JGSP **50** (2018) 79–95

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

2+1-MOULTON CONFIGURATION

NAOKO YOSHIMI AND AKIRA YOSHIOKA

Communicated by Jan J. Sławianowski

Abstract. We pose a new problem of collinear central configuration in Newtonian n-body problem. For a given two-body, we ask whether we can add a new body in a way such that i) the configuration of the total three-body is also collinear central with the configuration of the initial two-body being fixed and further ii) the initial two-body keeps its motion without any change during the process. We find three solutions to the above problem. We also consider a similar problem such that while the condition i) is satisfied but by modifying the condition ii) the motion of the initial two-body is not necessarily equal to the original one. We also find explicit solutions to the second problem.

MSC: 70F15
Keywords: celestial mechanics, collinear central configuration, three-body problem

1. Introduction

Leonard Euler had found the first solution of the three-body problem on a line, the collinear three-body problem [2]. In general, solutions of the n-body problem on a line, called a collinear n-body problem, become *collinear central configuration*, that is, the ratios of the distances of the bodies from the center of mass are constants. Moulton [5] proved that for a fixed mass vector $\mathbf{m} = (m_1, \ldots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration $\mathbf{q} = (q_1, \ldots, q_n)$ with mass $\mathbf{m} = (m_1, \ldots, m_n)$ (up to translation and scaling), where q_i denotes the position of the i th-body on a line $i = 1, \ldots, n$. The configuration is called a *Moulton Configuration*, which will be abbreviated as MC.

In this paper, we consider the following problem. We assume we are given a MC $\mathbf{q}_{_A} = (q_{_{A_1}}, q_{_{A_2}})$ of two bodies A_1, A_2 such that $q_{_{A_1}} < q_{_{A_2}}$ with mass $\mathbf{m}_{_A} = (m_{_{A_1}}, m_{_{A_2}})$. We consider to add a body B of position $q_{_B}$ with mass $m_{_B}$, to A_1, A_2 on the same line containing A_1, A_2 so that i) the configuration of A_1, A_2 and B is MC with the configuration of the initial two-body being fixed and ii) the motion of A_1, A_2 are kept invariant during the process. More precisely, let q_i denote one

doi: 10.7546/jgsp-50-2018-79-95

of the positions of A_1, A_2, B such that $q_1 < q_2 < q_3$ and m_i denote its mass, respectively,

Definition 1 (2+1-Moulton Configuration, Fig. 1). *We call* $\mathbf{q} = (q_1, q_2, q_3)$ *with mass vector* $\mathbf{m} = (m_1, m_2, m_3)$ *a* 2+1-Moulton Configuration *for two bodies* A_1, A_2 *when it satisfies the following conditions*:

i) A_1*,* A_2 *and* B *are in Moulton Configuration and the configuration of* A_1*,* A_2 *is equal to the original one* $\mathbf{q}_A$ *with* $\mathbf{m}_A$*.*
ii) *The center of mass and the angular velocity (see for details,* Definition 5 *below) of* A_1*,* A_2*,* B *are equal to the initial ones given by* A_1*,* A_2*, respectively.*

Figure 1. 2+1-Moulton Configuration.

Then we show in this paper

Theorem 2. *For a given* Moulton Configuration $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ *with* $\mathbf{m}_A = (m_{A_1}, m_{A_2})$

i) *there exist three* 2+1-Moulton Configurations *for* $\mathbf{q}_A$ *with* $\mathbf{m}_A$
ii) *the mass of the added body is zero.*

We also consider the situation such that only the condition i) of Definition 1 is satisfied, namely A_1, A_2 and B is in Moulton Configuration and the configuration of A_1, A_2 is the same as the original one, while their center of mass and the motion are not necessarily equal to the original ones.

Definition 3 (Weak-2+1-Moulton Configuration, Fig. 2)**.** *We call* $\mathbf{q}$ *with* $\mathbf{m}$ *a* weak-2+1-Moulton Configuration *for* $\mathbf{q}_A$ *with* $\mathbf{m}_A$ *when it satisfies only the condition* i) *of* Definition 1.

We also prove

Theorem 4. *Under the same assumption as in* Theorem 2*, we have the following.*

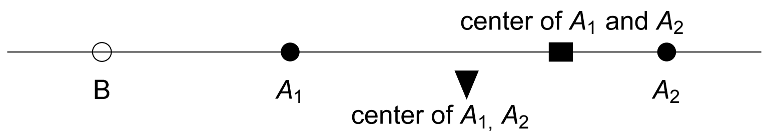

Figure 2. Weak-2+1-Moulton Configuration.

i) *When the masses of* A_1 *and* A_2 *are not equal, that is,* $m_{A_1} \neq m_{A_2}$*, we have intervals* $I_1 \subset (-\infty, q_{A_1})$*,* $I_2 \subset (q_{A_1}, q_{A_2})$ *and* $I_3 \subset (q_{A_2}, \infty)$ *such that if the position* q_B *of the added body* B *belongs to* I_i*,* $i = 1, 2, 3$*, the mass* m_B *is uniquely determined and is positive, and* B *gives a* weak-2+1-Moulton Configuration *for* $\mathbf{q}_A$ *with* $\mathbf{m}_A$*.*

ii) *When* $m_{A_1} = m_{A_2}$*, we have intervals* $I_1 \subset (-\infty, q_{A_1})$ *and* $I_3 \subset (q_{A_2}, \infty)$ *such that for every* $q_B \in I_i$*,* $i = 1, 3$*, we have the same results as in* i) *above. As to the interval* (q_{A_1}, q_{A_2})*, we also have a unique point* $q_B \in (q_{A_1}, q_{A_2})$ *and its positive mass* m_B *which is parametrized by angular momentum in some interval such that* B *gives a* weak-2+1-Moulton Configuration *for* $\mathbf{q}_A$ *with* $\mathbf{m}_A$*.*

In previous paper [6], the first author has considered 2+2-*Moulton Configuration* for two bodies and had obtained three solutions for a given $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ with $\mathbf{m}_A = (m_{A_1}, m_{A_2})$.

