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GENERALIZED SEIBERG-WITTEN EQUATIONS ON A RIEMANN SURFACE

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Abstract. In this paper we consider twice-dimensionally reduced, generalized Seiberg-Witten (S-W) equations, defined on a compact Riemann surface. A novel feature of the reduction technique is that the resulting equations produce an extra "Higgs field". Under suitable regularity assumptions, we show that the moduli space of gauge-equivalent classes of solutions to the reduced equations, is a smooth Kähler manifold and construct a pre-quantum line bundle over the moduli space of solutions.

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1. Introduction

Dimensional reduction of gauge-theories have been instrumental in the understanding of Topological QFTs (TQFT). As motivating examples, one can consider the vortex equations [7], which are the dimensional reduction of four-dimensional Yang-Mills equations, the dimensional reduction and quantization of three-dimensional Chern-Simons gauge theory.

Seiberg-Witten gauge theory has been of interest to mathematicians, for as a TQFT, it provides new topological invariants which may provide new directions leading towards the classification of smooth, four-dimensional manifolds. Dimensional reduction of Seiberg-Witten equations to two-dimensions has been studied by Martin & Restuccia [8], Saclioglua & Nergiza [12] and Dey [6]. Except for [6], the reduction does not involve any Higgs field.

In this paper, we construct a dimensional reduction of *generalized Seiberg-Witten equations* in four-dimensions. The central element of this generalization involves construction of a non-linear Dirac operator by replacing the spinor representation $\mathbb{H}$ with a hyperKähler manifold admitting certain symmetries. The generalization was introduced by Taubes [13] for dimension three and extended to dimension four by Pidstrygach [10].

The reduction technique we use is similar to the one in [6]. Namely, we first consider the generalized Seiberg-Witten equations on $\mathbb{R}^4$ and then project the equations on the complex plane. The resulting equations are conformally invariant and therefore can be defined on any manifold modelled on $\mathbb{R}^2$ by using conformal maps - namely Riemann surfaces.

Under suitable regularity conditions, the moduli space of solutions to the reduced equations is a smooth Kähler manifold. If the Kähler two-form is integral, we show that the Quillen determinant line-bundle on the configuration space, descends as the pre-quantum line bundle over the moduli space. Following [4], we regard the moduli space as the phase space and define its Hilbert space quantization as the space of holomorphic sections of the Quillen determinant line-bundle.

The article is organized as follows: we first review the requisite preliminaries on the hyperKähler manifolds in Section 2 and then proceed to a quick introduction to the non-linear Dirac operator in four-dimensions in subsection (2.1). Using this, we introduce the generalized Seiberg-Witten equations. Although the generalization makes sense for any four-dimensional manifold, for the sake of simplicity and with the further exposition in mind, we stick to the simplest case where the base manifold is $\mathbb{R}^4$. In Section 3, we describe a dimensional reduction technique and define the reduced equations on $\mathbb{R}^2$. Using the conformal invariance of the equations, we define them on an arbitrary compact, oriented, Riemann surface. In Section 4 we show that the moduli space of gauge-equivalent solutions is a smooth, Kähler manifold. In the final Section 5, we describe the Quillen determinant line bundle construction on the moduli space.

2. Definitions and Notations

A *hyperKähler manifold* (M, g^M, I_1, I_2, I_3) is a $4n$-dimensional Riemannian manifold, endowed with three complex structures satisfying quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1$, such that the metric g^M is Kähler with respect to each I_j, $j = 1, 2, 3$.

In fact, for any $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ such that $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$, $I_\xi := \xi_1 I_1 + \xi_2 I_2 + \xi_3 I_3 \in \mathrm{End}(TM)$ is again a Kähler structure on M. In other words, M carries a family of Kähler structures, parametrized by two-sphere S^2. In particular, a hyperKähler manifold is a symplectic manifold in many different ways.

Suppose that a Lie group G acts smoothly on M, preserving the hyperKähler structure. Namely, the action is isometric and fixes the two-sphere of complex structures. Then G preserves the Kähler forms $\omega_1, \omega_2, \omega_3$, associated to I_1, I_2, I_3 respectively. Additionally, if the three associated symplectic moment maps exist, then they can be combined into a single *hyperKähler moment map* $\mu : M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra of G. Such an action of G on M for which the hyperKähler moment map exists is said to be *tri-Hamiltonian*.

Example 1. *Let* $M = \mathbb{H}$*. Then* $T\mathbb{H} = \mathbb{H} \times \mathbb{H}$*. For* $(h, v) \in T\mathbb{H}$*, define the complex structures*

$$I_1(h, v) = (h, -v\mathrm{i}), \qquad I_2(h, v) = (h, -v\mathrm{j}), \qquad I_3(h, v) = (h, -v\mathrm{k}).$$

We have $\omega = \frac{1}{2}\, \mathrm{d}\bar{h} \wedge \mathrm{d}h$*. Consider the* $\mathrm{U}(1)$*-action on* $\mathbb{H}$ *given by* $\mathrm{U}(1) \times \mathbb{H} \ni (z, h) \mapsto zh \in \mathbb{H}$*. The action preserves the three Kähler structures and is tri-Hamiltonian, with the hyperKähler moment map* $\mu : \mathbb{H} \longrightarrow \mathfrak{sp}(1) \cong \mathfrak{sp}(1)^*$ *given by*

$$\mu(h) = \frac{1}{2}\, \bar{h}\mathrm{i}h.$$

2.1. Generalized Seiberg-Witten on $\mathbb{R}^4$

Consider the flat Euclidean space $\mathbb{R}^4 = \mathbb{H}$, with co-ordinates (x_0, x_1, x_2, x_3). Fix the constant Spin-structure $c : \mathbb{H} = T_x\mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{H}$, given by

$$c(\xi) = \begin{pmatrix} * & \gamma(\xi) \\ -\gamma(\xi) & 0 \end{pmatrix}, \qquad \gamma(\xi) = \begin{pmatrix} \xi_0 + \mathrm{i}\xi_1 & \xi_2 + \mathrm{i}\xi_3 \\ -\xi_2 + \mathrm{i}\xi_3 & \xi_0 - \mathrm{i}\xi_1 \end{pmatrix}.$$

Thus, $\gamma(e_0) = I,\ \gamma(e_j) = I_j$ for $j = 1, 2, 3$. The covariant derivative of a spinor $u : \mathbb{R}^4 \longrightarrow \mathbb{H}$ is given by

$$Du(e_i) = \frac{\partial u}{\partial x_i}\cdot$$

Composing this with Clifford multiplication c, we obtain the Dirac operator $\mathcal{D}$: $C^\infty(\mathbb{R}^4, \mathbb{H}) \longrightarrow C^\infty(\mathbb{R}^4, \mathbb{H})$ on the space of positive spinors

$$\mathcal{D}^+ = -\frac{\partial}{\partial x_0} + \mathrm{i}\frac{\partial}{\partial x_1} + \mathrm{j}\frac{\partial}{\partial x_2} + \mathrm{k}\frac{\partial}{\partial x_3}\cdot$$

We say that a smooth map $u : \mathbb{R}^4 \longrightarrow \mathbb{H}$ is *harmonic* if $\mathcal{D}^+u = 0$. Clearly, the Dirac operator (and hence also the harmonicity condition) can be easily generalized to the case where $\mathbb{H}$ is replaced by an arbitrary hyperKähler manifold (M, g^M, I_1, I_2, I_3). More precisely, for a hyperKähler manifold (M, g^M, I_1, I_2, I_3) and a smooth map $u : \mathbb{R} \longrightarrow M$

$$\mathcal{D}u = -\frac{\partial u}{\partial x_0} + I_1\frac{\partial u}{\partial x_1} + I_2\frac{\partial u}{\partial x_2} + I_3\frac{\partial u}{\partial x_3}\cdot$$

