JGSP **45** (2017) 95–110

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

$2+2$ MOULTON CONFIGURATION

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Communicated by Charles-Michel Marle

Abstract. We pose a new problem of collinear central configurations in Newtonian n-body problem. It is known that the configuration of two bodies moving along the Newtonian force is always a collinear central configuration. Can we add new two bodies on the straight line of initial two bodies without changing the move of the initial two bodies and the configuration of the four bodies is central, too? We call it 2+2 Moulton configuration.

We find three special solutions to this problem and find each mass of new two bodies is zero.

MSC: 70F10
Keywords: Celestial mechanics, collinear central configuration, four-body problem, inverse problem

1. Introduction

Euler found solutions of three-body problem on a line, collinear three problem [2], $(n, d) = (3, 1)$, for the first time in history. In general solutions of n-body problem on a line, called a collinear n-body-problem, become *collinear central configuration*, that is, the ratios of the distances of the bodies from the center of mass are constants [5]. F. Moulton [5] proved that for a fixed mass vector $\mathbf{m} = (m_1, \dots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration $\mathbf{q} = (q_1, \dots, q_n)$ with mass $\mathbf{m} = (m_1, \dots, m_n)$ (up to translation and scaling). The configuration is called a *Moulton Configuration*, which will be abbreviated as M.C.

Many papers about M.C. were published by many authors since then. For example, Albouy and Moeckel [1] consider the inverse problem: given a fixed collinear configuration $(q_1, \dots, q_n)$ of n bodies, the problems to find masses $m_1, m_2, \dots, m_n$ which make $(q_1, \dots, q_n)$ with $(m_1, \dots, m_n)$ central. They proved that for $n \leq 6$, each configuration $(q_1, \dots, q_n)$ determines a one-parameter family of masses $(m_1, \dots, m_n)$ which makes $(q_1, \dots, q_n)$ with $(m_1, \dots, m_n)$ central.

doi: 10.7546/jgsp-45-2017-95-110

In this paper, we consider the following problem. We assume we are given a M.C. of two bodies, A_1, A_2 of configuration $\mathbf{q} = (q_{A_1}, q_{A_2}) \in \mathbb{R}^2$, such that $q_{A_1} < q_{A_2}$ with mass $\mathbf{m}_A = (m_{A_1}, m_{A_2})$. We consider to add two bodies, B_1, B_2 of $\mathbf{q}_B = (q_{B_1}, q_{B_2})$ with $\mathbf{m}_B = (m_{B_1}, m_{B_2})$, to A_1, A_2 on the same line so that

i) the positions $q_{A_1}, q_{A_2}, q_{B_1}, q_{B_2}$ are mutually distinct and their configuration is M.C. with masses $m_{A_1}, m_{A_2}, m_{B_1}, m_{B_2}$

ii) the motion of A_1, A_2 are kept invariant during the process.

We call it $2+2$-*Moulton-Configuration* for two bodies $\mathbf{q}_A = (q_{A_1}, q_{A_2})$ with $\mathbf{m}_A = (m_{A_1}, m_{A_2})$. More precisely, we denote by $q_1, \ldots, q_4$ the positions of A_1, A_2, B_1, B_2 and $(m_1, \ldots, m_4)$ their mass, respectively, and we define

Definition 1 (2+2-Moulton-Configuration) *We call* $\mathbf{q} = (q_1, q_2, q_3, q_4)$ *with* $\mathbf{m} = (m_1, m_2, m_3, m_4)$ *"2+2 Moulton-Configuration" for two bodies,* $\mathbf{q}_A$ *with* $\mathbf{m}_A$ *when it satisfies the following conditions.*

i) *The configuration* $\mathbf{q}$ *with* $\mathbf{m}$ *is a Moulton Configuration without changing the initial configuration* $\mathbf{q}_A$ *with* $\mathbf{m}_A$

ii) *The center of mass of the systems* c *and center of mass of* A_1 *and* A_2 c_A *are the same, and the motion of* A_1 *and* A_2 *is invariant.*

The main result of this paper is the following.

Theorem 2 ($2+2$ Moulton Configuration) i) *There exist three* $2+2$ *Moulton Configurations for two bodies,* $\mathbf{q}_A$ *with* $\mathbf{m}_A$ *with order*

- $q_{B_1} < q_{A_1} < q_{B_2} < q_{A_2}$
- $q_{B_1} < q_{A_1} < q_{A_2} < q_{B_2}$
- $q_{A_1} < q_{B_1} < q_{A_2} < q_{B_2}$

ii) *The masses of added two bodies* m_{B_1} *and* m_{B_2} *are zero for each case.*

We can also set the $k+l$ Moulton Configuration problem in the same manner. We will consider this problem in forthcoming papers.

We will prove Theorem 1 by showing i), namely the positions of $\mathbf{q} = (q_1, q_2, q_3, q_4)$ first, and then by determining m_{B_1}, m_{B_2} to obtain ii). We call the problem i) the position problem and ii) the mass problem.

Then this paper is organized as follows. In Section 2, we deduce the equations for the position problem and also the mass problem. In Section 3, we solve position problem each case. In Section 4, we show that the mass of the added bodies is zero where any cases and in last section we show an example of $2+2$ central configuration.

2. Equations for $2+2$ Moulton Configuration

2.1. Collinear Central Configuration

We consider the d-dimensional Newtonian n-body problem

$$m_i\ddot{\mathbf{q}}_i(t) = \sum_{j=1\ i\neq j}^{n} \frac{m_i m_j(\mathbf{q}_j(t)-\mathbf{q}_i(t))}{\|\mathbf{q}_i(t)-\mathbf{q}_j(t)\|^3} = \frac{\partial}{\partial \mathbf{q}_i(t)}U(\mathbf{q}(t)), \qquad 1 \leq i \leq n \tag{1}$$

where $U(\mathbf{q})$ is the Newtonian potential function

$$U(\mathbf{q}) = \sum_{(i,j)\ i<j} \frac{m_i m_j}{\|\mathbf{q}_i-\mathbf{q}_j\|}, \qquad i,j=1,\cdots,n$$

$m_i \in \mathbb{R}^+ (i=1,2,\dots,n)$ are masses of the bodies and $\mathbf{q}(t)=(\mathbf{q}_1(t),\dots,\mathbf{q}_n(t)) \in (\mathbb{R}^d)^n$, $1\leq d\leq 3$ is their configuration. Here we except $\mathbf{q}_i(t)=\mathbf{q}_j(t)$ for some $i\neq j$.