The present paper is organized as follows. In Section 2, we define a Moulton manifold and give its parametrization for $n = 2$ and 3. In Section 3, we prove Theorems 1, 2, and in Section 4 we present examples of 2+1-MC and weak-2+1-MC.

2. Manifold of Moulton Configurations

2.1. Collinear Central Configuration

We consider the Newtonian n-body problem

$$m_i \ddot{\mathbf{q}}_i(t) = \sum_{j=1\ i\neq j}^{n} \frac{m_i m_j (\mathbf{q}_j(t) - \mathbf{q}_i(t))}{\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\|^3} = \frac{\partial}{\partial \mathbf{q}_i} U(\mathbf{q}(t)), \qquad 1 \leq i \leq n \quad (1)$$

where $U(\mathbf{q})$ is the Newtonian potential function

$$U(\mathbf{q}) = \sum_{(i,j)\ i<j} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}, \qquad i, j = 1, \cdots, n$$

in which $m_i \in \mathbb{R}^+, i = 1, 2, \ldots, n$ are the masses of the bodies and $\mathbf{q}(t) = (\mathbf{q}_1(t), \ldots, \mathbf{q}_n(t)) \in (\mathbb{R}^d)^n$, $1 \leq d \leq 3$ is their configuration. Here we assume that $\mathbf{q}_i(t) \neq \mathbf{q}_j(t)$ for $i \neq j$.

It is well-known that the equation (1) is *scale and translation invariant*. That is, for a solution $\mathbf{q}(t) = (\mathbf{q}_1(t), \mathbf{q}_2(t), \ldots, \mathbf{q}_n(t))$ of (1), a vector-valued function $\kappa\mathbf{q}(\kappa^{-3/2}t) + \tilde{\mathbf{u}}t + \tilde{\mathbf{v}} = (\kappa\mathbf{q}_1(\kappa^{-3/2}t) + \mathbf{u}t + \mathbf{v}, \ldots, \kappa q_n(\kappa^{-3/2}t) + \mathbf{u}t + \mathbf{v})$ is also a solution, where κ is a positive constant and $\mathbf{u} = (u^1, \ldots, u^d)$, $\mathbf{v} = (v^1, \ldots, v^d)$ are constant d-vectors.

If we consider a solution of the form $\mathbf{q}(t) = \tilde{\mathbf{c}} + \phi(t)(\mathbf{q} - \tilde{\mathbf{c}})$, we easily see that $\mathbf{q}$ satisfies the equation (2) below, where $\phi(t)$ is a scalar-valued function, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in (\mathbb{R}^d)^n$ is a constant vector, $\mathbf{c}$ is the center of mass of the system $\mathbf{c} = \sum_{i=1}^n m_i\mathbf{q}_i / \sum_{i=1}^n m_i$ and $\tilde{\mathbf{c}}$ is its diagonal vector $\tilde{\mathbf{c}} = (\mathbf{c}, \ldots, \mathbf{c})$. Then we naturally obtain the following concept.

Definition 5 (Central Configuration [4, Section 2.1.3]**).** *We call a configuration* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in (\mathbb{R}^d)^n$ *with mass* $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in (\mathbb{R}^+)^n$ *a central configuration if* $\mathbf{q}$ *satisfies*

$$\sum_{j=1}^{n} \frac{m_j(\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} + \lambda(\mathbf{q}_i - \mathbf{c}) = \mathbf{0}, \qquad i = 1, 2, \ldots, n \tag{2}$$

for some $\lambda \in \mathbb{R}$*, where* $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ *is a distance of two bodies.*

We easily see that the equations (5) yields $\lambda = U(\mathbf{q})/(2I) > 0$, where $I = \sum_{i=1}^n m_i\|\mathbf{q}_i - \mathbf{c}\|^2/2$. Thus, the motion of n-body is determined by the position of the center of mass and $\phi(t)$, hence by λ. Here we remark λ represents the square of the angular velocity (see [5]).

Conversely, we see that for a central configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ with mass $\mathbf{m} = (m_1, \ldots, m_n)$ and a real valued function $\phi(t)$ satisfying $\ddot{\phi} = -\lambda\phi/|\phi|^3$, the curve $\mathbf{q}(t) = \tilde{\mathbf{c}} + \phi(t)(\mathbf{q} - \tilde{\mathbf{c}})$ is a solution of the equation (1).

The invariance of the equation (1) naturally induces an equivariance of the equation (2) under the scaling and parallel transform. Let κ be a positive number and $\mathbf{u}$ be a vector in $\mathbb{R}^d$. For a solution $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ of the equation (2), we set $\hat{\mathbf{q}} = (\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \ldots, \hat{\mathbf{q}}_n) = \kappa\mathbf{q} + \tilde{\mathbf{u}} = (\kappa\mathbf{q}_1 + \mathbf{u}, \ldots, \kappa\mathbf{q}_n + \mathbf{u}) \in (\mathbb{R}^d)^n$. Then $\hat{\mathbf{q}}$ satisfies

$$\sum_{j=1\ i\neq j}^{n} \frac{m_j(\hat{\mathbf{q}}_j - \hat{\mathbf{q}}_i)}{\|\hat{\mathbf{q}}_i - \hat{\mathbf{q}}_j\|^3} + \hat{\lambda}(\hat{\mathbf{q}}_i - \hat{\mathbf{c}}) = 0, \qquad i = 1, 2, \ldots, n$$

where $\hat{\lambda} = \kappa^{-3}\lambda$ and $\hat{\mathbf{c}} = (\kappa\mathbf{c} + \mathbf{u}, \ldots, \kappa\mathbf{c} + \mathbf{u})$.