The second ingredient we need in order to define the generalized S-W equations is a hyper-Kähler moment map. Assume that M admits a tri-Hamiltonian action of a compact Lie group G. Consider $\mathbb{R}^4$ with basic co-ordinates (x_1, x_2, x_3, x_4) and let P denote the trivial product bundle $\mathbb{R}^4 \times \mathrm{G} \longrightarrow \mathbb{R}^4$. A connection on P is described by a Lie-algebra-valued one-form

$$\mathsf{a} = a_0\, \mathrm{d}x_0 + a_1\, \mathrm{d}x_1 + a_2\, \mathrm{d}x_2 + a_3\, \mathrm{d}x_3 \ \in\ \Omega^1(\mathbb{R}^4, \mathfrak{g})$$

where $a_i : \mathbb{R}^4 \longrightarrow \mathfrak{g}$ are smooth maps. The curvature of a is a $\mathfrak{g}$-valued two-form

$$F(\mathsf{a}) = \sum_{i<j} F_{\mathsf{a}}^{ij}\, \mathrm{d}x_i \wedge \mathrm{d}x_j \in \Omega^2(\mathbb{R}^4, \mathfrak{g})$$

in which

$$F_{\mathsf{a}}^{ij} = \Big(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\Big) + [a_i, a_j].$$

For a smooth map $u : \mathbb{R}^4 \longrightarrow M$ and a connection a on P, we define the *twisted* Dirac operator by

$$\mathcal{D}_{\mathsf{a}} u = -\left(\frac{\partial u}{\partial x_0} + L_u^M a_0\right) + \sum_{i=1}^{3} I_i \left(\frac{\partial u}{\partial x_i} + L_u^M a_i\right)$$

where $L_u^M a_i$ denotes the fundamental vector field generated by the infinitesimal action of G on M, at a point $u(\cdot)$ given by

$$(L_u^M a_0)(p) = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\, a_0(p)) \cdot u(p)\Big|_{t=0}, \qquad p \in \mathbb{R}^4.$$

The generalized S-W equations for a pair $(u, \mathsf{a}) \in C^\infty(\mathbb{R}^4, M) \times \Omega^1(\mathbb{R}^4, \mathfrak{g})$ are given by

$$\begin{aligned} F_{\mathsf{a}}^+ + \mu \circ u &= 0 \\ \mathcal{D}_{\mathsf{a}} u &= 0 \end{aligned} \tag{1}$$

where, $F_{\mathsf{a}}^+ \in \Omega^2(\mathbb{R}^4, \Lambda_+^2(\mathbb{R}^4)^* \otimes \mathfrak{g})$ is the self-dual part of the curvature F_{a}. In the first equation we use the identification $\Lambda_+^2(\mathbb{R}^4)^* \cong \mathbb{R}^3$ and $\mathfrak{g} \cong \mathfrak{g}^*$ using an ad-invariant metric on $\mathfrak{g}$.

Equivalently, we can write the equations as

$$\begin{aligned} F_{\mathsf{a}}^{01} + F_{\mathsf{a}}^{23} + \mu_1 \circ u &= 0 \\ F_{\mathsf{a}}^{02} + F_{\mathsf{a}}^{31} + \mu_2 \circ u &= 0 \\ F_{\mathsf{a}}^{03} + F_{\mathsf{a}}^{12} + \mu_3 \circ u &= 0 \\ -\left(\frac{\partial u}{\partial x_0} + L_u^M a_0\right) + \sum_{i=1}^{3} I_i \left(\frac{\partial u}{\partial x_i} + L_u^M a_i\right) &= 0 \end{aligned} \tag{2}$$

where $\{\mu_1, \mu_2, \mu_3\}$ are the moment maps associated with the Kähler two-forms $\omega_1, \omega_2, \omega_3$ respectively.

3. Dimensional Reduction

In this section, we generalise the approach in [6] for the equations (2). Namely, we consider the generalised Seiberg-Witten equations on $\mathbb{R}^4$ and then project them on the complex plane. The resulting equations are conformally invariant and so can be defined on Riemann surfaces.

For the rest of this article, we will focus our attention to the case when $G = \mathrm{U}(1)$.

Identify the Lie algebra $\mathrm{i}\mathbb{R} \cong \mathbb{R}$ and assume that the Lie-algebra-valued functions $\{a_i\}_{i=0}^3$ are independent of (x_2, x_3). Then a_0, a_1 define a connection $\mathsf{a} := a_0 \mathrm{d}x_0 +$

$a_1 \mathrm{d}x_1$ over $\mathbb{R}^2$. The maps a_2 and a_3, which we re-label as ϕ_1 and ϕ_2, define an auxillary field $\phi = \phi_1 + \mathrm{i}\phi_2$ (also known as *Higgs fields*) on $\mathbb{R}^2$. The first equation now reads

$$\begin{aligned} F_{\mathsf{a}} + (\mu_1 \circ u)\omega_\Sigma &= 0 \\ \left(\frac{\partial \phi_1}{\partial x_1} - \frac{\partial \phi_2}{\partial x_2}\right) + \mu_2 \circ u &= 0 \\ \left(\frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_2}{\partial x_1}\right) + \mu_3 \circ u &= 0. \end{aligned} \tag{3}$$

Here ω_Σ is the volume form of Σ. From a more co-ordinate independent point of view, we have a connection a on a principal $\mathrm{U}(1)$-bundle P over $\mathbb{R}^2$ together with an auxiliary field

$$\phi \in \Omega^0\left(\mathbb{R}^2, \mathbb{C}\right).$$

Set $z = x_0 + \mathrm{i}x_1$ and define the one-form

$$\Phi = \phi \mathrm{d}z \in \Omega^1\left(\mathbb{R}^2, \mathbb{C}\right).$$

The second and the third equations in (3) can be combined into a single equation

$$\overline{\partial}\Phi + (\mu_c \circ u)\omega_\Sigma = 0 \tag{4}$$

where, $\mu_c \circ u := \mu_2 \circ u + \mathrm{i}\mu_3 \circ u$. The equations (3) now read

$$\begin{aligned} F_{\mathsf{a}} + (\mu_1 \circ u)\omega_\Sigma &= 0 \\ \overline{\partial}\Phi + (\mu_c \circ u)\omega_\Sigma &= 0. \end{aligned} \tag{5}$$

The fourth equation in (2) can be re-written as

$$-\left[\left(\frac{\partial u}{\partial x_0} + L_u^M a_0\right) - I_1\left(\frac{\partial u}{\partial x_1} + L_u^M a_1\right)\right] + \left(I_2 L_u^M \phi_1 + I_3 L_u^M \phi_2\right) = 0. \tag{6}$$

Let $I_{\mathbb{R}^2}$ denote the standard complex structure on $\mathbb{R}^2$, given by $\frac{\partial}{\partial x_1} = I_{\mathbb{R}^2}\left(\frac{\partial}{\partial x_0}\right)$ and $\frac{\partial}{\partial x_0} = -I_{\mathbb{R}^2}\left(\frac{\partial}{\partial x_1}\right)$. Observe that

$$\begin{aligned} a_0 &= \mathsf{a}\left(\frac{\partial}{\partial x_0}\right) \\ a_1 &= \mathsf{a}\left(\frac{\partial}{\partial x_1}\right) = \mathsf{a}\left(I_{\mathbb{R}^2}\left(\frac{\partial}{\partial x_0}\right)\right). \end{aligned}$$