If we consider a solution of the form $\mathbf{q}(t) = \tilde{\mathbf{c}} + \phi(t)(\mathbf{q}-\tilde{\mathbf{c}})$, we easily see $\mathbf{q}$ satisfies the equation (2) below, where $\phi(t)$ is a scalar-valued function, $\mathbf{q} = (\mathbf{q}_1,\dots,\mathbf{q}_n)$ is a constant vector, $\mathbf{c}$ is the center of mass of the system $\mathbf{c} = \sum_{i=1}^n m_i\mathbf{q}_i / \sum_{i=1}^n m_i$ and $\tilde{\mathbf{c}} = (\mathbf{c},\dots,\mathbf{c})$. Then we naturally obtain the following concept.

Definition 3 (Central Configuration [4] Section 2.1.3**)** *We call $\mathbf{q}=(\mathbf{q}_1,\mathbf{q}_2,\dots,\mathbf{q}_n) \in (\mathbb{R}^d)^n$ a central configuration with $\mathbf{m}=(m_1,m_2,\dots,m_n)\in(\mathbb{R}^+)^n$ if $\mathbf{q}$ satisfies*

$$\sum_{j=1}^{n} \frac{m_j(\mathbf{q}_j-\mathbf{q}_i)}{r_{ij}^3} + \lambda(\mathbf{q}_i-\mathbf{c}) = \mathbf{0}, \qquad i=1,2,\dots,n \tag{2}$$

for some $\lambda\in\mathbb{R}$, where $r_{ij}=\|\mathbf{q}_i-\mathbf{q}_j\|$ is a distance of two bodies.

We easily see that the λ automatically satisfies $\lambda = U(\mathbf{q})/(2I)$, where $I = \sum_{i=1}^n m_i\|\mathbf{q}_i-\mathbf{c}\|^2/2$. We remark here λ is positive.

Conversely for a central configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ with mass $\mathbf{m} = (m_1, \ldots, m_n)$, and for a real valued function $\phi(t)$ which satisfies $\ddot{\phi} = -\lambda\phi/|\phi|^3$, if we put $\mathbf{q}(t) = \tilde{\mathbf{c}} + \phi(t)(\mathbf{q} - \tilde{\mathbf{c}})$, then $\mathbf{q}(t)$ is a solution of the equation (1).

Now we consider $d = 1$, which means that all bodies lie on a straight line, that is, collinear. We call it a *collinear central configuration*, or a *Molton Configuration*. Since the configuration of the bodies are collinear, the equation (2) is rewritten in the form

$$A \cdot {}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = \mathbf{0} \qquad \text{for some} \qquad \lambda \in \mathbb{R}$$

where $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$ and A is a skew-symmetric matrix defined by $A = (a_{ij}),\ a_{ij} = (q_i - q_j)^{-2}$ for $i < j$, and $a_{ii} = 0$, $a_{ji} = -a_{ij}$. One can easily see A is regular when n is even. Then for even n we have

$$^t\mathbf{m} = -\lambda A^{-1} \cdot {}^t(\mathbf{q} - \tilde{\mathbf{c}}). \tag{3}$$

Remark 4. *It is known that any two-body problem is always reduced to a collinear central configuration.*

2.2. Key Lemma

When we consider conditions for A_1, A_2, B_1, B_2 to become $2 + 2$ Moulton Configuration for two bodies $\mathbf{q}_A$ with $\mathbf{m}_A$, we get naturally the following

Proposition 5. *If we have a* $2 + 2$ *Moulton Configuration, then we have*

$$(m_{B_1} q_{B_1} + m_{B_2} q_{B_2})/(m_{B_1} + m_{B_2}) = c_A.$$

In what follows we take the origin of coordinates as the center of mass of the system A_1, A_2, that is, $c_A = 0$, for simplicity. We can set the distance of q_{A_1} and q_{A_2} is unit by scaling. Then we set $(q_{A_1}, q_{A_2}) = (u - 1, u),\ u \in (0, 1)$. Since two bodies are always a collinear central configuration, or Moulton Configuration, then the equation (3) gives

$$\begin{pmatrix} m_{A_1} \\ m_{A_2} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}^{-1} \begin{pmatrix} q_{A_1} \\ q_{A_2} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u-1 \\ u \end{pmatrix} = \lambda \begin{pmatrix} u \\ 1-u \end{pmatrix} \tag{4}$$

because $a_{12} = (q_{A_1} - q_{A_2})^{-2} = 1$.

Then we add new two bodies B_1, B_2 of configuration $\mathbf{q}_B = (q_{B_1}, q_{B_2})$, $q_{B_1} < q_{B_2}$ with $\mathbf{m}_B = (m_{B_1}, m_{B_2})$. We denote by $\mathbf{q} = (q_1, \ldots q_4)$ the configuration of

$\{A_1, A_2, B_1, B_2\}$ and by $\mathbf{m} = (m_1, \ldots, m_4)$ their masses, respectively. When $\{A_1, A_2, B_1, B_2\}$ is M.C., the equation (3) also gives

$$\begin{aligned} m_1 &= (\tilde{a}_{23}q_4 - \tilde{a}_{24}q_3 + \tilde{a}_{34}q_2)\tilde{\lambda}/P_4 \\ m_2 &= (\tilde{a}_{14}q_3 - \tilde{a}_{13}q_4 - \tilde{a}_{34}q_1)\tilde{\lambda}/P_4 \\ m_3 &= (\tilde{a}_{12}q_4 - \tilde{a}_{14}q_2 + \tilde{a}_{24}q_1)\tilde{\lambda}/P_4 \\ m_4 &= (\tilde{a}_{13}q_2 - \tilde{a}_{12}q_3 - \tilde{a}_{23}q_1)\tilde{\lambda}/P_4 \end{aligned} \tag{5}$$

where P_4 is a Pfaffian of the matrix $A = (\tilde{a}_{ij})$, i.e., $P_4 = \tilde{a}_{12}\tilde{a}_{34} - \tilde{a}_{13}\tilde{a}_{24} + \tilde{a}_{14}\tilde{a}_{23}$, $A = (\tilde{a}_{ij})$ is the coefficient matrix for (q_1, q_2, q_3, q_4), and $\tilde{\lambda}$ is a certain constant. It is easy to see $P_4 > 0$ because $\tilde{a}_{12} > \tilde{a}_{13}$, $\tilde{a}_{34} > \tilde{a}_{24}$ then $\tilde{a}_{12}\tilde{a}_{34} > \tilde{a}_{13}\tilde{a}_{24}$.

