generalize that definition to the current context. We use the usual identification of observables in quantum mechanics as self-adjoint operators acting in some Hilbert space.

Definition 1. *We say that two self-adjoint operators S and T acting in $L^2(\mathbb{R}, m_\mu)$ are* Accardi complementary *if for any pair of bounded Borel subsets A and B of $\mathbb{R}$ we have that the operator $E^S(A)E^T(B)$ is trace class with trace given by*

$$\mathrm{Tr}\left(E^S(A)E^T(B)\right) = m_\mu(A)m_\mu(B).$$

Here E^S is the projection-valued measure on $\mathbb{R}$ associated with the self-adjoint operator S by the spectral theorem, and similarly for E^T. So, $E^S(A)E^T(B)$ is clearly a bounded operator acting on $L^2(\mathbb{R}, m_\mu)$. But whether it is also a trace class is another matter. And, given that it is a trace class, it is a further matter to determine if the trace can be written as the product of two measures, as indicated. Accardi's result in [1] (which is also discussed in detail and proved in [3]) is that $Q \equiv Q_0$ and $P \equiv P_0$ are Accardi complementary. Accardi also conjectured that this property of Q and P characterized this pair of operators acting on $L^2(\mathbb{R}, m_0)$. It turns out (see [3]) that this is not so.

2. The Main Result

We now ask whether the operators Q_μ and P_μ are Accardi complementary. To begin this analysis, we will use the following intertwining relation between these operators given by the μ-deformed Fourier transform $\mathcal{F}_\mu$, which is proved in [8]: $P_\mu = \mathcal{F}_\mu^* Q_\mu \mathcal{F}_\mu$. This implies the corresponding intertwining relation between their associated projection valued measures, that is $E^{P_\mu}(B) = \mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu$ for every Borel subset B of $\mathbb{R}$. We wish to calculate the trace of $E^{Q_\mu}(A)E^{P_\mu}(B) = E^{Q_\mu}(A)\mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu$, where A and B are bounded, Borel subsets of $\mathbb{R}$. To aid us we define an auxiliary operator

$$K := M_e \mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu M_e \ : \ L^2(\mathbb{R}, m_\mu) \to L^2(\mathbb{R}, m_\mu)$$

where $(M_e\psi)(x) := e(x)\psi(x)$ is the multiplication operator by any C^∞ function of compact support $e : \mathbb{R} \to \mathbb{R}$ satisfying $e(x) = 1$ for all $x \in A$. Such a function exists since A is bounded. Note that K depends on A, B and the choice of the function e.

We will now calculate the action of K on $\psi \in L^2(\mathbb{R}, m_\mu)$. We let $\chi_B : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of B. So for any $x \in \mathbb{R}$ we have that

$$\begin{aligned}
K\psi(x) &= (M_e \mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu M_e\psi)(x) = e(x)(\mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu M_e\psi)(x) \\
&= e(x) \int_{\mathbb{R}} \mathrm{d}m_\mu(k) \exp_\mu(\mathrm{i}kx) \left(E^{Q_\mu}(B)\mathcal{F}_\mu M_e\psi\right)(k) \\
&= e(x) \int_{\mathbb{R}} \mathrm{d}m_\mu(k) \exp_\mu(\mathrm{i}kx)\chi_B(k) \left(\mathcal{F}_\mu M_e\psi\right)(k) \\
&= e(x) \int_{\mathbb{R}} \mathrm{d}m_\mu(k) \exp_\mu(\mathrm{i}kx)\chi_B(k) \int_{\mathbb{R}} \mathrm{d}m_\mu(y) \exp_\mu(-\mathrm{i}ky) \left(M_e\psi\right)(y) \\
&= e(x) \int_{\mathbb{R}} \mathrm{d}m_\mu(k) \exp_\mu(\mathrm{i}kx)\chi_B(k) \int_{\mathbb{R}} \mathrm{d}m_\mu(y) \exp_\mu(-\mathrm{i}ky) e(y)\psi(y) \\
&= \int_{\mathbb{R}} \mathrm{d}m_\mu(y) \left[\int_{\mathbb{R}} \mathrm{d}m_\mu(k) \exp_\mu(\mathrm{i}kx)\chi_B(k) \exp_\mu(-\mathrm{i}ky) e(x)e(y) \right] \psi(y).
\end{aligned}$$