Now we consider the case $d = 1$, which means that all bodies lie on a straight line, that is, collinear. Then we call a solution $\mathbf{q}$ of (2) a *collinear central configuration*, or a *Moulton Configuration*. Since the configuration of the bodies are collinear, the equation (2) is rewritten in the form

$$A\,{}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = {}^t\mathbf{0} \quad \text{for some } \lambda \in \mathbb{R} \tag{3}$$

where $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}^n$ and A is a skew-symmetric matrix defined by $A = (a_{ij}),\ a_{ij} = (q_i - q_j)^{-2}$ for $i < j$, and $a_{ii} = 0$, $a_{ji} = -a_{ij}$.

Remark 6. *It is known that any solution of two-body problem is always reduced to a collinear central configuration.*

2.2. Moulton Manifold

In this subsection, we consider the equation (3) of Moulton Configuration of n-bodies in a geometric way.

Let us consider a $2n+2$-dimensional Euclidean spase $\mathbb{R}^{2n+2}$ with coordinates $(\mathbf{q}, \lambda, c, \mathbf{m}) = (q_1, \dots, q_n, \lambda, c, m_1, \dots, m_n)$ and consider the open domain

$$\begin{aligned}\mathcal{O}_n = \{(q_1, \dots, q_n, &\lambda, c, m_1, \dots, m_n) \in \mathbb{R}^{2n+2};\\ &q_1 < q_2 < \cdots < q_n, \lambda > 0, q_1 < c < q_n, m_1, \dots, m_n > 0\}.\end{aligned}$$

Then the equation (3) defines manifold

$$\mathcal{M}_n = \{(\mathbf{q}, \lambda, c, \mathbf{m}) \in \mathcal{O}_n\,;\ A\,{}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = {}^t\mathbf{0}\}$$

called an *n-Moulton manifold*, which can be regarded as the set of all Moulton Configurations of n-bodies. The manifold $\mathcal{M}_n$ has a parametrization whose expression depends on the case where n is even or n is odd (cf. [1], [5]). In this paper we discuss the cases $n = 2$ and $n = 3$, and the general cases for $n = 2k$, $n = 2k+1$ is given in a similar way.

2-Moulton manifold. The equation (3) for $n = 2$ shows that 2-Moulton manifold is given by

$$\mathcal{M}_2 = \left\{(q_{A_1}, q_{A_2}, \lambda_A, c_A, m_{A_1}, m_{A_2}) \in \mathcal{O}_2\ ;\ \begin{pmatrix} m_{A_1} \\ m_{A_2} \end{pmatrix} = \frac{\lambda_A}{a_{12}} \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}\right\}.$$

We consider the open domain

$$\mathcal{D}_2 = \{(q_{A_1}, q_{A_2}, c_A, \lambda_A) \in \mathbb{R}^4\ ;\ q_{A_1} < c_A < q_{A_2}, \lambda_A > 0\}$$

and define the maps $m_{A_i} : \mathcal{D}_2 \to \mathbb{R}_+$, $m_{A_i} = m_{A_i}(q_{A_1}, q_{A_2}, \lambda_A, c_A)$, $i = 1, 2$

$$\begin{pmatrix} m_{A_1} \\ m_{A_2} \end{pmatrix} = \frac{\lambda_A}{a_{12}} \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}$$

which are naturally derived from (3). Then the 2-Moulton manifold $\mathcal{M}_2$ is given as the graph of the map $\mathbf{m}_A = (m_{A_1}, m_{A_2})$, i.e., $\mathcal{M}_2 = \mathbf{m}_A(\mathcal{D}_2)$.

The scale and translation invariance of the n-body problem naturally induces the action of a positive number κ and the real number μ on $\mathcal{M}_2$ as

$$(q_{A_1}, q_{A_2}, c_A, \lambda_A, m_{A_1}, m_{A_2}) \mapsto (\kappa q_{A_1} + \mu, \kappa q_{A_2} + \mu, \kappa^{-3}\lambda_A, \kappa c_A + \mu, m_{A_1}, m_{A_2}).$$

3-Moulton manifold. The parametrization of 3-Moulton manifold $\mathcal{M}_3$ is slightly different from $\mathcal{M}_2$. For $n = 3$, the matrix A in the equation (3) is not invertible. Regarding (3) as an equation with respect to $\mathbf{m} = (m_1, m_2, m_3)$ we consider an augmented matrix of the equation

$$\begin{pmatrix} 0 & a_{12} & a_{13} & -\lambda(q_1 - c) \\ -a_{12} & 0 & a_{23} & -\lambda(q_2 - c) \\ -a_{13} & -a_{23} & 0 & -\lambda(q_3 - c) \end{pmatrix}$$

where $a_{ij} = (q_i - q_j)^{-2}$ $(i < j)$ and we obtain by the sweep-out method

$$\begin{pmatrix} a_{12} & 0 & -a_{23} & -\lambda(q_2 - c) \\ 0 & a_{12} & a_{13} & -\lambda(q_1 - c) \\ 0 & 0 & 0 & * \end{pmatrix}$$

where $* = \lambda(-a_{12}(q_3 - c) + a_{13}(q_2 - c) - a_{23}(q_1 - c))$. Then the equation $* = 0$, that is

$$c = (a_{12}q_3 - a_{13}q_2 + a_{23}q_1)/P \tag{4}$$

where $P = a_{12} - a_{13} + a_{23}$ is the necessary and sufficient condition for equation (3) to have a solution. Thus the equation (3) reduces to the system

$$\begin{aligned} a_{12}m_2 + a_{13}m_3 + \lambda(q_1 - c) &= 0 \\ -a_{12}m_1 + a_{23}m_3 + \lambda(q_2 - c) &= 0. \end{aligned}$$