If $\{A_1, A_2, B_1, B_2\}$ is also a $2+2$ Moulton Configuration for A_1, A_2, the masses corresponding to A_1, A_2 are the same as in equation (4). Then the point is that Definition 1 i) yields the following.

Lemma 6 (Key Lemma). *The equation* (4), i.e., $(m_{A_1}, m_{A_2}) = \lambda(u, 1-u)$ coincides with the corresponding equations in (5) above.

Using this relation we will solve the problems i) and ii). As a possibility we have the following cases:

Case 1: $q_{B_1} < q_{B_2} < q_{A_1} < q_{A_2}$ Case 2: $q_{B_1} < q_{A_1} < q_{B_2} < q_{A_2}$
Case 3: $q_{B_1} < q_{A_1} < q_{A_2} < q_{B_2}$ Case 4: $q_{A_1} < q_{B_1} < q_{B_2} < q_{A_2}$
Case 5: $q_{A_1} < q_{B_1} < q_{A_2} < q_{B_2}$ Case 6: $q_{A_1} < q_{A_2} < q_{B_1} < q_{B_2}$.

We easily obtain that $m_{B_1}q_{B_1} + m_{B_2}q_{B_2} \neq 0$ in Cases 1 and 6 because both q_{B_1} and q_{B_2} are negative in Case 1 and those are positive in Case 6. Then the Cases 1 and 6 give no solutions.

3. Position Problem

We prove Theorem 2 i) by determining the positions of added bodies B_1 and B_2 for every given $u \in (0,1)$ in Cases 2, 3 and 5. Then we show that Case 4 allows no solutions.

3.1. Case 2

We set $(q_1,\ q_2,\ q_3,\ q_4) = (q_{B_1}, q_{A_1}, q_{B_2}, q_{A_2})$, in which $q_1 < q_2 < q_3 < q_4$ with $(m_1, m_2, m_3, m_4) = (m_{B_1}, m_{A_1}, m_{B_2}, m_{A_2})$ (See Fig. 1). Then we have (m_2, m_4)

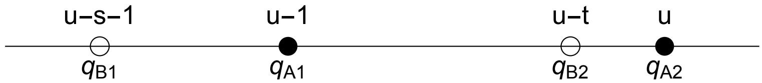

Figure 1. Case 2.

$= (m_{A_1}, m_{A_2})$. Since the motion of the system is determined by the solution of $\ddot{\phi} = -\lambda\phi/|\phi|^3$, Definition 1 (ii) gives $\lambda = \tilde{\lambda}$ in equation (5). Then we obtain from (4) and (5)

$$\frac{1}{P_4}\begin{pmatrix}\tilde{a}_{14}q_3 - \tilde{a}_{13}q_4 - \tilde{a}_{34}q_1\\ \tilde{a}_{13}q_2 - \tilde{a}_{12}q_3 - \tilde{a}_{23}q_1\end{pmatrix} = \begin{pmatrix}u\\ 1-u\end{pmatrix}. \tag{6}$$

Since $q_2 = q_{A_1}$, $q_4 = q_{A_2}$, we put $(q_1,\ q_2,\ q_3,\ q_4) = (u-s-1,\ u-1,\ u-t,\ u)$ such that $s > 0$, $t \in (0,1)$ for every $u \in (0,1)$. Then $\tilde{a}_{12} = s^{-2}$, $\tilde{a}_{13} = (1+s-t)^{-2}$, $\tilde{a}_{14} = (1+s)^{-2}$, $\tilde{a}_{23} = (1-t)^{-2}$, $\tilde{a}_{24} = 1$, $\tilde{a}_{34} = t^{-2}$ and (6) is equivalent to

$$\left(\frac{s-u+1}{t^2} + \frac{u-t}{(s+1)^2} - \frac{u}{(s-t+1)^2}\right)\frac{1}{P_4} \ = \ u \tag{7}$$

$$-\left(\frac{u-t}{s^2} + \frac{1-u}{(s-t+1)^2} + \frac{-s+u-1}{(t-1)^2}\right)\frac{1}{P_4} \ = \ 1-u \tag{8}$$

where

$$P_4 = \frac{1}{s^2t^2} + \frac{1}{(s+1)^2(t-1)^2} - \frac{1}{(s-t+1)^2}.$$

Then we see easily that (7), (8) are equivalent (9), (10) below respectively.

$$\frac{(s+1)^2}{s^2}(s^3 + (1-u)s^2 - u) = \frac{t^3}{(t-1)^2}(t^2 - (u+2)t + 2u + 1) \tag{9}$$

$$\frac{s^3}{(s+1)^2}(s^2 + (3-u)s + 3 - 2u) = \frac{(1-t)^3}{t^2}(t^2 + (1-u)t + 1 - u). \tag{10}$$

The positions of B_1 and B_2 are decided by the solutions s and t of (9), (10) for a given $u \in (0,1)$.

Lemma 7. *For every $u \in (0,1)$ which there is a unique solution (s,t) of simultaneous equation* (9), (10).

Proof: Let us put the left hand side of (9) as $f_1(s)$ and the right hand side as $g_1(t)$. Similarly, we set the left hand side of (10) as $f_2(s)$ and the right hand as $g_2(t)$. (See Fig. 2 and Fig. 3).