This exhibits K as an integral kernel operator with kernel given by

$$K(x,y) := e(x)e(y) \int_{\mathbb{R}} \mathrm{d}m_\mu(k) \exp_\mu(\mathrm{i}kx)\chi_B(k) \exp_\mu(-\mathrm{i}ky)$$

for $x, y \in \mathbb{R}$. (We use the same symbol for the operator and its kernel and let context indicate the meaning.) Clearly, we have that K is C^∞ with compact support in $\mathbb{R} \times \mathbb{R}$. Moreover, on the diagonal we have

$$K(x,x) = e(x)^2 \int\limits_{\mathbb{R}} \mathrm{d}m_\mu(k) |\exp_\mu(\mathrm{i}kx)|^2 \chi_B(k) \geq 0.$$

We now can do the central calculation for A, B bounded Borel sets

$$\begin{aligned}
\operatorname{Tr}\big(E^{Q_\mu}(A)E^{P_\mu}(B)\big) &= \operatorname{Tr}\big(E^{Q_\mu}(A)\mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu\big) \\
&= \operatorname{Tr}\big(E^{Q_\mu}(A)\mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu E^{Q_\mu}(A)\big) \\
&= \operatorname{Tr}\big(E^{Q_\mu}(A)M_e\mathcal{F}_\mu^* E^{Q_\mu}(B)\mathcal{F}_\mu M_e E^{Q_\mu}(A)\big) \\
&= \operatorname{Tr}\big(E^{Q_\mu}(A)K E^{Q_\mu}(A)\big) \\
&= \operatorname{Tr}\big(E^{Q_\mu}(A)K\big) = \int\limits_A \mathrm{d}m_\mu(x) K(x,x) \\
&= \int\limits_A \mathrm{d}m_\mu(x) e(x)^2 \int\limits_{\mathbb{R}} \mathrm{d}m_\mu(k) |\exp_\mu(\mathrm{i}kx)|^2 \chi_B(k) \\
&= \int\limits_A \mathrm{d}m_\mu(x) \int\limits_B \mathrm{d}m_\mu(k) |\exp_\mu(\mathrm{i}kx)|^2.
\end{aligned}$$

The step where we evaluated the trace by the (obvious) integral can be justified using Lemma 1 of [3], provided that $0 \notin \bar{A}$ (the closure of A) and e is chosen so that $0 \notin \operatorname{supp}(e)$. (Take $X = \mathbb{R} \setminus \{0\}$ in [3], so that K has compact support in $X \times X$ and the density of m_μ in X is C^∞ and strictly positive. Lemma 1 in [3] also asserts that $E^{Q_\mu}(A)K$ is trace class.) Of course, we find Accardi's result as the special case $\mu = 0$ of this formula, since then the integrand is identically equal to 1, and so the right hand side reduces to $m_0(A)m_0(B)$. (When $\mu = 0$, the technical hypothesis $0 \notin \bar{A}$ is not needed.)

We are now ready to state our main result.

Theorem 2. *Let A and B be bounded Borel subsets of $\mathbb{R}$ with $0 \notin \bar{A}$. Then $E^{Q_\mu}(A)E^{P_\mu}(B)$ is a trace class operator in $L^2(\mathbb{R}, m_\mu)$ for any $\mu > -1/2$ with*

$$0 \leq \operatorname{Tr}\big(E^{Q_\mu}(A)E^{P_\mu}(B)\big) = \int\limits_A \mathrm{d}m_\mu(x) \int\limits_B \mathrm{d}m_\mu(k) |\exp_\mu(\mathrm{i}kx)|^2 < \infty. \quad (1)$$

Moreover, if $\mu > 0$ and $m_\mu(A) \neq 0 \neq m_\mu(B)$, then we have that

$$\operatorname{Tr}\big(E^{Q_\mu}(A)E^{P_\mu}(B)\big) < m_\mu(A)m_\mu(B). \quad (2)$$

In particular, the operators Q_μ and P_μ are not Accardi complementary if $\mu > 0$.