In order to parametrize the solutions (m_1, m_2, m_3), we introduce a parameter $M = m_1 + m_2 + m_3$ which represents the total mass. We consider the system

$$\begin{aligned} a_{12}m_2 + a_{13}m_3 + \lambda(q_1 - c) &= 0 \\ -a_{12}m_1 + a_{23}m_3 + \lambda(q_2 - c) &= 0 \\ m_1 + m_2 + m_3 &= M. \end{aligned}$$

Then the solution is unique for each M and using (4) we have

$$\begin{aligned} m_1 &= (a_{23}M + \lambda(q_2 - q_3))/P \\ m_2 &= (-a_{13}M + \lambda(q_3 - q_1))/P \\ m_3 &= (a_{12}M + \lambda(q_1 - q_2))/P. \end{aligned} \tag{5}$$

Thus, the parameter space of $\mathcal{M}_3$ can be taken as

$$\mathcal{D}_3 = \{(q_1, q_2, q_3, \lambda, M) \in \mathbb{R}^5 \ ; \ q_1 < q_2 < q_3, \ \lambda, M > 0\}$$

and a map

$$\mu : \mathcal{D}_3 \to \mathcal{M}_3, \qquad \mu(\mathbf{q}, \lambda, M) = (\mathbf{q}, \lambda, c(\mathbf{q}), \mathbf{m}(\mathbf{q}, \lambda, M))$$

where $\mathbf{m}(\mathbf{q}, \lambda, M) = (m_1, m_2, m_3)$ is given by (5) and $c(\mathbf{q})$ by (4), gives the parametrization of the 3-Moulton manifold $\mathcal{M}_3$.

3. Proof of Theorems

Now suppose we are given a two-body A_1, A_2 which is a Moulton Configuration $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ and mass $\mathbf{m}_A = (m_{A_1}, m_{A_2})$ such that $q_{A_1} < q_{A_2}, \ m_{A_1}, m_{A_2} > 0$. We consider to add a body B with a mass m_B in the same line of A_1, A_2 so that A_1, A_2 and B form a 2+1-Moulton Configuration for two bodies A_1 and A_2.

We set the distance of q_{A_1} and q_{A_2} is the unit such that $q_{A_2} - q_{A_1} = 1$ by scaling for simplicity and then the parametrization map is written as

$$\begin{pmatrix} m_{A_1} \\ m_{A_2} \end{pmatrix} = \lambda_A \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix} \tag{6}$$

because $a_{12} = (q_{A_2} - q_{A_1})^{-2} = 1$.

As a possibility, we have three distinct cases. Case 1: $q_B < q_{A_1} < q_{A_2}$, Case 2: $q_{A_1} < q_B < q_{A_2}$ and Case 3: $q_{A_1} < q_{A_2} < q_B$. We will present proofs for each case.

3.1. Case 1 and Case 3

Case 1. We set $(q_1, \ q_2, \ q_3) = (q_B, q_{A_1}, q_{A_2}), \ q_1 < q_2 < q_3$ with $(m_1, m_2, m_3) = (m_B, m_{A_1}, m_{A_2})$, see Fig. 3.

Figure 3. Case 1.

Now we consider the condition i) of Definition 1. Note that $q_{A_2} - q_{A_1} = 1$ by scaling. Then A_1, A_2, B are in MC, (5) gives

$$\begin{aligned} m_B &= (M-\lambda)/P \\ m_{A_1} &= -a_{13}M + \lambda(q_{A_2} - q_B))/P \\ m_{A_2} &= a_{12}M + \lambda(q_B - q_{A_1}))/P \end{aligned}$$

since $a_{23} = (q_{A_1} - q_{A_2})^{-2} = 1$, $q_2 - q_3 = q_{A_1} - q_{A_2} = -1$. The identity (4) shows c can be expressed as a function of q_B which is denoted by $c_1(q_B)$ such that

$$c = c_1(q_B) = (a_{12}(q_B)q_{A_2} - a_{13}(q_B)q_{A_1} + q_B)/P \tag{7}$$

where $a_{12} = (q_B - q_{A_1})^{-2}$, $a_{13} = (q_B - q_{A_2})^{-2}$, $P = a_{12} - a_{13} + a_{23} = a_{12} - a_{13} + 1$. Since the configuration of A_1, A_1 is equal to the original one, m_{A_1}, m_{A_2} satisfy the equation (6) and then we have the equation for M, λ, q_B such that

$$\begin{aligned} (-a_{13}M + \lambda(q_{A_2} - q_B))/P &= \lambda_A(q_{A_2} - c_A) \\ (a_{12}M + \lambda(q_B - q_{A_1}))/P &= \lambda_A(c_A - q_{A_1}) \end{aligned}$$

or

$$K_1 \begin{pmatrix} M \\ \lambda \end{pmatrix} = \lambda_A P \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}, \quad \text{where} \quad K_1 = \begin{pmatrix} -a_{13} & q_{A_2} - q_B \\ a_{12} & q_B - q_{A_1} \end{pmatrix}.$$

We have

$$\det K_1 = \Big(-a_{13}(q_B - q_{A_1}) - a_{12}(q_{A_2} - q_B)\Big) = \left(\frac{q_{A_1} - q_B}{(q_B - q_{A_2})^2} - \frac{q_{A_2} - q_B}{(q_B - q_{A_1})^2}\right) < 0$$

since $q_B < q_{A_1} < q_{A_2}$. Thus for each $q_B < q_{A_1}$, M, λ are expressed as functions of q_B such that

$$\begin{pmatrix} M \\ \lambda \end{pmatrix} = \frac{\lambda_A P}{\det K_1} \begin{pmatrix} q_B - q_{A_1} & q_B - q_{A_2} \\ -a_{12} & -a_{13} \end{pmatrix} \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}.$$