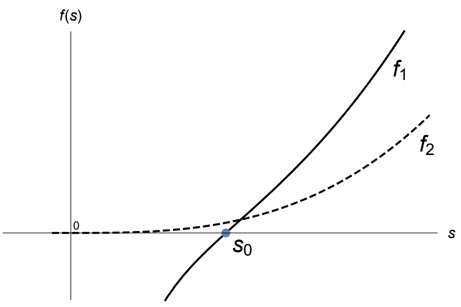

Figure 2. $(s, f(s))$-plane in Case 2.

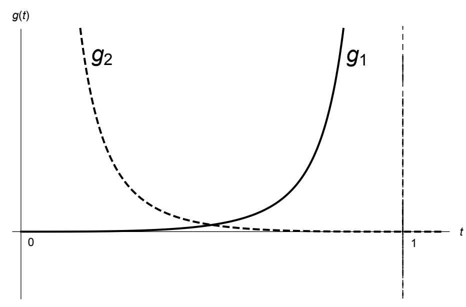

Figure 3. $(t, g(t))$-plane in Case 2.

Firstly we see easily that f_1 and g_1 are strictly monotone increasing satisfying $\lim_{s\to 0} f_1 = -\infty$, $\lim_{s\to\infty} f_1 = \infty$ and $\lim_{t\to 0} g_1 = 0$, $\lim_{t\to 1} g_1 = +\infty$. We set s_0 such that $f_1(s_0) = 0$. Then for every $t \in (0,1)$ there exists $s_1 = s_1(t)$ such that $f_1(s_1(t)) = g_1(t)$. The function $s_1(t)$ satisfies $s_1(t) > s_0$ and is continuous, monotone increasing in $t \in (0,1)$.

Secondly, $f_2(s)$ is strictly monotone increasing for $s > 0$, and $\lim_{s\to 0} f_2 = 0$, $\lim_{s\to+\infty} f_2 = +\infty$, while $g_2(t)$ is a strictly monotone decreasing function on $t \in (0,1)$, and $\lim_{t\to 0} g_2 = \infty$, $g_2(1) = 0$. Therefore there is a unique $s_2 = s_2(t)$ for every $t \in (0,1)$ such that $f_2(s_2(t)) = g_2(t)$, and then $s_2(t)$ is a continuous monotony decreasing function of t.

Notice the functions $s_1(t)$, $s_2(t)$ satisfy that as $t \to 0$, $s_1(t) \to s_0 > 0$ and $s_2(t) \to +\infty$, and as $t \to 1_{-0}$, $s_1(t) \to +\infty$ and $s_2(t) \to 0$. Then there exists a unique $t = \bar{t} \in (0,1)$ such that $s_1(\bar{t}) = s_2(\bar{t})$, which gives the solution $(s_1, \bar{t})$ for (9), (10). ■

3.2. Case 3

We set $(q_1,\ q_2,\ q_3,\ q_4) = (q_{B_1}, q_{A_1}, q_{A_2}, q_{B_2}),\ q_1 < q_2 < q_3 < q_4$ with $(m_1, m_2, m_3, m_4) = (m_{B_1}, m_{A_1}, m_{A_2}, m_{B_2})$ (See Fig. 4). Then similarly to the Case 2 we obtain from (4) and (5) the equation

$$\begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \frac{\tilde{\lambda}}{P_4} \begin{pmatrix} \tilde{a}_{14}q_3 - \tilde{a}_{13}q_4 - \tilde{a}_{34}q_1 \\ \tilde{a}_{12}q_4 - \tilde{a}_{14}q_2 + \tilde{a}_{24}q_1 \end{pmatrix} = \lambda \begin{pmatrix} u \\ 1-u \end{pmatrix}. \tag{11}$$

Here we also assume $\tilde{\lambda} = \lambda$. We put $(q_1,\ q_2,\ q_3,\ q_4) = (u-s-1,\ u-1,\ u,\ u+t)$, such that $s, t > 0$ for every $u \in (0,1)$. Then $\tilde{a}_{12} = s^{-2}$, $\tilde{a}_{13} = (1+s)^{-2}$, $\tilde{a}_{14} = (1+s+t)^{-2}$, $\tilde{a}_{23} = 1$, $\tilde{a}_{24} = (1+t)^{-2}$ and $\tilde{a}_{34} = t^{-2}$. Therefore we can

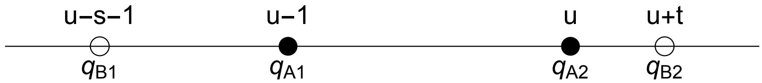

Figure 4. Case 3

rewrite the equation (11) in the form

$$\frac{(s+1)^2}{s^2}(s^3+(1-u)s^2-u)=\frac{t^3}{(t+1)^2}(t^2+(u+2)t+2u+1) \tag{12}$$

$$\frac{s^3}{(s+1)^2}(s^2+(3-u)s+3-2u)=\frac{(t+1)^2}{t^2}(t^3+ut^2+u-1). \tag{13}$$

Let us put the left hand side of (12) as $f_3(s)$ and the right hand side as $g_3(t)$, similarly, we set the left hand side of (13) as $f_4(s)$ and the right hand side as $g_4(t)$. (See Fig. 5 and Fig. 6)

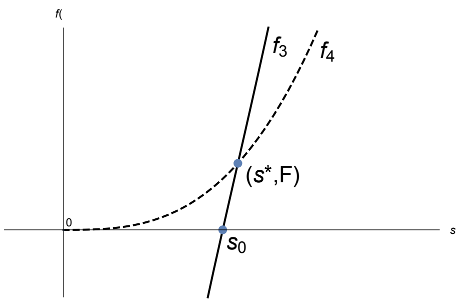

Figure 5. $(s, f(s))$-plane in Case 3.