Then the left hand side of (6) can be written as

$$\begin{aligned}\left(\frac{\partial u}{\partial x_0}+L_u^M a_0\right)-I_1\left(\frac{\partial u}{\partial x_1}+L_u^M a_1\right) &= \mathrm{d}u\left(\frac{\partial}{\partial x_0}\right)+L_u^M\left(\mathsf{a}\left(\frac{\partial}{\partial x_0}\right)\right)\\ &\quad - I_1\left(\mathrm{d}u\left(\frac{\partial}{\partial x_1}\right)+L_u^M\left(\mathsf{a}\left(\frac{\partial}{\partial x_1}\right)\right)\right)\\ &= D_{\mathsf{a}}u\left(\frac{\partial}{\partial x_0}\right)-I_1 D_{\mathsf{a}}u\left(I_{\mathbb{R}^2}\left(\frac{\partial u}{\partial x_0}\right)\right)\\ &= \left(D_{\mathsf{a}}u - I_1 D_{\mathsf{a}}u\circ I_{\mathbb{R}^2}\right)\left(\frac{\partial}{\partial x_0}\right)\\ &:= \partial_{\mathsf{a}}u\left(\frac{\partial}{\partial x_0}\right).\end{aligned}$$

On the other hand, observe now that

$$\phi_1=\Phi\left(\frac{\partial}{\partial x_0}\right)\qquad\text{and}\qquad \phi_2=\Phi\left(-\frac{\partial}{\partial x_1}\right).$$

The right-hand side can be expressed as

$$\begin{aligned}\left(I_2L_u^M\phi_1+I_3L_u^M\phi_2\right) &= I_2L_u^M\left(\Phi\left(\frac{\partial}{\partial x_0}\right)\right)+I_3L_u^M\left(\Phi\left(-\frac{\partial}{\partial x_1}\right)\right)\\ &= I_2L_u^M\left(\Phi\left(\frac{\partial}{\partial x_0}\right)\right)-I_3L_u^M\left(\Phi\left(I_{\mathbb{R}^2}\left(\frac{\partial}{\partial x_0}\right)\right)\right)\\ &= I_2L_u^M\left(\Phi\left(\frac{\partial}{\partial x_0}\right)\right)-I_1\left(I_2L_u^M\left(\Phi\left(I_{\mathbb{R}^2}\left(\frac{\partial}{\partial x_0}\right)\right)\right)\right)\\ &= \left(I_2L_u^M\Phi - I_1\left(I_2L_u^M\Phi\circ I_{\mathbb{R}^2}\right)\right)\left(\frac{\partial}{\partial x_0}\right)\\ &= \left(X_\Phi(u)\right)^{1,0}\left(\frac{\partial}{\partial x_0}\right).\end{aligned}$$

Combining this together with (5), we get the reduced equations on $\mathbb{R}^2$

$$\begin{aligned} F_{\mathsf{a}}+(\mu_1\circ u)\omega_\Sigma &= 0\\ \partial_{\mathsf{a}}u-\left(X_\Phi(u)\right)^{1,0} &= 0\\ \overline{\partial}\Phi+(\mu_c\circ u)\omega_\Sigma &= 0.\end{aligned}\tag{7}$$

The equations are conformally invariant and hence can be defined on manifolds modelled locally on $\mathbb{R}^2$, namely, Riemann surfaces.

3.1. Generalized Seiberg-Witten on a Riemann Surface

Let $(\Sigma, g_\Sigma, J_\Sigma)$ be a compact, oriented Riemann surface of genus g, with a conformal metric $ds^2 = h^2 \mathrm{d}z \otimes \mathrm{d}\bar{z}$. Let $\pi_P : P \to \Sigma$ be a principal $\mathrm{U}(1)$-bundle over Σ. Let (M, g^M, I_1, I_2, I_3) be a hyperKähler manifold endowed with a tri-Hamiltonian action of $\mathrm{U}(1)$.

Let $u \in C^\infty(P, M)^{\mathrm{U}(1)}$ be a smooth, $\mathrm{U}(1)$-equivariant map to M. Then the covariant derivative of u with respect to a connection a on P is given by

$$D_{\mathsf{a}} u = \mathrm{d}u + L_u^M \mathsf{a} \in \Omega^1(P, u^*TM)^{\mathrm{U}(1)}_{\mathrm{hor}}$$

where the subscript "hor" denotes that the one-form is horizontal. This therefore descends to a one form on Σ with values in $u^*TM/\mathrm{U}(1)$. The complex structure I_1 determines a $\mathrm{U}(1)$-invariant complex structure on $u^*TM \longrightarrow P$ and hence also on $u^*TM/\mathrm{U}(1) \longrightarrow \Sigma$. We denote by $\partial_{\mathsf{a}} u$ the $(1,0)$-part of the one-form $D_{\mathsf{a}} u$, with respect to I_1. More precisely

$$\partial_{\mathsf{a}} u = \frac{1}{2} \left(D_{\mathsf{a}} u - I_1 \circ D_{\mathsf{a}} u \circ J_\Sigma \right).$$

Denote the space of smooth $U(1)$-equivariant maps $u : P \to M$ by $C^\infty(P, M)^{\mathrm{U}(1)}$, the space of smooth connections on P by $\mathcal{A}(P)$.

Define the configuration space

$$\mathcal{C} = \mathcal{A}(P) \times C^\infty(P, M)^{\mathrm{U}(1)} \times \Omega^{1,0}(\Sigma, \mathbb{C}).$$

The space $\mathcal{C}$ is an infinite-dimensional Frechét manifold with an action of the gauge group $\mathcal{G} = C^\infty(P, \mathrm{U}(1))$ given by

$$g \cdot (\mathsf{a}, u, \Phi) \longmapsto \left(\mathsf{a} + g^{-1} \mathrm{d}g,\ g \cdot u\ , \Phi \right).$$

Note that the gauge group does not act on the Higgs field!

For $(\mathsf{a}, u, \Phi) \in \mathcal{C}$, we define the dimensional reduction of generalized Seiberg-Witten equations on Σ by

$$\begin{aligned} F_{\mathsf{a}} + (\mu_1 \circ u)\, \omega_\Sigma &= 0 \\ \partial_{\mathsf{a}} u - (X_\Phi(u))^{1,0} &= 0 \\ \overline{\partial} \Phi + (\mu_c \circ u)\, \omega_\Sigma &= 0. \end{aligned} \tag{8}$$

The equations (8) are invariant under the action of $\mathcal{G}$. The first and third equations require some explanation. For the first equation, we consider $F_{\mathsf{a}} \in \Omega^2(P, \mathbb{R})_{hor}$. For the third equation, observe that $\mu_c \circ u : P \longrightarrow \mathbb{C}$ is $\mathrm{U}(1)$-invariant and therefore descends to a complex-valued map on Σ, which we again denote by $\mu_c \circ u$.

4. Moduli Space

In this section, we show that the moduli space of gauge equivalent solutions to (8) can be realised as a Marsden-Weinstein reduction of a certain submanifold of the irreducible configuration space. In other words, we identify the solution to equations (8) with the zero locus of moment map and the one defining the submanifold. Further, we also show that the L^2-metric on the moduli space is a Kähler metric. The techniques we use are fairly similar to those of Hitchin used to study moduli spaces of vortices and Higgs bundles.

For the rest of the section we assume that the tri-Hamiltonian $\mathrm{U}(1)$-action on M is semi-free; i.e., outside the set of fixed points $M^{\mathrm{U}(1)}$, the action is free.

4.1. Abstract Setup

We will not describe the Sobolev completion of fibre-bundles here, but rather refer the interested reader to [14, Subsection 4.1 of Appendix B] for details. Set $k \geq 1$ and $p > 2$ satisfy $k - \frac{2}{p} > 0$ so that $W^{k,p} \hookrightarrow W^{1,2} \bigcap C^0$ by Sobolev embedding.