From the equation (7) we obtain $\det K_1 = P(q_B - c_1(q_B))$ and then we can write the above equation as

$$\lambda = \frac{\lambda_A P}{q_B - c_1(q_B)}(c_A - c_1(q_B)) + \lambda_A \frac{q_B - c_A}{q_B - c_1(q_B)} \tag{8}$$

$$M = \lambda_A \frac{q_B - c_A}{q_B - c_1(q_B)}. \tag{9}$$

Then we obtain

$$m_B = (M - \lambda)/P = -\frac{\lambda_A}{q_B - c_1(q_B)}(c_A - c_1(q_B)) = -\lambda_A \frac{c_1(q_B) - c_A}{c_1(q_B) - q_B}. \quad (10)$$

Now we consider the condition ii) of Definition 1. We investigate the equation $c_1(q_B) = c_A$. It is easy to see that

$$c_1(q_B) = q_{A_1} + \frac{a_{12}(q_B) - q_{A_1} + q_B}{P}.$$

Then since $q_{A_1} < c_A < q_{A_2}$, it is necessary that $q_{A_1} < c_1(q_B) < q_{A_2} = q_{A_1} + 1$ which is equivalent to the inequality $q_{A_1} - q_B < a_{12}$ and $a_{13} < q_{A_1} + 1 - q_B$ so that we obtain $q_{A_1} - 1 < q_B < q_{A_1}$. In this interval we have

$$\begin{aligned} c_1'(q_B) &= \frac{1}{P^2}\Big((a_{12}' + 1)P - (a_{12} + q_B - q_{A_1})P'\Big) \\ &= \frac{1}{P^2}\Big(a_{12}'(1 - a_{13} + q_{A_1} - q_B) + a_{13}'(a_{12} - (q_{A_1} - q_B)) + P\Big) > 0 \end{aligned}$$

because $a_{12}', a_{13}' > 0$, $1 - a_{13} > 0$, and $a_{12} - (q_{A_1} - q_B) > 0$ in $q_{A_1} - 1 < q_B < q_{A_1}$, $P > 0$, where $c_1'(q_B) = \mathrm{d}c_1(q_B)/\mathrm{d}q_B$. Then we have

Lemma 7. *There exists a unique q_B satisfying $c_1(q_B) = c_A$ in $(q_{A_1} - 1,\ q_{A_1})$.*

Proof: The function $c_1(q_B)$ is increasing monotonically. Since $\lim c_1(q_B)_{q_B \to q_{A_1} - 1} = q_{A_1}$ and $\lim c_1(q_B)_{q_B \to q_{A_1}} = q_{A_1} + 1$, there exists a unique solution $q_B = q_B^0$ which satisfies the equation $c_1(q_B) = c_A$. ■

Now we are in a position to give the proofs of our Theorems 2 and 4. The condition i) of Definition 1 is satisfied by (8), (9) for each $q_B < q_{A_1}$, and moreover, there exists unique $q_B = q_B^0 < q_{A_1}$ such that the center $c = c_1(q_B)$ is equal to c_A and also $\lambda = \lambda_A$ holds by (8), then the motion of the three bodies A_1, A_2 and B, hence its part A_1, A_2, is equal to the original one, and in addition, we have $m_B = 0$ by (10). Then for the Case 1 we obtain Theorem 2.

As to Theorem 2, we consider the following way. In addition to the fact that the condition i) of Definition 1 is satisfied by (8), (9) for each $q_B < q_{A_1}$, the equation (10) yields that if $q_B < q_B^0$, m_B is positive because $c_1(q_B) - q_B > 0$ and $c_1(q_B) - c_A < 0$. Thus, for all $q_B < q_B^0$, A_1, A_2, B give the weak-2+1 Moulton Configuration for two bodies A_1, A_2 where the mass of B is positive. Then we have Theorem 2 for the Case 1.

Figure 4. Case 3.

Case 3. We consider it in a parallel manner to the Case 1. Let $(q_1,\ q_2,\ q_3) = (q_{A_1}, q_{A_2}, q_B)$, $q_1 < q_2 < q_3$ with $(m_1,\ m_2,\ m_3) = (m_{A_1}, m_{A_2}, m_B)$. Then by scaling we have in (5), $a_{12} = (q_{A_1} - q_{A_2})^{-2} = 1$, $q_1 - q_2 = q_{A_1} - q_{A_2} = -1$. If A_1, A_2, B satisfy the condition i) of Definition 1, we have as well

$$\begin{aligned} m_{A_1} &= (a_{23}M + \lambda(q_{A_2} - q_B))/P \\ m_{A_2} &= (-a_{13}M + \lambda(q_B - q_{A_1}))/P \\ m_B &= (M - \lambda)/P \end{aligned}$$

and the equation (4) reads

$$c = c_3(q_B) = \frac{1}{P}\left(-a_{13}q_{A_2} + a_{23}q_{A_1} + q_B\right)$$

in which $a_{13} = (q_{A_1} - q_B)^{-2}$, $a_{23} = (q_{A_2} - q_B)^{-2}$, $P = a_{12} - a_{13} + a_{23} = a_{23} - a_{13} + 1$. Then similarly to Case 1 we have the following relation

$$\begin{aligned} (a_{23}M + \lambda(q_{A_2} - q_B))/P &= \lambda_A(q_{A_2} - c_A) \\ (-a_{13}M + \lambda(q_B - q_{A_1}))/P &= \lambda_A(c_A - q_{A_1}). \end{aligned}$$

Therefore, for each $q_{A_2} < q_B$, we obtain

$$\begin{pmatrix} M \\ \lambda \end{pmatrix} = \frac{\lambda_A P}{D_3}\begin{pmatrix} q_B - q_{A_1} & q_B - q_{A_2} \\ a_{13} & a_{23} \end{pmatrix}\begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}$$

where D_3 is a determinant of the coefficient matrix and we see $D_3 = a_{23}(q_B - q_{A_1}) + a_{13}(q_{A_2} - q_B) > 0$. We easily see that $D_3 = P(q_B - c_3(q_B))$. Then we have the following result similarly to the Case 1.