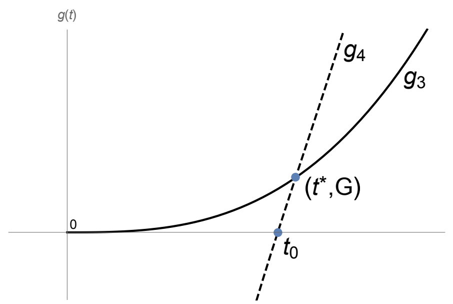

Figure 6. $(t, g(t))$-plane in Case 3.

It is clear $f_3(s)$, $f_4(s)$, $g_3(t)$ and $g_4(t)$ are strictly monotone increasing functions. Moreover $\lim_{s\to 0} f_3 = -\infty$, $\lim_{s\to\infty} f_3 = +\infty$, $g_3(0) = 0$, and $\lim_{t\to\infty} g_3 = \infty$, while $f_4(0) = 0$, $\lim_{s\to\infty} f_4 = +\infty$, $\lim_{t\to 0} g_4 = -\infty$ and $\lim_{t\to\infty} g_4 = \infty$. Then we obtain a unique solution $s = s_3(t) > s_0$ such that $f_3(s_3(t)) = g_3(t)$ and for every $t > 0$, and $s = s_4(t)$ such that $f_4(s_4(t)) = g_4(t)$ for every $t > t_0$, respectively, where the constants s_0, t_0 are given by $f_3(s_0) = 0$, $g_4(t_0) = 0$, respectively.

Lemma 8. *There exists uniquely* $t = \bar{t}$ *such that* $s_3(\bar{t}) = s_4(\bar{t})$.

Proof: It is easy to see that the graphs of f_3 and f_4 intersect once at a certain $s = s^* > 0$ because $f_3(s) - f_4(s)$ is monotone increasing and $f_3(2) - f_4(2) = (556 - 277u)/36 > 0$. We set $F = f_3(s^*) = f_4(s^*)$. Similarly, we see easily that the graphs of $g_3(t)$ and $g_4(t)$ intersect at a unique point $t = t^* > 0$ and $g_4(t) > g_3(t)$ for $t > t^*$.

Firstly we see that $s_3(t_0) > s_4(t_0)$ because $s_3(t_0) > s_0 > 0$, and $s_4(t_0) = 0$ since $g_4(t_0) = 0$ (see Fig. 5). Secondly we show that $s_3(t) < s_4(t)$ for large t. Take a sufficiently large $\check{t}$ such that $g_4(\check{t}) > g_3(\check{t}) > F$. Then $f_3(s_4(\check{t})) > f_4(s_4(\check{t}))$ because $s_4(\check{t}) > s^*$. On the other hand $f_4(s_4(\check{t})) > f_3(s_3(\check{t}))$ because $g_4(\check{t}) > g_3(\check{t})$ and $f_i(s_i(\check{t})) = g_i(\check{t})$ $(i = 1, 2)$. Then we have $f_3(s_4(\check{t})) > f_3(s_3(\check{t}))$. Since $f_3(s)$ is monotone increasing $s_4(\check{t}) > s_3(\check{t})$.

Then we have $s_3(t_0) > s_4(t_0)$ and $s_3(\check{t}) < s_4(\check{t})$, hence there exists $\bar{t} > t_0$ such that $s_3(\bar{t}) = s_4(\bar{t})$. ■

3.3. Case 5

Let $(q_1,\ q_2,\ q_3,\ q_4) = (q_{A_1}, q_{B_1}, q_{A_2}, q_{B_2})$, and $(m_1,\ m_2,\ m_3,\ m_4) = (m_{A_1}, m_{B_1}, m_{A_2}, m_{B_2})$ (See Fig. 7). Then we obtain

$$\begin{pmatrix} m_1 \\ m_3 \end{pmatrix} = \frac{\tilde{\lambda}}{P_4} \begin{pmatrix} \tilde{a}_{23}q_4 - \tilde{a}_{24}q_3 + \tilde{a}_{34}q_2 \\ \tilde{a}_{12}q_4 - \tilde{a}_{14}q_2 + \tilde{a}_{24}q_1 \end{pmatrix} = \lambda \begin{pmatrix} u \\ 1-u \end{pmatrix}. \tag{14}$$

Here we assume $\tilde{\lambda} = \lambda$.

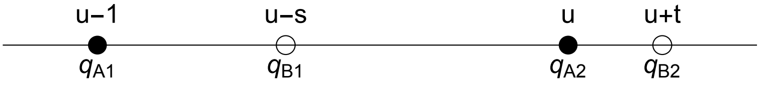

Figure 7. Case 5.

We set $(q_1,\ q_2,\ q_3,\ q_4) = (u-1,\ u-s,\ u,\ u+t)$, such that $s \in (0,1)$, $t > 0$ for every $u \in (0,1)$. Then $\tilde{a}_{12} = (1-s)^{-2}$, $\tilde{a}_{13} = 1$, $\tilde{a}_{14} = (1+t)^{-2}$, $\tilde{a}_{23} = s^{-2}$, $\tilde{a}_{24} = (s+t)^{-2}$ and $\tilde{a}_{34} = t^{-2}$. Therefore we can write the equation (14) in the form

$$\frac{s^3}{(s-1)^2}(s^2 - (u+2)s + 2u + 1) = \frac{t^3}{(t+1)^2}(t^2 + (u+2)t + 2u + 1) \tag{15}$$

$$\frac{(s-1)^3}{s^2}(s^2 + (1-u)s - u + 1) = -\frac{(t+1)^2}{t^2}(t^3 + ut^2 + u - 1). \tag{16}$$

Lemma 9. *For every $u \in (0,1)$, there is a unique solution (s,t) of simultaneous equations* (15) and (16).

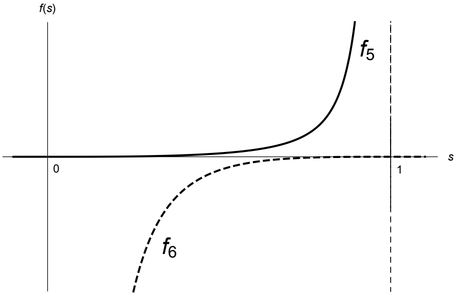

Figure 8. $(s, f(s))$-plane in Case 5.