Fix a smooth fiducial connection A_0 on P and define $\mathcal{A}^{1,p}(P)$ to be the completion of $\mathcal{A}(P)$ with respect to A_0 in the $W^{1,p}$-Sobolev norm. For $E := (P \times_{\mathrm{U}(1)} M)/\mathrm{U}(1)$, denote by $W^{1,p}(P, M)^{\mathrm{U}(1)} \cong W^{1,p}(\Sigma, E)$ the Sobolev completion of $C^\infty(P, M)^{\mathrm{U}(1)}$, Lastly, let $W^{1,p}(\Sigma, \Lambda^{1,0}\Sigma \otimes \mathbb{C})$ denote the Sobolev completion of $\Omega^1(\Sigma, \mathbb{C})$ in $W^{1,p}$.

Since $kp > 2$, the Sobolev multiplication theorem $W^{1,p} \otimes W^{1,p} \longrightarrow L^p$ implies $\partial_{\mathsf{a}} u \in L^p\left(\Sigma,\ \Lambda^{1,0}\Sigma \otimes E_u\right)$, where $E_u := u^*TM/\mathrm{U}(1)$. Also, for $kp > 2$, the Sobolev composition law holds. Consequently, $\mu_1 \circ u \in W^{1,p}(\Sigma, \mathbb{R})$ and $\mu_c \circ u \in W^{1,p}(\Sigma, \mathbb{C})$.

Finally, let $\mathcal{G}^{2,p}$ denote the Sobolev completion of $\mathcal{G}$ in the $W^{2,p}$-norm. Then $\mathcal{G}^{2,p}$ is a Banach Lie group acting smoothly on $\mathcal{A}^{1,p}$. The Lie algebra is given by $\mathrm{Lie}(\mathcal{G}^{2,p}) = W^{2,p}(\Sigma, \mathbb{R})$.

Consider the infinite dimensional Banach manifold

$$\mathcal{C}^{1,p} = \mathcal{A}^{1,p}(P) \times W^{1,p}(\Sigma, E) \times W^{1,p}(\Sigma, \Lambda^{1,0}\Sigma \otimes \mathbb{C}).$$

The tangent to $\mathcal{C}^{1,p}$ at a point $q := (\mathsf{a}, u, \Phi) \in \mathcal{C}^{1,p}$ is given by

$$T_q\mathcal{C}^{1,p} = W^{1,p}(\Sigma, \Lambda^1\Sigma) \times W^{1,p}(\Sigma, E_u) \times W^{1,p}(\Sigma, \Lambda^{1,0}\Sigma \otimes \mathbb{C}).$$

Consider the infinite-dimensional vector bundle $\mathcal{E}^p \longrightarrow \mathcal{C}^{1,p}$, with fibre at a point $q \in \mathcal{C}^{1,p}$ being given by

$$\mathcal{E}^p_q = L^p(\Sigma, \Lambda^2\Sigma) \times L^p(\Sigma, \Lambda^{1,0}\Sigma \otimes E_u) \times L^p(\Sigma, \Lambda^2\Sigma \otimes \mathbb{C}).$$

Observe that the action of the gauge group $\mathcal{G}^{2,p}$ on $\mathcal{C}^{1,p}$ lifts to an action on $\mathcal{E}^p$. Define the equivariant section

$$\begin{aligned} &\mathcal{F} : \mathcal{C}^{1,p} \longrightarrow \mathcal{E}^p \\ &\mathcal{F}(\mathsf{a}, u, \Phi) = \Big(F_{\mathsf{a}} + (\mu_1 \circ u)\, \omega_\Sigma,\ \partial_{\mathsf{a}} u - (X_\Phi(u))^{1,0},\ \overline{\partial}\Phi + (\mu_c \circ u)\, \omega_\Sigma \Big). \end{aligned} \tag{9}$$

Then the solutions to (8) are the zeroes of $\mathcal{F}$.

4.2. Linearized Operator

The linearization of the equations (8) at a zero $q = (\mathsf{a}, u, \Phi) \in \mathcal{C}^{1,p}$ of $\mathcal{F}$ gives the operator

$$D_q : T_q\mathcal{C}^{1,p} \longrightarrow \mathcal{E}^p_q$$

$$D_q \begin{pmatrix} \alpha \\ \xi \\ \eta^{1,0} \end{pmatrix} \longmapsto \begin{pmatrix} \mathrm{d}\alpha + \mathrm{d}\mu_1(\xi)\, \omega_\Sigma \\ D_{\mathsf{a},u,\Phi}\xi + (L_u\alpha)^{1,0} - (X_\eta)^{1,0} \\ \overline{\partial}\eta^{1,0} + \mathrm{d}\mu_c(\xi)\, \omega_\Sigma \end{pmatrix}.$$

Here $D_{\mathsf{a},\Phi}\xi = (\nabla^{\mathsf{a}}\xi)^{1,0} + \left(\nabla_\xi X_\Phi\right)^{1,0}$, where ∇ is the Levi-Civita connection on (M, g^M). The induced connection on u^*TM is given by $\nabla^{\mathsf{a}}\xi + \nabla_\xi X_\Phi(u)$, where $\nabla^{\mathsf{a}}\xi = \nabla\xi + \nabla_\xi K^M_{\mathsf{a}}$.

The equivariance of the section $\mathcal{F} : \mathcal{C}^{1,p} \longrightarrow \mathcal{E}^p$ under the action of the gauge group $\mathcal{G}^{2,p}$ implies that we have the following complex

$$0 \to W^{2,p}(\Sigma, \mathbb{R}) \xrightarrow{\mathrm{d}_1} T_q\mathcal{C}^{1,p} \xrightarrow{\mathrm{d}_2} \mathcal{E}^p_q \to 0 \tag{10}$$

where $\mathrm{d}_1(\alpha) = (L_u\alpha,\ \mathrm{d}\alpha,\ 0) \in T_q\mathcal{C}^{k,2}$ and $\mathrm{d}_2\left(\alpha, \xi, \eta^{1,0}\right) = D_q\left(\alpha, \xi, \eta^{1,0}\right)$. Note that if we deform the complex by a homotopy, so as to get rid of the zeroeth order terms, the Euler characteristic or the symbols of the operators remain unchanged. In other words, the complex (10) can be written as a sum of three complexes

$$0 \to W^{2,p}(\Sigma, \mathbb{R}) \xrightarrow{\mathrm{d}} W^{1,p}(\Sigma, \Lambda^1\Sigma) \xrightarrow{\mathrm{d}_{\mathsf{a}}} L^p(\Sigma, \Lambda^2\Sigma) \to 0 \tag{11}$$

$$0 \to W^{1,p}(\Sigma, E_u) \xrightarrow{D_q} L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes E_u) \to 0 \tag{12}$$

$$0 \to W^{2,p}(\Sigma, \mathbb{C}) \xrightarrow{\partial} W^{1,p}(\Sigma, \Lambda^{1,0}\Sigma \otimes \mathbb{C}) \xrightarrow{\overline{\partial}} L^p(\Sigma, \Lambda^2\Sigma) \to 0. \tag{13}$$

Clearly, each of the above complexes is elliptic and consequently, (10) is an elliptic complex.

Let δ denote the equivariant map $\delta : P \longrightarrow E\mathrm{U}(1)$ which is a lift of the classifying map $\tilde{\delta} : \Sigma \longrightarrow B\mathrm{U}(1)$. Then $(u, \delta) : P \longrightarrow M \times E\mathrm{U}(1)$ descends to a map

$$\bar{u} : \Sigma \longrightarrow M_{\mathrm{U}(1)} := M \times_{\mathrm{U}(1)} E\mathrm{U}(1).$$

Define $[u] \in H_2(M_{\mathrm{U}(1)}, \mathbb{Z})$ to be the push-forward of the fundamental class of $[\Sigma]$ under the map $\bar{u}$.