Lemma 8. *There exists a unique q_B in $(q_{A_2}, q_{A_2} + 1)$ satisfying $c_3(q_B) = c_A$.*

The proof is obtained by similar manner in Lemma 7 by considering

$$c_3(q_B) = q_{A_2} + \frac{q_B - a_{23}(q_B) - q_{A_2}}{P} \tag{11}$$

which is strictly increasing for $q_{A_2} < q_B < q_{A_2} + 1$. Moreover we have

$$\lambda = -\lambda_A P \frac{c_3(q_B) - c_A}{q_B - c_3(q_B)} + \lambda_A \frac{q_B - c_A}{q_B - c_3(q_B)}$$
$$M = \lambda_A \frac{q_B - c_A}{q_B - c_3(q_B)}.$$

and

$$m_B(q_B) = (M - \lambda)/P = \lambda_A \frac{c_3(q_B) - c_A}{q_B - c_3(q_B)}.$$

Thus $m_B(q_B) = 0$ for q_B satisfying $c_3(q_B) = c_A$. We also have $\lambda = \lambda_A$ when $c_3(q_B) = c_A$. Then we have Theorem 2 for Case 3.

We set $q_B = q_B^0 \in (q_{A_2}, q_{A_2} + 1)$ satisfying $c_3(q_B) = c_A$ given in Lemma 8. Therefore, if $q_B < q_B{}^0$, m_B is positive because $q_B - c_3(q_B) > 0$ for $q_{A_2} < q_B < q_{A_2} + 1$ and $c_3(q_B) - c_A > 0$ when $q_B < q_B{}^0$. Then similarly to the Case 1 we have Theorem 2 for Case 3.

3.2. Case 2

We set $(q_1,\ q_2,\ q_3) = (q_{A_1}, q_B, q_{A_2})$, $q_1 < q_2 < q_3$ with $(m_1, m_2, m_3) = (m_{A_1}, m_B, m_{A_2})$, see Fig. 5.

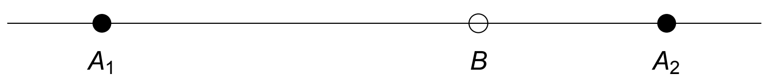

Figure 5. Case 2.

Now we consider the condition i) of Definition 1. Since A_1, B, A_2 are in Moulton Configuration, we have

$$m_{A_1} = a_{23}M + \lambda(q_B - q_{A_2}))/P$$
$$m_B = (\lambda - M)/P$$
$$m_{A_2} = a_{12}M + \lambda(q_{A_1} - q_B))/P$$

and (4) yields

$$c = c_2(q_B) = \frac{1}{P}\left(a_{12}q_{A_2} + a_{23}q_{A_1} - q_B\right) \tag{12}$$

where $a_{12} = (q_{A_1} - q_B)^{-2}$, $a_{23} = (q_B - q_{A_2})^{-2}$, $P = a_{12} - a_{13} + a_{23} = a_{12} + a_{23} - 1$. By the condition i) of Definition 1, the configuration of A_1, A_2 is the same as the original one and then we obtain the following

$$\begin{aligned} (a_{23}M + \lambda(q_B - q_{A_2}))/P &= \lambda_A(q_{A_2} - c_A) \\ (a_{12}M + \lambda(q_{A_1} - q_B))/P &= \lambda_A(c_A - q_{A_1}). \end{aligned} \tag{13}$$

Similarly to Case 1, we write the equation as

$$K_2 \begin{pmatrix} M \\ \lambda \end{pmatrix} = \lambda_A P \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}, \qquad K_2 = \begin{pmatrix} a_{23} & q_B - q_{A_2} \\ a_{12} & q_{A_1} - q_B \end{pmatrix}.$$

From equation (12) we obtain $\det K_2 = a_{23}(q_{A_1} - q_B) - a_{12}(q_B - q_{A_2}) = P(c_2(q_B) - q_B)$. We remark here that the difference from the previous cases, Case 2 is that one has a point where the determinant of the matrix K_2 vanishes.

The singular point is given as follows. By a direct calculation, we have

$$c_2(q_B) - q_B = -(q_B - q_A^0)\left(1 + \frac{a_{12}a_{23} + 1}{P}\right) \tag{14}$$

where q_A^0 is the midpoint of A_1 and A_2, namely, $q_A^0 = (q_{A_1} + q_{A_2})/2 = q_{A_1} + 1/2$. Here we set

$$R(q_B) = 1 + \frac{a_{12}a_{23} + 1}{P}$$

and this obviously satisfies $R(q_B) \geq 1$. Thus $\det K_2$ vanishes at $q_B = q_A^0$, and then for each q_B such that $q_{A_1} < q_B < q_{A_2}$ and $q_B \neq q_A^0$, the variables M, λ can be found as functions of q_B such that

$$\begin{pmatrix} M \\ \lambda \end{pmatrix} = \frac{\lambda_A P}{\det K_2} \begin{pmatrix} q_{A_1} - q_B & q_{A_2} - q_B \\ -a_{12} & a_{23} \end{pmatrix} \begin{pmatrix} q_{A_2} - c_A \\ c_A - q_{A_1} \end{pmatrix}$$

or

$$\begin{aligned} \lambda &= \lambda_A P \frac{c_A - c_2(q_B)}{c_2(q_B) - q_B} + \lambda_A \frac{c_A - q_B}{c_2(q_B) - q_B} \\ M &= \lambda_A \frac{c_A - q_B}{c_2(q_B) - q_B}. \end{aligned} \tag{15}$$

Then m_B is given as a function of q_B such that

$$m_B(q_B) = (\lambda - M)/P = \lambda_A \frac{c_A - c_2(q_B)}{c_2(q_B) - q_B} \tag{16}$$

for $q_{A_1} < q_B < q_{A_2}$ and $q_B \neq q_A^0$. Here by (14) we have also

$$\frac{c_A - c_2(q_B)}{c_2(q_B) - q_B} = \frac{1}{R(q_B)} \frac{c_2(q_B) - c_A}{q_B - q_A^0}.$$

Let us notice that $c_2(q_A^0) = q_A^0$, then $c_2(q_A^0) - c_A \neq 0$ if $q_A^0 \neq c_A$, hence m_B is divergent when $c_2(q_B) - q_B = 0$, or $q_B = q_A^0$. It is easy to see $c_A \neq q_A^0$ is equivalent to $m_{A_1} \neq m_{A_2}$.