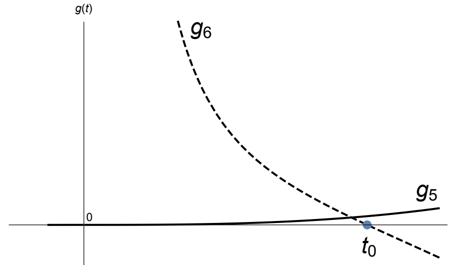

Figure 9. $(t, g(t))$-plane in Case 5.

Proof: Let us put the left hand side of (15) as $f_5(s)$ and the right hand side as $g_5(t)$ and similarly, we set the left hand side of (16) as $f_6(s)$ and the right hand side as $g_6(t)$. (See Fig. 8 and Fig. 9)

First $f_5(s)$ and $g_5(s)$ are monotone increasing for $s \in (0,1)$ and $t > 0$, respectively, and $\lim_{s\to 0} f_5(s) = 0$, $\lim_{s\to 1} f_5(s) = +\infty$, and $\lim_{t\to 0} g_5(t) = 0$, $\lim_{t\to+\infty} g_5(t) = +\infty$. Therefore, for every $t > 0$ there exists $s_5 = s_5(t)$ such that $f_5(s_5(t)) = g_5(t)$. Since both f_5 and g_5 are monotone increasing, so is $s_5(t)$ such that $\lim_{t\to+0} s_5(t) = +0$ and $\lim_{t\to+\infty} s_5(t) = +\infty$.

Secondly $f_6(s)$ is strictly monotone increasing for $s \in (0,1)$ such that $\lim_{s\to 0} f_6 = -\infty$, $\lim_{s\to 1} f_6 = 0$, and $g_6(t)$ is strictly monotone decreasing for $t > 0$ such that $\lim_{t\to 0} g_6 = +\infty$, $\lim_{t\to+\infty} g_6 = -\infty$. We set $t_0 > 0$ such that $g_6(t_0) = 0$, then there exists $s_6 = s_6(t) > 0$ such that $f_6(s_6) = g_6(t)$ for every $t > t_0$. Since f_6 is increasing and g_6 is decreasing, $s_6(t)$ is decreasing such that $\lim_{t\to t_0+} s_6(t) = 1$, $\lim_{t\to+\infty} s_6(t) = +0$. We notice $\lim_{t\to t_0+} s_5(t) = \tilde{s}$, where $f_5(\tilde{s}) = g_5(t_0)$ and $\tilde{s} < 1$. Then there exists unique $t = \bar{t} \in (t_0, \infty)$ such that $s_5(\bar{t}) = s_6(\bar{t})$, which gives the solution $(s_5(\bar{t}), \bar{t})$ for (15), (16). ∎

3.4. Case 4

In this subsection, we show that there are no solutions for the Case 4. Let $(q_1,\ q_2,\ q_3,\ q_4) = (q_{A_1}, q_{B_1}, q_{B_2}, q_{A_2})$, and $(m_1,\ m_2,\ m_3,\ m_4) = (m_{A_1}, m_{B_1}, m_{B_2}, m_{A_2})$, then we obtain

$$\begin{pmatrix} m_1 \\ m_4 \end{pmatrix} = \frac{\tilde{\lambda}}{P_4} \begin{pmatrix} \tilde{a}_{23}q_4 - \tilde{a}_{24}q_3 + \tilde{a}_{34}q_2 \\ \tilde{a}_{13}q_2 - \tilde{a}_{12}q_3 - \tilde{a}_{23}q_1 \end{pmatrix} = \lambda \begin{pmatrix} u \\ 1-u \end{pmatrix}. \tag{17}$$

Here we set $\bar{\lambda} = \lambda$. We put $(q_1,\ q_2,\ q_3,\ q_4) = (u-1,\ u-s-t,\ u-t,\ u)$, such that $s, t > 0$, $s+t \in (0,1)$ for every $u \in (0,1)$. Then $\tilde{a}_{12} = (1-s-t)^{-2}$, $\tilde{a}_{13} = (1-t)^{-2}$, $\tilde{a}_{14} = 1$, $\tilde{a}_{23} = s^{-2}$, $\tilde{a}_{24} = (s+t)^{-2}$ and $\tilde{a}_{34} = t^{-2}$. Therefore we write the equation (17) in the form

$$t^2\Big(t-u-\frac{u}{(t-1)^2}\Big)=(s+t)^2\Big(s+t-u-\frac{u}{(s+t-1)^2}\Big) \tag{18}$$

$$(t-1)^2\Big(u-t+\frac{1-u}{t^2}\Big)=(s+t-1)^2\Big(u-s-t+\frac{1-u}{(s+t)^2}\Big). \tag{19}$$

We can rewrite the equations (19) and (18), $f(t)=f(s+t)$ and $g(t)=g(s+t)$, respectively. Moreover $f(t)$ and $g(t)$ are monotone decreasing. Therefore, obviously, we have no solutions.

Remark 10. *We can prove that there are no solutions* (s,t) *for Case* 1 *and Case* 6 *by the similar manner as Case* 4.

4. Mass Problem

We will show that masses $m_{B_1},\ m_{B_2}$of added bodies are zero in Case 2, namely, s,t,u which satisfy the equations (9) (10) give $m_{B_1}=m_{B_2}=0$ by (5). We get also $m_{B_1}=m_{B_2}=0$ in Case 3 and Case 5.

In Section 3.1 we have shown that for each $u\in(0,1)$, there exists unique solution (s,t) of the equation (9), (10), which gives a curve denoted by C in the space

$$S=\{(s,t,u);\ 0<s,\ 0<t<1,\ 0<u<1\}$$

Hence the curve C is a real algebraic curve determined by the equation (9), (10).