Proposition 2. *The operator* $\mathrm{d}_1^* + \mathrm{d}_2 : T_q\mathcal{C}^{1,p} \longrightarrow \Omega^0(\Sigma, \mathbb{R})_{L^p} \oplus \mathcal{E}_q^p$ *is a Fredholm operator for every solution* $(\mathsf{a}, u, \Phi) \in \mathcal{C}^{1,p}$ *of* (8) *and has a real index given by*

$$\textit{Index}\ (\mathrm{d}_1^* + \mathrm{d}_2) = (2n-1)\chi(\Sigma) + 2\left\langle c_1^{\mathrm{U}(1)}(TM), [u]\right\rangle + 2g \qquad (14)$$

where $c_1^{\mathrm{U}(1)}(TM)$ *is the equivariant first Chern class of* TM.

Proof: The ellipticity of the complex (10) has the consequence that the operator $\mathrm{d}_1^* + \mathrm{d}_2 : T_q\mathcal{C}^{1,p} \longrightarrow \Omega^0(\Sigma, \mathbb{R})_{L^p} \oplus \mathcal{E}_q^p$ is Fredholm and therefore has a well-defined index. Since the complex (10) decomposes into three complexes, the index of (10) is the sum of indices of the complexes (11), (12), (13). The index for the operator

$$\Omega^1(\Sigma, \mathbb{R}) \longrightarrow \Omega^0(\Sigma, \mathbb{R}) \oplus \Omega^0(\Sigma, \mathbb{R}) : \quad \alpha \mapsto (\mathrm{d}^*\alpha, *\mathrm{d}\alpha)$$

is given by $-\chi(\Sigma)$. By Riemann-Roch theorem, the index of the complex (12) is given by $2\langle c_1^{\mathrm{U}(1)}(TM), [u]\rangle + 2n\chi(\Sigma)$. Finally, the index for the third complex (13) is $2g$. It is a simple observation now that $\mathrm{d}_1^* + \mathrm{d}_2$ is a compact perturbation of the elliptic operators. Therefore, the index of $\mathrm{d}_1^* + \mathrm{d}_2$ is given by

$$\text{Index}\ (\mathrm{d}_1^* + \mathrm{d}_2) = (2n-1)\chi(\Sigma) + 2\left\langle c_1^{\mathrm{U}(1)}(TM), [u]\right\rangle + 2g. \qquad (15)$$

The statement follows. ■

Note: Our computation is standard. For the case of the symplectic vortices, the index is just $(n - \dim(\mathrm{G}))\,\chi(\Sigma) + 2\left\langle c_1^{\mathrm{U}(1)}(TM), [u]\right\rangle$ where the dimension of the target symplectic manifold is $2n$. For our case, there is an additional contribution $2g$ due to Higgs field.

4.3. Transversality

In order to prove that the moduli space of solutions is a smooth Banach manifold, we need to establish transversality. Namely, we need to prove that $\mathcal{F}$ is transverse to the zero section $\iota : \mathcal{C}^{1,p} \longrightarrow \mathcal{E}^p$. The transversality condition in needed to ensure that the zero-set of the section, i.e., the space of solutions to the reduced equations, is a submanifold of the configuration space. In other words, this means that zero is a regular value of $\mathcal{F}$. We can achieve transversality by suitably perturbing the section $\mathcal{F}$. For a suitable class of such perturbations, the action of the gauge group on the space of solutions to the perturbed equations is free. We thus obtain the structure of a smooth manifold on the moduli space of solutions to the reduced equations.

The techniques used for this are almost verbatim to the ones used in [3, 9], for the case of symplectic vortices. So we skip mentioning the proofs.

Define the space of perturbations

$$\mathcal{P} := \{(\sigma_1, \sigma_2, \sigma_3) \in \mathcal{C} \; ; \; g \cdot (\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \sigma_2, \sigma_3), g \in \mathcal{G}\} .$$

For $c \in \mathbb{R}$, consider the perturbed equations

$$\begin{aligned} *F_{\mathsf{a}} - \mu_1 \circ u &= c + \sigma_1 \\ \partial_{\mathsf{a}} u - (X_\Phi(u))^{1,0} &= \sigma_2 \\ *\overline{\partial}\Phi^{1,0} + \mu_c \circ u &= \sigma_3. \end{aligned} \tag{16}$$

Fix a cohomology class $B \in H^2(M_{\mathrm{U}(1)}, \mathbb{Z})$ and for a fixed $\sigma \in \mathcal{P}$, define the solution space

$$N_\sigma(B, c) := \left\{(\mathsf{a}, u, \Phi) \in \mathcal{C}^{1,p} \; ; \; [u] = B \text{ and } (\mathsf{a}, u, \Phi) \text{ satisfy (16)}\right\} .$$

Observe that $N_\sigma(B, c)$ is invariant under the action of the gauge group $\mathcal{G}^{2,p}$. Let $M^{\mathrm{U}(1)}$ denote the fixed points of the U(1)-action on the hyperKähler manifold M. Define

$$C_0 = \mu_1 \left(M^{\mathrm{U}(1)}\right) - 2\pi \frac{\deg(P)}{\mathrm{Vol}(\Sigma)} \subset \mathbb{R}.$$

Define $\mathcal{S}_*^{1,p} := \left\{u \in \mathcal{S}^{1,p} \; ; \; [u] = B, \; u(P) \nsubseteq M^{\mathrm{U}(1)}\right\}$.

The following lemma follows almost verbatim from [9, Lemma 3.4.1, Cor. 3.4.2].

Lemma 3.

1. *Let $c \in \mathbb{R} \setminus C_0$ and define $\mathcal{P}_c = \{\sigma \in \mathcal{P} \; ; \; |\sigma_1| < |c - C_0|\}$. Then, for $\sigma \in \mathcal{P}_c$, if (a, u, Φ) satisfy (16), then $u(P) \nsubseteq M^{\mathrm{U}(1)}$.*
2. *If $\sigma \in \mathcal{P}_c$, the action of $\mathcal{G}^{2,p}$ on $N_\sigma(B, c)$ is free.*

Define the moduli space of solutions to be the quotient

$$\mathcal{M}_\sigma(B, c) := N_\sigma(B, c)/\mathcal{G}^{2,p}. \tag{17}$$

The following theorem follows almost verbatim from [9, Theorem 3.4.4].

Theorem 4. *Let $c \in \mathbb{R} \setminus C_0$. Then for any $\sigma \in \mathcal{P}_c$, the moduli space $\mathcal{M}_\sigma(B, c)$ is a smooth manifold of real dimension*

$$(2n-1)\chi(\Sigma) + 2\left\langle c_1^{\mathrm{U}(1)}(TM), B\right\rangle + 2g.$$

4.4. Kähler Structure on Moduli Space

In this section we show that the moduli space can be realized as a Marsden-Weinstein quotient of a submanifold of the configuration space $\mathcal{C}^{1,p}$. The arguments in this section follow the work in [1] on moduli space of Seiberg-Witten equations on Kähler surfaces.

Recall that for a finite dimensional symplectic manifold (M, ω), with a Hamiltonian action of a Lie group G, there exists a moment map $\mu : M \longrightarrow \mathfrak{g}^*$, which is unique up to addition by constants in the centre of $\mathfrak{g}$. The equivariance of the moment map implies that the zero-locus of μ is G-invariant. If 0 is a regular point of μ, then $\mu^{-1}(0)/G$ is again a symplectic manifold. This follows from the well-known Marsden-Weinstein reduction theorem. If M is Kähler and the group action preserves both the metric and the symplectic form, then the quotient is a Kähler manifold. If M is a hyperKähler manifold and the group action preserves the metric and all the three Kähler forms, then the quotient is again a hyperKähler manifold.