Proof: We have shown the equality in (1), so we only have to show that the integral is finite. But this follows since the integrand is continuous and the domain of integration is bounded.

We next claim that $|\exp_\mu(\mathrm{i}kx)| \leq 1$ for $\mu > 0$ and that this inequality is strict if $kx \neq 0$. First for $\mu > 0$ note that $\exp_\mu(z) = \int_{-1}^{1} \mathrm{d}\eta_\mu(t)\mathrm{e}^{zt}$ for all $z \in \mathbb{C}$ by formula (2.3.5) in [8], where $\mathrm{d}\eta_\mu(t) = B(1/2, \mu)^{-1}(1-t)^{\mu-1}(1+t)^\mu \mathrm{d}t$ is a probability measure on $[-1, 1]$. Here the normalization constant involves $B(1/2, \mu)$, a value of the beta function (see [4]). Then, it follows that for all real $s \neq 0$ we have

$$
\begin{aligned}
|\exp_\mu(\mathrm{i}s)|^2 &= \left(\int_{-1}^{1} \mathrm{d}\eta_\mu(t)\cos(st)\right)^2 + \left(\int_{-1}^{1} \mathrm{d}\eta_\mu(t)\sin(st)\right)^2 \\
&< \int_{-1}^{1} \mathrm{d}\eta_\mu(t)\cos^2(st) + \int_{-1}^{1} \mathrm{d}\eta_\mu(t)\sin^2(st) = \int_{-1}^{1} \mathrm{d}\eta_\mu(t) = 1
\end{aligned} \tag{3}
$$

where the above inequality is an application of the strict form of Jensen's inequality [5], provided that the integrands are not constant. Clearly, $|\exp_\mu(\mathrm{i}s)| = 1$ if $s = 0$. But since $(A \times B) \setminus (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$ has positive $m_\mu \times m_\mu$ measure and $(A \times B) \cap (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$ has zero $m_\mu \times m_\mu$ measure, (2) now follows from (1) and (3). ∎

Given that there are other inequalities in μ-deformed analysis which hold in one direction for $\mu > 0$ and in the reverse direction when $-1/2 < \mu < 0$ and are equalities for $\mu = 0$, we conjecture that this holds here too, namely, that

$$
\mathrm{Tr}\left(E^{Q_\mu}(A)E^{P_\mu}(B)\right) > m_\mu(A)m_\mu(B) \tag{4}
$$

for A, B bounded Borel sets of positive m_μ measure and $-1/2 < \mu < 0$. If this is conjecture is true, then Q_μ and P_μ are not Accardi complementary for $-1/2 < \mu < 0$. Of course, Accardi showed the case of equality for $\mu = 0$ in [1].

We suppose that the technical hypothesis $0 \notin \bar{A}$ in this theorem can be dropped without changing the result.

For the rest of this note we would like to discuss the possibility of getting a more revealing formula for the integral in (1), for example something that would help us to prove the conjecture (4). Or can (1) be written in general as the product

$\nu_\mu(A)\nu_\mu(B)$ for *some* measure ν_μ? (This can be done, of course, for $\mu = 0$.) Therefore we wish to analyze the above integrand $|\exp_\mu(\mathrm{i}kx)|^2$ in the general case $\mu > -1/2$. First we introduce the following definitions from Rosenblum [8].