On the other hand, the identity (5) yields in Case 2 that the mass of B_1 and B_2 is given by

$$\begin{aligned} m_{B_1} &= m_1 \ = (\tilde a_{23}q_4-\tilde a_{24}q_3+\tilde a_{34}q_2)\tilde\lambda/P_4\\ m_{B_2} &= m_3 \ = (\tilde a_{12}q_4-\tilde a_{14}q_2+\tilde a_{24}q_1)\tilde\lambda/P_4. \end{aligned}$$

In §3.1 we put $(q_1,\ q_2,\ q_3,\ q_4)=(u-s-1,\ u-1,\ u-t,\ u)$ such that $s>0$, $t\in(0,1)$ for every $u\in(0,1)$. Then $\tilde a_{12}=s^{-2}$, $\tilde a_{13}=(1+s-t)^{-2}$, $\tilde a_{14}=(1+s)^{-2}$, $\tilde a_{23}=(1-t)^{-2}$, $\tilde a_{24}=1$, $\tilde a_{34}=t^{-2}$ and then we have

$$\begin{aligned} m_{B_1} &= \frac{\lambda}{P_4}\left(\frac{u-1}{t^2}+\frac{u}{(t-1)^2}+t-u\right)\\ m_{B_2} &= \frac{\lambda}{P_4}\left(\frac{u}{s^2}-\frac{u-1}{(s+1)^2}-s+u-1\right) \end{aligned}$$

where $P_4=1/s^2t^2+1/(s+1)^2(t-1)^2-1/(s-t+1)^2$.

Now let us consider an algebraic curve C_1 in S given by

$$m_{B_1} = m_{B_2} = 0. \tag{20}$$

In what follows, we sill show C and C_1 coincide.

First we solve u in (9), which is denoted by u_1 and also we solve u in (10), denoted by u_2 respectively as

$$u_1 = \frac{s^2(t-1)^2\left(s^3+3s^2+3s-t^3+1\right)}{s^2\left(-t^4+2t^3+2t^2-4t+2\right)+\left(s^4+2s^3+2s+1\right)(t-1)^2} \tag{21}$$

$$u_2 = \frac{\left(s^2+3s+3\right)s^3t^2+(s+1)^2(t-1)^3\left(t^2+t+1\right)}{(s+2)s^3t^2+(s+1)^2(t-1)^3(t+1)}. \tag{22}$$

Then the curve C is given by the equation

$$u_1 = u_2. \tag{23}$$

Secondly we consider the curve C_1 given by (20). Since $P_4 \neq 0$, the equations $m_{B_1} = 0$ and $m_{B_2} = 0$ yield

$$u = u_{m_{B_1}} = \frac{(t-1)^3\left(t^2+t+1\right)}{t^4-2t^3-t^2+2t-1} \tag{24}$$

$$u = u_{m_{B_2}} = \frac{s^3\left(s^2+3s+3\right)}{s^4+2s^3+s^2+2s+1} \tag{25}$$

respectively. Then the curve C is given by $H(s,t) = u_1 - u_2 = 0$ and the curve C_1 is given by $h(s,t) = u_{m_{B_1}} - u_{m_{B_2}} = 0$, respectively.

From (21), (22), we have $H(s,t) = H_n(s,t)/H_d(s,t)$, where the numerator H_n is given by

$$\begin{aligned}
H_n = s^7 &\left(-\left(-2t^6+6t^5-3t^4-4t^3+7t^2-4t+1\right)\right)\\
&- s^6\left(2t^7-14t^6+33t^5-23t^4-9t^3+26t^2-16t+4\right)\\
&- s^5\left(6t^7-33t^6+72t^5-59t^4-2t^3+37t^2-24t+6\right)\\
&- s^4(t-1)^2\left(7t^5-21t^4+24t^3+2t^2-6t+3\right)\\
&- s^3(t-1)^2\left(8t^5-15t^4+12t^3-5t^2+6t-3\right)\\
&- s^2(t-1)^3\left(7t^4-5t^3+t^2-6t+6\right)\\
&- 4s(t-1)^5\left(t^2+t+1\right)-(t-1)^5\left(t^2+t+1\right)
\end{aligned}$$

and from (24), (25), $h(s,t) = h_n(s,t)/h_d(s,t)$, where the numerator h_n is given by

$$\begin{aligned} h_n = s^5\left(-t^4+2t^3+t^2-2t+1\right) + s^4\left(t^5-5t^4+7t^3+2t^2-4t+2\right) \\ + s^3\left(2t^5-7t^4+8t^3+t^2-2t+1\right) + s^2(t-1)^3\left(t^2+t+1\right) \\ + 2s(t-1)^3\left(t^2+t+1\right) + (t-1)^3\left(t^2+t+1\right) \end{aligned}$$

respectively. A direct calculation gives

Lemma 11. *We have the identity*

$$H_n(s,t) = \left(s^2(-2t^2+2t-1) - (2s+1)(t-1)^2\right) h_n(s,t).$$

Therefore $H_n(s,t)=0$ is equivalent to $h_n(s,t)=0$ because $s^2\left(-2t^2+2t-1\right) - (2s+1)(t-1)^2 < 0$ for $s>0$, $t\in(0,1)$. Then we see $m_{B_1} = m_{B_2} = 0$ on C.

Similarly, in Case 3 we have

$$u_1 = \frac{s^2(t+1)^2\left(s^3+3s^2+3s-t^3+1\right)}{s^4(t+1)^2+2s^3(t+1)^2+s^2\left(t^4+2t^3+2t^2+4t+2\right)+2s(t+1)^2+(t+1)^2}$$

$$u_2 = \frac{s^5t^2+3s^4t^2+3s^3t^2-s^2(t+1)^2\left(t^3-1\right)-2s(t+1)^2\left(t^3-1\right)-(t+1)^2\left(t^3-1\right)}{s^4t^2+2s^3t^2+s^2(t+1)^2\left(t^2+1\right)+2s(t+1)^2\left(t^2+1\right)+(t+1)^2\left(t^2+1\right)}$$

$$u_{m_1} = -\frac{(t+1)^2\left(t^3-1\right)}{t^4+2t^3+t^2+2t+1}, \qquad u_{m_4} = \frac{s^3\left(s^2+3s+3\right)}{s^4+2s^3+s^2+2s+1}$$