Lemma 9. *$c_2(q_B)$ is a strictly monotone decreasing function of q_B, $q_{A_1} < q_B < q_{A_2}$ such that* $\lim_{q_B \to q_{A_1}} c_2(q_B) = q_{A_2}$, $\lim_{q_B \to q_{A_2}} c_2(q_B) = q_{A_1}$.

Proof: We have

$$c_2(q_B) = q_{A_1} + \frac{a_{12} - (q_B - q_{A_1})}{P}$$

which gives the limit of $c_2(q_B)$ for $q_B \to q_{A_1}$ and $q_B \to q_{A_2}$, respectively. We have also

$$c_2'(q_B) = \frac{1}{P^2}\Big(a_{12}'(a_{23} - 1 + q_B - q_{A_1}) - a_{23}'(a_{12} - (q_B - q_{A_1})) - P\Big).$$

Since

$$a_{12}' = -\frac{2}{(q_B - q_{A_1})^3} < 0, \; a_{23}' = -\frac{2}{(q_B - q_{A_1} - 1)^3} > 0$$

for $q_{A_1} < q_B < q_{A_1} + 1$, and $a_{23} - 1 + q_B - q_{A_1} > 0$, $a_{12} - (q_B - q_{A_1}) > 0$, $P > 0$, we obtain $c_2'(q_B) < 0$. ■

Since $q_{A_1} < c_A < q_{A_2}$, Lemma 9 gives the unique q_B^0 such that $q_{A_1} < q_B < q_{A_2}$ satisfying $c_2(q_B^0) = c_A$, and also shows that $q_B^0 \neq q_A^0$ when $c_A \neq q_A^0$.

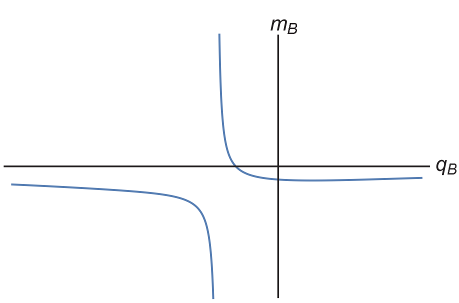

Figure 6. $q_B^0 \in (q_A^0, q_{A_2})$.

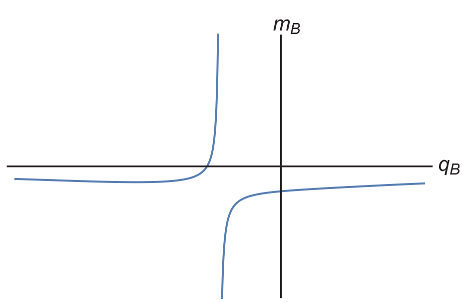

Figure 7. $q_B^0 \in (q_{A_1}, q_A^0)$.

Now we prove Theorems 2 and 4. Theorem 2 can be proven for the case $c_A \neq q_A^0$ in the following way. For each q_B such that $q_{A_1} < q_B < q_{A_2}$ and $q_B \neq q_A^0$, the condition i) of Definition 1 is satisfied since q_B gives λ, M by the equations (15). Moreover, the identity (16) yields, if $c_A \neq q_A^0$, that the graph of the function m_B is given in Fig. 6, for the case where $c_A < q_A^0$, and Fig. 7, for $c_A > q_A^0$, respectively, from which we obtain an interval where m_B is positive, and then follows Theorem 4.

Further we consider the condition ii) of Definition 1. It is easy to see that for $q_B = q_B^0$, it holds $c_2(q_B^0) = c_A$ and $\lambda = \lambda_A$ which yields that the motion is not changed, and then we obtain Theorem 1.

For the exceptional case $c_A = q_A^0$, we go back to the equation (13) and we have

$$
\begin{aligned}
(a_{23}M + \lambda(q_B - q_{A_2}))/P &= \lambda_A(q_{A_2} - c_A) = \lambda_A/2 \\
(a_{12}M + \lambda(q_{A_1} - q_B))/P &= \lambda_A(c_A - q_{A_1}) = \lambda_A/2.
\end{aligned}
$$

Then taking into account that $a_{23} - a_{12} = 2(q_B - q_A^0)a_{12}a_{23}$ we have

$$
0 = (a_{23} - a_{12})M + 2\lambda(q_B - q_A^0) = 2(q_B - q_A^0)(a_{12}a_{23}M + \lambda).
$$

Since $M, \lambda > 0$ we obtain $q_B = q_A^0$.

Remark that $c_2(q_B^0) = c_A = q_A^0$ and the monotone property of the function $c_2(q_B)$ shows $q_B^0 = q_A^0$. Then substituting $q_B = q_A^0$ gives $a_{12} = a_{23} = 4$ and $P = 7$. Then the equation (13) is equivalent to the relation

$$
M = \frac{1}{8}(\lambda + 7\lambda_A)
$$

which yields

$$
m_B = \frac{1}{8}(\lambda - \lambda_A).
$$

Thus, for the exceptional case, the configuration $(q_{A_1}, q_B^0, q_{A_2})$ with mass (m_{A_1}, m_B, m_{A_2}) is the weak-2+1-Moulton Configuration for two bodies A_1, A_2 and the mass of B is given as a function of λ such that $m_B = (\lambda - \lambda_A)/8$ and is positive for every $\lambda > \lambda_A$. Thus we obtain Theorem 4. Further putting $\lambda = \lambda_A$, we obtain $m_B = 0$ and we have the 2+1-Moulton Configuration $(q_{A_1}, q_B^0, q_{A_2})$ with (m_{A_1}, m_B, m_{A_2}) for two bodies A_1 and A_2 such that $m_B = 0$, and thus we have Theorem 2.