We now turn to the infinite-dimensional analogue of the above quotient constructions, namely for an action of the gauge group $\mathcal{G}^{2,p}$ on the configuration space $\mathcal{C}^{1,p}$. The configuration space $\mathcal{C}^{1,p}$ carries a hyper-Kähler structure, given by

$$\mathcal{I}_1 = \begin{pmatrix} * & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -* \end{pmatrix}, \qquad \mathcal{I}_2 = \begin{pmatrix} 0 & 0 & * \\ 0 & J & 0 \\ * & 0 & 0 \end{pmatrix}, \qquad \mathcal{I}_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & K & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The L^2-metric on $\mathcal{C}^{1,p}$, defined by

$$g^{\mathcal{C}}(X, Y) = \frac{1}{2}\int_\Sigma \alpha_1 \wedge *\alpha_2 + \frac{1}{2}\int_\Sigma g_u^M(\xi_1, \xi_2)\,\omega_\Sigma + \frac{1}{2}\int_\Sigma \eta_1 \wedge *\eta_2$$

where, $X = (\alpha_1, \xi_1, \eta_1)$, $Y = (\alpha_2, \xi_2, \eta_2) \in T_q\mathcal{C}^{1,p}$. Here, the pull-back metric $g_u^M : u^*TM \otimes u^*TM \longrightarrow \mathbb{R}$ is defined by

$$g_u^M((p, v), (p, w)) = g_{u(p)}^M(v, w), \ \ (p, v), (p, w) \in u^*TM \subset P \times TM.$$

The metric $g^{\mathcal{C}}$ is $\mathcal{G}^{2,p}$-invariant and is Hermitian with respect to all three complex structures; i.e., $g^{\mathcal{C}}(\mathcal{I}_i X, \mathcal{I}_i Y) = g^{\mathcal{C}}(X, Y)$ for all $i = 1, 2, 3$. Therefore, the associated two-forms $\Omega_i(\cdot, \cdot) = g^{\mathcal{C}}(\mathcal{I}_i(\cdot), \cdot)$ are clearly non-degenerate, closed and therefore Kähler. Thus the natural L^2-metric on the configuration space $\mathcal{C}^{1,p}$ is a hyperKähler metric. The action of the gauge group $\mathcal{G}^{2,p}$ on $\mathcal{C}^{1,p}$ preserves the metric and the hyperKähler structure and is therefore hyperHamiltonian.

The real and the complex moment maps for the gauge action are given by first and the third equations of (8) respectively. This can indeed be seen as follows:

The fundamental vector field for the infinitesimal action of the gauge group $\mathcal{G}^{2,p}$, at a point $q \in \mathcal{C}^{1,p}$ is given by $L_q^{\mathcal{C}}\gamma = (\mathrm{d}\gamma,\ L_u\gamma,\ 0)$. Define

$$\tilde{\mu}_{\mathcal{I}_1} : \mathcal{C}^{1,p} \longrightarrow \mathrm{Lie}(\mathcal{G}^{2,p})^*, \qquad \langle \tilde{\mu}_{\mathcal{I}_1}, \gamma \rangle(q) = \frac{1}{2}\int_\Sigma F_{\mathsf{a}} \cdot \gamma + \langle \gamma, \mu_1 \circ u \rangle\, \omega_\Sigma \quad (18)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing. Therefore, for $X = (\alpha, \xi, \eta) \in T_q\mathcal{C}^{1,p}$

$$\begin{aligned} \langle \mathrm{d}\tilde{\mu}_{\mathcal{I}_1}(X), \gamma \rangle(q) &= \frac{1}{2}\int_\Sigma \mathrm{d}\alpha \cdot \gamma + \langle \gamma, \mathrm{d}(\mu_1 \circ u)(\xi) \rangle\ \omega_\Sigma \\ &= \frac{1}{2}\int_\Sigma -\mathrm{d}\gamma \wedge \alpha + g_u^M(I_1 L_u^M \gamma, \xi)\, \omega_\Sigma \\ &= \iota_{L_q^{\mathcal{C}}\gamma}\Omega_1(X). \end{aligned}$$

Denote by $\mathcal{C}' \subset \mathcal{C}^{1,p}$, the subspace of all the solutions to second and third equations in (8)

$$\mathcal{C}' = \left\{ \left(\mathsf{a}, u, \Phi \in \mathcal{C}_*^{1,p}\right)\ ;\ \begin{pmatrix} \partial_{\mathsf{a}} u - (X_\Phi)^{1,0} \\ *\bar{\partial}\eta^{1,0} - \mu_c \circ u \end{pmatrix} = 0 \right\}. \tag{19}$$

Lemma 5. *$\mathcal{C}'$ is a complex submanifold of $\mathcal{C}^{1,p}$ with respect to the complex structure $\mathcal{I}_1$.*

Proof: We only need to show that $T\mathcal{C}' \subset T\mathcal{C}^{1,p}$ is a complex sub-bundle. Abbreviate $D_{\mathsf{a},u,\Phi} := D$ for simplicity. The tangent space at a point $q := (\mathsf{a}, u, \Phi) \in \mathcal{C}'$ is given by

$$T_q\mathcal{C}' = \left\{ \begin{pmatrix} \xi \\ \alpha \\ \eta \end{pmatrix}\ ;\ \begin{pmatrix} D\xi + (L_u\alpha)^{1,0} - (X_\eta)^{1,0} \\ *\bar{\partial}\eta^{1,0} - \mathrm{d}\mu_c(\xi) \end{pmatrix} = 0 \right\}. \tag{20}$$

To see that $\mathcal{I}_1$ preserves $T_q\mathcal{C}'$, observe that

$$\mathcal{I}_1 \begin{pmatrix} \alpha \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} *\alpha \\ -I_1\xi \\ -*\eta^{1,0} \end{pmatrix}.$$

But since D is a Cauchy-Riemann operator and $(L_u(*\alpha))^{1,0} = -I_1(L_u\alpha)^{1,0}$ and similarly $(X_{*\eta})^{1,0} = -I_1\,(X_\eta)^{1,0}$, we get

$$D(-I_1)\xi + (L_u(*\alpha))^{1,0} - (X_{*\eta})^{1,0} = (-I_1)\left(D\xi + (L_u\alpha)^{1,0} - (X_\eta)^{1,0}\right) = 0.$$

Since, $\eta^{1,0} \in \Omega^{1,0}(\Sigma,\mathbb{C})$, $*\eta^{1,0} = -\mathrm{i}\eta$. Also, $\mathrm{d}\mu_c(-I_1\xi) = -\mathrm{i}\mathrm{d}\mu_c(\xi)$. Clearly then,

$$**\,\overline{\partial}*\eta^{1,0} - \mathrm{d}\mu_c(I_1\xi) = -\mathrm{i}\left(*\overline{\partial}\eta^{1,0} - \mathrm{d}\mu_c(\xi)\right) = 0.$$

Therefore $\mathcal{I}_1$ preserves $T\mathcal{C}' \subset T\mathcal{C}^{1,p}$. ■

The L^2-metric restricts to a Kähler metric on $\mathcal{C}'$. The induced action of the gauge group $\mathcal{G}^{2,p}$ preserves the induced metric and the symplectic two-form Ω_1. $\mathcal{C}'$ also admits a momentum map $\mu'_{\mathcal{I}_1}$ which is just a restriction of the momentum map $\tilde{\mu}_{\mathcal{I}_1}$ on the configuration space. We denote this restriction by $\tilde{\mu}_{\mathcal{I}_1}$ itself. The solutions to the dimensionally reduced generalized Seiberg-Witten equations now correspond to the quotient of the zero locus of the momentum map $\tilde{\mu}_{\mathcal{I}_1}$ by the gauge group $\mathcal{G}^{2,p}$. Using the standard arguments in Kähler geometry, we now show that the L^2-metric that is induced on the quotient $\mathcal{M} := \tilde{\mu}_{\mathcal{I}_1}^{-1}\{0\}/\mathcal{G}^{2,p}$ is a Kähler metric.