then

$$\begin{aligned} H_n = s^7\left(2t^5+5t^4+4t^3+5t^2+4t+1\right) \\ + s^6\left(4t^6+19t^5+31t^4+23t^3+22t^2+16t+4\right) \\ + s^5\left(2t^7+19t^6+56t^5+71t^4+46t^3+35t^2+24t+6\right) \\ + s^4(t+1)^2\left(5t^5+21t^4+24t^3+4t^2+6t+3\right) \\ + s^3(t+1)^2\left(4t^5+15t^4+12t^3-t^2-6t-3\right) \\ + s^2(t+1)^2\left(5t^5+12t^4+6t^3-5t^2-12t-6\right) \\ + 4s(t+1)^4\left(t^3-1\right) + (t+1)^4\left(t^3-1\right) \end{aligned}$$

$$\begin{aligned} h_n = s^5\left(-\left(t^4+2t^3+t^2+2t+1\right)\right) \\ - s^4\left(t^5+5t^4+7t^3+2t^2+4t+2\right) \\ - s^3\left(2t^5+7t^4+8t^3+t^2+2t+1\right) - s^2(t+1)^2\left(t^3-1\right) \\ - 2s(t+1)^2\left(t^3-1\right) - (t+1)^2\left(t^3-1\right). \end{aligned}$$

Similarly we have

$$H_n(s,t) = -\left(s^2(2t+1) + 2s(t+1)^2 + (t+1)^2\right) h_n(s,t)$$

and $s^2(2t+1) + 2s(t+1)^2 + (t+1)^2 > 0$ for $s, t > 0$.

Also in Case 5 we see

$$u_1 = \frac{(s-1)^2(t+1)^2\left(s^3 - t^3\right)}{s^4(t+1)^2 - 2s^3(t+1)^2 + s^2t^3(t+2) - 2st^3(t+2) + t^3(t+2)}$$
$$u_2 = -\frac{s^5t^2 - 2s^4t^2 + s^3t^2 + s^2\left(t^5 + 2t^4 + t^3 - 2t^2 - 2t - 1\right) + 2st^2 - t^2}{s^4\left(-t^2\right) + 2s^3t^2 + s^2(t+1)^2\left(t^2+1\right) - 2st^2 + t^2}$$
$$u_{m_2} = -\frac{(t+1)^2\left(t^3-1\right)}{t^4 + 2t^3 + t^2 + 2t + 1}, \qquad u_{m_4} = \frac{(s-1)^3\left(s^2+s+1\right)}{s^4 - 2s^3 - s^2 + 2s - 1}$$

then

$$\begin{aligned} H_n = s^7 &\left(2t^6 + 6t^5 + 7t^4 + 8t^3 + 7t^2 + 4t + 1\right) \\ &- s^6\left(-2t^7 + 9t^5 + 14t^4 + 25t^3 + 23t^2 + 12t + 3\right) \\ &- s^5\left(6t^7 + 9t^6 - 10t^4 - 32t^3 - 28t^2 - 12t - 3\right) \\ &- s^4t^2\left(-3t^5 + 2t^4 + 16t^3 + 18t^2 + 21t + 9\right) \\ &+ s^3t^2\left(4t^5 + 19t^4 + 32t^3 + 21t^2 + 12t + 3\right) \\ &- s^2t^3\left(7t^4 + 23t^3 + 28t^2 + 9t + 3\right) \\ &+ 4st^5\left(t^2 + 3t + 3\right) - t^5\left(t^2 + 3t + 3\right) \end{aligned}$$

$$\begin{aligned} h_n = -s^5&\left(t^4 + 2t^3 + t^2 + 2t + 1\right) + s^4\left(-t^5 + 3t^3 + 3t^2 + 6t + 3\right) \\ &+ s^3\left(2t^5 + 3t^4 - 3t^2 - 6t - 3\right) + s^2t^3\left(t^2 + 3t + 3\right) \\ &- 2st^3\left(t^2 + 3t + 3\right) + t^3\left(t^2 + 3t + 3\right). \end{aligned}$$

It holds

$$H_n(s,t) = \left(-s^2\left(2t^2 + 2t + 1\right) + 2st^2 - t^2\right) h_n(s,t)$$

and $-s^2\left(2t^2 + 2t + 1\right) + 2st^2 - t^2 < 0$ for $s \in (0,1)$, $t > 0$. Thus we have $m_{B_1} = m_{B_2} = 0$ for both Case 3 and Case 5.

5. Procyon – an Example

Procyon is the α star in Canis Minor. It is a couple of binary star, Procyon A and B, which are the heavenly bodies where two fixed stars work on an orbit around the center of gravity of both. The mass of Procyon A is $1.42 \pm 0.04 M_s$ and of Procyon B is $0.575 \pm 0.017 M_s$ [3], where M_s is the weight of the sun, i.e., $1M_s = 1.989 \times 10^{30} kg$.

We consider a system of Procyon A and B as a initial two bodies A_1, A_2 of M.C.. We suppose A_1 and A_2 are not under the influence from other celestial bodies. If we assume $m_{A_1} = 1.42$ and $m_{A_2} = 0.575$, then we obtain $u \fallingdotseq 0.712$ and $\lambda = 1.995$ by solving simultaneous equations (4), namely $m_{A_1} = u\lambda = 1.42$ and $m_{A_2} = (1-u)\lambda = 0.575$,

Using Mathematica we obtain the solution of (9), (10) is $(s, t) \fallingdotseq (0.830, 0.408)$. Therefore the position and mass of each bodies are as follows in Table 1. Similarly, we can get their positions and mass in Case 3 and Case 5 (See Table 1).

Table 1. The positions and mass in each case Procyon.

		q_1	q_2	q_3	q_4
Case 2	position	-1.118	-0.288	0.304	0.712
	mass(M_s)	0	0.575	0	1.42
Case 3	position	-1.118	-0.288	0.712	1.259
	mass(M_s)	0	0.575	1.42	0
Case 5	position	-0.288	0.304	0.712	1.259
	mass(M_s)	0.575	0	1.42	0

Acknowledgements

My thanks go to Professor Akira Yosioka, who guided me in my research with great passion. I am grateful to be able to get guidance from such a great professor.

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