4. Example

4.1. 2+1-Moulton Configuration – Procyon –

Procyon is the α star in Canis Minor. It is a binary star system consisting of, Procyon A and B, which are heavenly bodies. The two stars work on an orbit around the center of gravity of both. The mass of Procyon A is $1.42 \pm 0.04 M_s$ and of Procyon B is $0.575 \pm 0.017 M_s$ [3], where M_s is a symbol denoting the weight of the sun i.e., $M_s = 1.989 \times 10^{30}$ kg.

We consider a system of Procyon A and B as a two-body MC such that we assume Procyon A to be A_1, Procyon B to be A_2. We suppose it not to come under an influence from others. We assume $q_{A_1} = 1$, then we obtain $c_A \fallingdotseq 1.29$, $\lambda_A \fallingdotseq 2.00$ by solving simultaneous equations (6), $m_{A1} = \lambda_A(q_{A_2} - c_A) = 1.42$ and $m_{A_2} = \lambda_A(c_A - q_{A_1}) = 0.575$, where we put the mass of A is $1.42 M_s$ and one of B is $0.575 M_s$.

We calculate using Mathematica then we obtain $q_B \fallingdotseq 0.171$ from the equation of (4), $c = c_A$ in case 1. Similarly, we can get their positions in the Cases 2 and 3 (see Table 1).

Table 1. 2+1-Moulton Configuration.

		q_1	q_2	q_3
case 1	position	0.171	1.00	2.00
	mass (M_s)	0.00	0.575	1.42
case 2	position	1.00	1.59	2.00
	mass (M_s)	0.575	0.00	1.42
case 3	position	1.00	2.00	2.55
	mass (M_s)	0.575	1.42	0.00

Table 2. Weak-2+1-Moulton Configuration.

	q_B	m_B	c	M	λ
case 1	1.6	0.42	2.67	3.92	3.10
case 2	3.05	1.66	2.88	5.16	17.2
case 3	4.3	1.11	3.16	4.61	2.10

4.2. Weak-2+1 Moulton Configuration

We show an example for weak-2+1 Moulton Configuration in each case. Firstly we give $(q_{A_1}, c_A, \lambda_A) = (2.5, 2.8, 3.5)$ so that we easily compare each case, then we obtain $(m_{A_1}, m_{A_2}) = (2.45, 1.05)$. Secondly we solve the equation $c = c_A$ on q_B. Then we select the value of q_B so that it makes m_B be positive. Moreover we calculate c, M and λ, where we rounded off to fourth figure. We present the result as Table 2 and Figs. 8, 9 and 10).

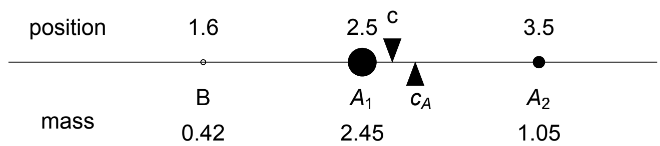

Figure 8. Case 1 $(c, M, \lambda) = (2.67, 3.92, 3.10)$.

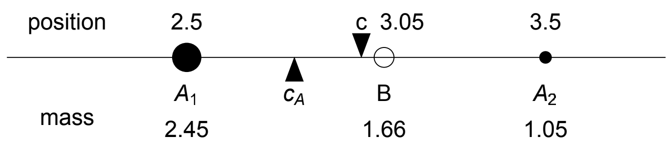

Figure 9. Case 2 $(c, M, \lambda) = (2.88, 5.16, 17.2)$.

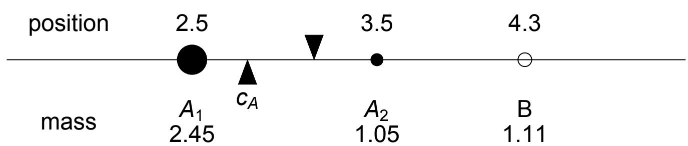

Figure 10. Case 3 $(c, M, \lambda) = (3.16, 4.61, 2.10)$.

Acknowledgements

The authors would like to thank Professor Ivaïlo Mladenov for helpful discussion and encouragement. This work was supported by JSPS KAKENHI Grant Number 18K03286.

References

[1] Albouy A. and Moeckel R., *The Inverse Problem For Collinear Central Configurations*, Celestial Mech. Dyn. Astronomy **77** (2000) 77–91.

[2] Euler L., *De Moth Rectilineo trim Corporum se Mutuo Attrahentium*, Commentrarii Academiae Scientiarum Peteropolitanae **11** (1767) 144–151.

[3] Kervella P., Thevenin F., Morel P., Berthomieu G., Borde P. and Provost J., *The Diameter and Evolutionary State of Procyon A*, Astronomy and Astrophysics **413** (2004) 251–256.

[4] Meyer K., Hall G. and Offin D., *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, 2$^{\rm nd}$ Edn, Springer, New York 2009.

[5] Moulton F., *The Straight Line Solutions of the Problem of N Bodies*, Ann. Math. **2** (1910) 1–17.

[6] Yoshimi N., *2+2 Moulton Configuration*, J. Geom. Symmetry Phys. **45** (2017) 95–110.

Naoko Yoshimi
Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku-ku
Tokyo 162-8601, JAPAN
E-mail address: `1114704@ed.tus.ac.jp`

Akira Yoshioka
Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku-ku
Tokyo 162-8601, JAPAN
E-mail address: `yoshioka@rs.kagu.tus.ac.jp`