Theorem 6. *Let Σ be a connected, compact, oriented, Riemannian surface and let $\mu'_{\mathcal{I}_1} : \mathcal{C}' \longrightarrow \mathbb{R}$ denote the restriction of the moment map $\tilde{\mu}_{\mathcal{I}_1}$ as in* (18) *for the action of the gauge group $\mathcal{G}^{2,p}$ on $\mathcal{C}'$. Then the metric induced on the quotient $(\mu'_{\mathcal{I}_1})^{-1}(0)/\mathcal{G}^{2,p} := \mathcal{M}$ is a Kähler metric.*

Proof: The submanifold $(\mu'_{\mathcal{I}_1})^{-1}(0) \subset \mathcal{C}' \subset \mathcal{C}^{1,p}$ carries a natural L^2-metric, induced from the L^2-metric on $\mathcal{C}^{1,p}$. The action of the gauge group is by isometries, which implies that there exists a unique Riemannian metric on the quotient such that $\pi : (\mu'_{\mathcal{I}_1})^{-1}(0) \longrightarrow \mathcal{M}$ is a Riemannian submersion. Let $X, Y \in \Gamma(\mathcal{M}, T\mathcal{M})$ and $\widetilde{X}, \widetilde{Y}$ denote the horizontal lifts to $(\mu'_{\mathcal{I}_1})^{-1}(0)$. Then the covariant derivative of $\widetilde{Y}$ with respect to $\widetilde{X}$ is given by

$$\nabla^{\mathcal{M}}_X Y = \pi_*\left(\nabla^{(\mu'_{\mathcal{I}_1})^{-1}(0)}_{\widetilde{X}}\widetilde{Y}\right)$$

where $\nabla^{(\mu'_{\mathcal{I}_1})^{-1}(0)}$ denotes the restriction of the Levi-Civita connection on the configuration space to $(\mu'_{\mathcal{I}_1})^{-1}(0)$. We identify the pull-back of $T\mathcal{M}$ with the horizontal sub-bundle of $T\left((\mu'_{\mathcal{I}_1})^{-1}(0)\right)$

$$\pi^*(TM) \cong \mathcal{H}\left((\mu'_{\mathcal{I}_1})^{-1}(0)\right) = \mathcal{H}(\mathcal{C}')|_{(\mu'_{\mathcal{I}_1})^{-1}(0)} \bigcap T\left((\mu'_{\mathcal{I}_1})^{-1}(0)\right).$$

But the restriction to $(\mu'_{\mathcal{I}_1})^{-1}(0)$ of $T\mathcal{C}'$ splits L^2-orthogonally as

$$\begin{aligned} T\mathcal{C}'|_{(\mu'_{\mathcal{I}_1})^{-1}(0)} &= \mathcal{H}\left((\mu'_{\mathcal{I}_1})^{-1}(0)\right) \oplus \left(\mathcal{H}\left((\mu'_{\mathcal{I}_1})^{-1}(0)\right)\right)^{\perp} \\ &\cong \pi^* T\mathcal{M} \oplus \left(\mathcal{H}\left((\mu'_{\mathcal{I}_1})^{-1}(0)\right)\right)^{\perp} \\ &= \pi^* T\mathcal{M} \oplus \operatorname{im}(T_0) \oplus \left(\ker(\mu'_{\mathcal{I}_1})^{\perp}\right) \end{aligned}$$

where T_0 denotes the linearization of the orbit map. In order to define the complex structure on $T\mathcal{M}$, it now suffices to show that $(\pi^* T\mathcal{M})^{\perp}$ is preserved by the induced complex structure $\mathcal{I}'_1$ and

- Note that the image of an element $\xi \in \Omega^0(\Sigma, \mathbb{R})$ under the linearization of the orbit map through $q = (\mathsf{a}, u, \Phi)$ is the same as the fundamental vector field $L_q\xi$. We have

  $$\langle \mathcal{I}_1 L_q\xi,\ Z\rangle = \Omega_1(L_q\xi, Z) = \langle \mathrm{d}\mu'_{\mathcal{I}_1}(Z), \xi\rangle.$$

  This implies that $\mathcal{I}_1 L_q\xi \perp \ker(\mathrm{d}\mu'_{\mathcal{I}_1})$.

- Let $Z \in \ker(\mathrm{d}\mu'_{\mathcal{I}_1})$ and $\mathcal{I}_1 Z \perp \operatorname{im}(T_0)$. Then for $\xi \in \Omega^0(\Sigma, \mathbb{R})$,

  $$0 = \langle \mathcal{I}_1 Z,\ L_q\xi\rangle = -\Omega_1(L_q\xi, Z) = -\langle \mathrm{d}\mu'_{\mathcal{I}_1}(Z), L_q\xi\rangle.$$

  Therefore $Z \in \ker(\mathrm{d}\mu'_{\mathcal{I}_1}) \bigcap \ker(\mathrm{d}\mu'_{\mathcal{I}_1})^{\perp} = \{0\}$.

Hence the complex structure preserves the splitting and defines a complex structure $\mathcal{I}^{\mathcal{M}}$ on $\mathcal{M}$.

It only remains to show that the complex structure is parallel. But this follows directly from the fact that

- The complex structure $\mathcal{I}'_1$ on $\mathcal{C}'$ is parallel.
- The Levi-Civita connection on $\mathcal{M}$ is given by the projection of the Levi-Civita connection on $(\mu'_{\mathcal{I}_1})^{-1}(0)$.
- Projection commutes with the complex structures $\mathcal{I}'_1$ and $\mathcal{I}^{\mathcal{M}}$.

Thus we have proved that the induced complex structure on $\mathcal{M}$ is parallel with respect to the Levi-Civita connection for the induced L^2-metric on $\mathcal{M}$. Therefore the L^2-metric on $\mathcal{M}$ is Kähler. ■

5. Pre-Quantum Line-Bundle on the Moduli Space Under the Assumption of Integrality Condition

Geometric Quantization

Given a symplectic manifold (M, ω), with ω integral (i.e., its cohomology class is in $H^2(M, \mathbb{Z})$), one can construct a Hermitian line bundle with a connection (called the pre-quantum line bundle) whose curvature Ω is proportional to the symplectic form ω. One can assign to functions $f \in C^\infty(M)$, an operator, $\hat{f} = -\mathrm{i}\nabla_{X_f} + f$ acting on the Hilbert space of square integrable sections of L (the wave functions). Here $\nabla = \mathrm{d} - \mathrm{i}\theta$ where locally $\omega = \mathrm{d}\theta$ and X_f is defined by $\omega(X_f, \cdot) = -\mathrm{d}f(\cdot)$. We have taken $\hbar = 1$. This assignment has the property that the Poisson bracket (induced by the symplectic form), correspond to an operator proportional to the commutator, i.e., if $f_3 = \{f_1, f_2\}_{PB}$ then $\hat{f}_3$ is proportional to $[\hat{f}_1, \hat{f}_2]$ for any two functions f_1, f_2.

The Hilbert space of pre-quantization is usually too big for most purposes. Geometric quantization involves construction of a polarization of the symplectic manifold such that we now take polarized sections of the line bundle, yielding a finite dimensional Hilbert space in most cases. However, $\hat{f}$ does not map the polarized Hilbert space to the polarized Hilbert space in general. Thus only a few observables from the set of all $f \in C^\infty(M)$ are quantizable.

This method of quantization was developed by Kostant and Souriau and discussed at length in Woodhouse [16].