Definition 3. *The* μ**-deformed binomial coefficient** *is defined for all non-negative integers* k *and* j *by* $\left(\begin{array}{c} k \\ j \end{array}\right)_\mu := \frac{\gamma_\mu(k)}{\gamma_\mu(k-j)\gamma_\mu(j)}$. *The* k-th μ-deformed binomial polynomial *is defined by* $p_{k,\mu}(x,y) := \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right)_\mu x^j y^{k-j}$, *where* $x, y \in \mathbb{C}$.

Next we take $s \in \mathbb{R}$ and find that

$$\begin{aligned}
0 \leq |\exp_\mu(\mathrm{i}s)|^2 = \exp_\mu(\mathrm{i}s)\exp_\mu(-\mathrm{i}s) &= \sum_{l=0}^{\infty} \frac{1}{\gamma_\mu(l)} \mathrm{i}^l s^l \sum_{m=0}^{\infty} \frac{1}{\gamma_\mu(m)} (-\mathrm{i})^m s^m \\
&= \sum_{k=0}^{\infty} \frac{1}{\gamma_\mu(k)} \mathrm{i}^k s^k \sum_{m=0}^{k} \frac{\gamma_\mu(k)}{\gamma_\mu(k-m)\gamma_\mu(m)} (-1)^m \\
&= \sum_{k=0}^{\infty} \frac{1}{\gamma_\mu(k)} \mathrm{i}^k s^k \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array}\right)_\mu (-1)^m 1^{(k-m)} \\
&= \sum_{k=0}^{\infty} \frac{1}{\gamma_\mu(k)} \mathrm{i}^k s^k p_{k,\mu}(-1,1) = \sum_{j=0}^{\infty} \frac{1}{\gamma_\mu(2j)} \mathrm{i}^{2j} s^{2j} p_{2j,\mu}(-1,1) \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{\gamma_\mu(2j)} p_{2j,\mu}(-1,1) s^{2j}.
\end{aligned}$$

We used here the identity $p_{k,\mu}(-1,1) = 0$ for k odd.
Substituting this formula into the result for the trace we obtain

$$\begin{aligned}
\mathrm{Tr}\left(E^{Q_\mu}(A)E^{P_\mu}(B)\right) &= \int_A \mathrm{d}m_\mu(x) \int_B \mathrm{d}m_\mu(k) \sum_{j=0}^{\infty} \frac{(-1)^j}{\gamma_\mu(2j)} p_{2j,\mu}(-1,1)(kx)^{2j} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{\gamma_\mu(2j)} p_{2j,\mu}(-1,1) \left(\int_A \mathrm{d}m_\mu(x) x^{2j}\right) \left(\int_B \mathrm{d}m_\mu(k) k^{2j}\right).
\end{aligned}$$

We note the following formulas for the μ-deformed binomial polynomials:

$$\begin{aligned} p_{0,\mu}(-1,1) &= 1 \\ p_{2n-1,\mu}(-1,1) &= 0 \\ p_{4n-2,\mu}(-1,1) &= \mu\,\frac{2^{2n-1}\prod_{k=n+1}^{2n-1}(\mu+k-1)}{\prod_{k=1}^{n}(\mu+k-1/2)} \\ p_{4n,\mu}(-1,1) &= \mu\,\frac{2^{2n}\prod_{k=n+1}^{2n-1}(\mu+k)}{\prod_{k=1}^{n}(\mu+k-1/2)} \\ p_{2n,\mu}(-1,1) &= \frac{2\mu}{n}\sum_{k=0}^{n-1}\left(\begin{array}{c} 2n \\ 2k+1 \end{array}\right)_{\mu}. \end{aligned}$$

In all of these $n \geq 1$ is an integer.

The first two are readily proved, and the next three we have checked empirically in a number of cases, and so we believe them to be true. However, we have not been able to use these to arrive at a more enlightening form of the integral (and hence the trace) in formula (1).

3. Conclusion

As a concluding remark, we would like to draw attention again to the conjectured inequality (4) and its immediate consequence that Q_μ and P_μ are not Accardi complementary for $-1/2 < \mu < 0$.

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