



STURMIAN SPIRALS IN LORENTZ-MINKOWSKI PLANE

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Abstract. The Sturm spirals which can be introduced as those plane curves whose curvature radius is equal to the distance from the origin are embedded in to one parameter family of curves. In this paper, we consider the spacelike and timelike Sturmian spirals in Lorentz-Minkowski plane.

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1. Introduction

The general theory of curves in Euclidean space (more generally in a Riemannian manifold) has been developed long time ago and we have a deep knowledge of their local and global geometries [7]. When the ambient space is a Lorentz-Minkowski

space (and more generally a semi-Riemannian space or a semi-Riemannian manifold), our knowledge is definitely more restricted in comparison with the Euclidean case.

Nevertheless the fundamental existence and uniqueness theorem in the theory of plane curves which states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [1] and [12]) is still valid. However, again the simplicity of the situation is elusive as in many cases it is impossible to find the curve explicitly.

One should notice also that while some classes of space curves in the Lorentz-Minkowski space like helices, Bertrand and Mannheim curves (see [3–5], [8]), are well documented as in the Euclidean case, the studies on planar curves in pseudo-Euclidean plane are not so many. For example, up to our knowledge, spiral curves are not studied in the Lorentz-Minkowski plane. They are representatives of the plane curves whose curvature depends solely on the distance from the origin in the Euclidean plane. These class of curves was already studied in some detail in [2, 13] and [14] where it was proven that they could be reconstructed via quadratures.

Also, Singer [13] proved that the problem of determining a curve whose curvature is $\kappa(r)$, where r is the distance from the origin, is solvable by quadratures. It is known that Sturmian spirals provide examples of such curves. These curves and their generalizations were studied in [10]. Here we take the same problem in Lorentz-Minkowski plane and consider both spacelike and timelike plane curves. The paper is organized as follows: Section 2 reviews some basic notions in Lorentz-Minkowski plane. In Section 3, we consider the spacelike Sturmian spirals and their generalization. Section 4 deals with their timelike analogues. All cases are presented by analytic formulas and illustrated graphically.

2. Preliminaries

The Lorentz-Minkowski plane \mathbb{E}_1^2 is the Euclidean plane \mathbb{E}^2 equipped with indefinite flat metric given by

$$g = -dx^2 + dz^2$$

where (x, z) are the rectangular coordinates on \mathbb{E}_1^2 . Recall that a vector $v \in \mathbb{E}_1^2 \setminus \{0\}$ is spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$. In particular, the vector $v = 0$ is said to be a spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in \mathbb{E}_1^2 , can locally be spacelike, timelike, if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike.

A spacelike or timelike curve α is parametrized by the arclength parameter s if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([11]).

A spacelike curve in Lorentz-Minkowski plane can be parametrized ([6], [7], [9]) as

$$\mathbf{x}(s) = \left(\int_0^s \sinh \Phi(s) ds, \int_0^s \cosh \Phi(s) ds \right) \quad (1)$$

with the Frenet vector fields

$$\mathbf{T}(s) = (\sinh \Phi(s), \cosh \Phi(s)), \quad \mathbf{N}(s) = (\cosh \Phi(s), \sinh \Phi(s)) \quad (2)$$

and the curvature function

$$\kappa(s) = g\left(\frac{d\mathbf{T}(s)}{ds}, \mathbf{N}(s)\right) = -\frac{d\Phi(s)}{ds}. \quad (3)$$

The Frenet vectors fields satisfy the relations

$$\frac{d\mathbf{x}}{ds} = \mathbf{T}, \quad \frac{d\mathbf{T}}{ds} = -\kappa\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}. \quad (4)$$

A timelike curve in Lorentz-Minkowski plane can be parametrized [6] respectively as

$$\mathbf{x}(s) = \left(\int_0^s \cosh \Phi(s) ds, \int_0^s \sinh \Phi(s) ds \right) \quad (5)$$

with the Frenet vector fields

$$\mathbf{T}(s) = (\cosh \Phi(s), \sinh \Phi(s)), \quad \mathbf{N}(s) = (\sinh \Phi(s), \cosh \Phi(s)) \quad (6)$$

and the curvature function

$$\kappa(s) = g\left(\frac{d\mathbf{T}(s)}{ds}, \mathbf{N}(s)\right) = \frac{d\Phi(s)}{ds}. \quad (7)$$