We now come back to the situation at hand. We construct a prequantum line bundle over the configuration space. It will descend to the moduli space as long as the symplectic form on the moduli space is integral. In the following we assume that the form is integral.

Let ρ denote the local Kähler potential for the first symplectic form ω_1 of M, our target hyperKähler manifold. Local potentials exist for any Kähler form. Let $p \in P$. Then $u(p) \in M$. Let V_p be a neighbourhood of $u(p)$ such that $\rho(u(p))$ is local a Kähler potential for ω_1 in V_p. $u^{-1}(V_p)$ is a covering of P which has a finite covering, namely $u^{-1}(V_{p_i})$, $i = 1, 2..., N$. Let ϕ_i , $i = 1, ..., N$ be a partition of unity subordinate to this finite covering of P. Let ρ_i, $i = 1, ..., N$ be the local Kähler potential for V_{p_i} for the form ω_1. Let $z = \pi(p)$ be a point on the Riemann surface Σ. Define

$$\rho_0(u) = \int_\Sigma \Sigma_{i=1}^N \rho_i(u(\pi^{-1}(z)))\phi_i(\pi^{-1}(z))\omega_\Sigma.$$

Let us define on the configuration space parametrized by the triple (a, u, Φ) a Quillen determinant bundle $\mathcal{Q} = \det(\bar{\partial}_{\mathsf{a}})$, [11], i.e., a line bundle whose the fiber over (a, u, Φ) is given by

$$\wedge^{\text{top}}(\ker \partial_{\mathsf{a}})^* \otimes \wedge^{\text{top}}(\text{coker } \partial_{\mathsf{a}}).$$

Following the idea in [2], we modify the Quillen metric $\exp(-\zeta_A'(0))$ by multiplying it with $\exp(\frac{\mathrm{i}}{4\pi}(\rho_0(u))$ and $\exp\left(\frac{\mathrm{i}}{8\pi}\int_\Sigma \Phi \wedge *\Phi\right)$, where $\Phi = \phi \mathrm{d}z - \bar{\phi}\mathrm{d}\bar{z}$. From the metric, one can calculate the curvature by the formula $\delta_w \delta_{\bar{w}} log||\sigma||$ where w is the holomorphic coordinate on the configuration space and σ is the canonical section of the determinant bundle [11]. The holomorphic coordinates on the configuration space is given by $(\mathsf{a}^{0,1}, u, \Phi)$ w.r.t. the complex structure $\mathcal{I}_1$. The first term in the metric, namely $\exp(-\zeta_{\mathsf{a}}'(0))$ contributes to the curvature by a term $\frac{\mathrm{i}}{2\pi}\left(-\frac{1}{2}\int_\Sigma \pi_!\left(\alpha_1 \wedge \alpha_2\right)\right)$, [4, 11], which is the first term in $\Omega_1(X, Y)$. The second term in the metric contributes to the second term in Ω_1. This can be seen as follows. Let $u(p) = (u_1(p), ..., u_n(p))$, in some local coordinate system centered at $u(p) \in M$ where $n = \dim M$. Once again $z = \pi(p)$.

$$\begin{aligned}
\gamma &= \delta_u \delta_{\bar{u}} \rho_0(u) = \int_\Sigma \Sigma_{i=1}^N \delta_u \delta_{\bar{u}} \rho_i(u(\pi^{-1}(z)))\phi_i(\pi^{-1}(z)))\omega_\Sigma \\
&= \int_\Sigma \Sigma_{i=1}^N \Sigma_{j=1}^{\dim M} \delta_{u_j} \delta_{\bar{u_j}} \rho_i(u(\pi^{-1}(z)))\phi_i(\pi^{-1}(z)))\omega_\Sigma \\
&= \frac{1}{2}\int_\Sigma \Sigma_{i=1}^N g_u^M\left(I_1 \cdot, \cdot\right)\phi_i(\pi^{-1}(z))\omega_\Sigma = \frac{1}{2}\int_\Sigma g_u^M\left(I_1 \cdot, \cdot\right)\omega_\Sigma
\end{aligned}$$

where we have used the fact that since ρ_i is a Kähler potential for ω_1 on V_{p_i}, $\Sigma_{j=1}^{\dim M} \delta_{u_j} \delta_{\bar{u_j}} \rho_i(u(\pi^{-1}(z))) = g_u^M\left(I_1 \cdot, \cdot\right)$ and $\Sigma_{i=1}^N \phi_i(\pi^{-1}(z)) = 1$. Then $\gamma(\xi_1, \xi_2)$ $= \int_\Sigma g_u^M(I_1\xi_1, \xi_2)\omega_\Sigma$. The third term in the metric contributes to the third term in Ω_1. This can be seen as follows: Recall $\Phi = \phi \mathrm{d}z - \bar{\phi}\mathrm{d}\bar{z}$, $*\Phi = \bar{\phi}\mathrm{d}\bar{z} + \phi \mathrm{d}z$ so that $\exp\left(\frac{\mathrm{i}}{8\pi}\int_\Sigma \Phi \wedge *\Phi\right) = \exp\left(\frac{\mathrm{i}}{4\pi}\int_\Sigma (\phi\bar{\phi})\mathrm{d}z \wedge \mathrm{d}\bar{z}\right)$. Let

$$\begin{aligned}
\tau &= \delta_\phi \delta_{\bar{\phi}}\left(\log\left(\exp\left(\frac{\mathrm{i}}{4\pi}\int_\Sigma (\phi\bar{\phi})\mathrm{d}z \wedge \mathrm{d}\bar{z}\right)\right)\right) \\
&= \frac{\mathrm{i}}{4\pi}\int_\Sigma \delta_\phi \delta_{\bar{\phi}}(\phi\bar{\phi})\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{\mathrm{i}}{4\pi}\int_\Sigma \left(\delta\phi \otimes \delta\bar{\phi} - \delta\bar{\phi} \otimes \delta\phi\right)\mathrm{d}z \wedge \mathrm{d}\bar{z}.
\end{aligned}$$

Then, $\tau(\eta_1, \eta_2) = \frac{\mathrm{i}}{4\pi}\int_\Sigma \eta_1 \wedge \eta_2$. The three terms combined gives us the following proposition:

Proposition 7. *On the configuration space, the Quillen bundle $\mathcal{Q}$ equipped with the modified metric mentioned above has curvature* $\frac{\mathrm{i}}{2\pi}\Omega_1$.

As in [4, 5], it can be shown that this line bundle descends to the moduli space as long as the descendent of Ω_1 is integral.

Proposition 8. *If the symplectic form (i.e., the descendent of Ω_1) on the moduli space is integral, the Quillen bundle $\mathcal{Q}$ equipped with the modified metric mentioned descends to the moduli space and has curvature the descendent of* $\frac{\mathrm{i}}{2\pi}\Omega_1$.

It is holomorphic and is a prequantum bundle since its curvature is proportional to the symplectic form with the proportionality constant $\frac{\mathrm{i}}{2\pi}\cdot$ Since the moduli space is Kahler and the line bundle is holomorphic, one can take holomorphic square integrable sections of this bundle as the Hilbert space of quantization.

6. Summary and Discussion

The dimensional reduction technique mentioned gives us a generalization of the Symplectic vortex equations ($\Phi = 0$). It is well-known that the invariants for Hamiltonian group actions on a symplectic manifold are related to Gromov-Witten invariants for its symplectic reduction [17]. Assuming that we at least have the moduli space of finite volume, for $\Phi = 0$, we must get equivalence between invariants for Hamiltonian group action on $\mu_c \circ u = 0$ and Gromov-Witten invariants for hyperKähler reduction of M. The latter are known to be trivial! This gives rise to an interesting question as to whether the presence of a non-zero Higgs-field helps us define non-trivial, Gromov-Witten like invariants for hyper-Kähler manifolds.

References

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