In this case the Frenet vectors fields satisfy the relations

$$\frac{d\mathbf{x}}{ds} = \mathbf{T}, \quad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = \kappa\mathbf{T}. \quad (8)$$

3. Spacelike Spirals in Lorentz-Minkowski Plane

In this section, we consider the spacelike Sturmian spirals and the generalized spacelike Sturmian spirals by using the following relation

$$\kappa(s) = \frac{\sigma}{r}, \quad r = \sqrt{|g(\mathbf{x}(s), \mathbf{x}(s))|}. \quad (9)$$

Let \mathbf{x} is a spacelike curve and the position vector \mathbf{x} be specified as

$$\mathbf{x}(s) = \xi(s)\mathbf{T}(s) + \eta(s)\mathbf{N}(s) \quad (10)$$

where ξ and η are differentiable functions of s . Differentiating (10) with respect to s and using (4), we get

$$\mathbf{T} = (\xi' - \eta\kappa)\mathbf{T} + (\eta' - \xi\kappa)\mathbf{N}. \quad (11)$$

From (10) and (11), we have

$$\xi' = 1 + \eta\kappa, \quad \eta' = \xi\kappa, \quad \kappa = \frac{\sigma}{r} = \frac{\sigma}{\sqrt{\varepsilon(\xi^2 - \eta^2)}} \quad (12)$$

where $\varepsilon = \pm 1$ is chosen so $\varepsilon(\xi^2 - \eta^2) \geq 0$.

By multiplying the first equation in (12) by ξ , the second one by η and summing the so obtained expressions, we find

$$\xi = \varepsilon r r'. \quad (13)$$

Substituting back (13) into the second equation in (12) and integrating the so obtained expression we end up with the formula

$$\eta = \varepsilon \int r \kappa(r) dr + c \quad (14)$$

where c is the integration constant.

3.1. Spacelike Sturmian Spirals

By the very definition [15] of these curve, they have the property that at each point their curvature radius \mathfrak{R} is equal to the distance r from the origin. Thus, κ is given by the third equation in (12) and $\sigma = 1$.

From (13) and (14), we get

$$\eta = \varepsilon r + c, \quad r' = \frac{\varepsilon \sqrt{(1 + \varepsilon)r^2 + 2\varepsilon cr + c^2}}{r}. \quad (15)$$

Now, we consider $\varepsilon = 1$ and $\varepsilon = -1$ cases separately.

i) Let $\varepsilon = 1$. Then we have

$$\eta = r + c, \quad r' = \frac{\sqrt{2r^2 + 2cr + c^2}}{r}, \quad c > 0. \quad (16)$$

It is convenient to perform the respective integration in (16) by switching to a new independent variable t defined by

$$\frac{ds}{dt} = r. \quad (17)$$

This leads to the following results

$$\begin{aligned} \xi(t) &= \frac{c}{\sqrt{2}} \cosh(\sqrt{2}t), & \eta(t) &= \frac{c}{2} (\sinh(\sqrt{2}t) + 1) \\ r(t) &= \frac{c}{2} (\sinh(\sqrt{2}t) - 1), & \Phi &= -t. \end{aligned} \quad (18)$$

By rewriting (10) in its components, we obtain the relations

$$x = \xi \sinh \Phi + \eta \cosh \Phi, \quad z = \xi \cosh \Phi + \eta \sinh \Phi \quad (19)$$

which combined with the above findings provide the sought parameterization of the Sturmian spirals

$$\begin{aligned} x &= -\frac{c}{\sqrt{2}} \cosh(\sqrt{2}t) \sinh t + \frac{c}{2} (\sinh(\sqrt{2}t) + 1) \cosh t \\ z &= \frac{c}{\sqrt{2}} \cosh(\sqrt{2}t) \cosh t - \frac{c}{2} (\sinh(\sqrt{2}t) + 1) \sinh t. \end{aligned} \quad (20)$$

ii) Let $\varepsilon = 1$ and $c = 0$. Then we have

$$\begin{aligned} \xi(s) &= 2s + \sqrt{2}d, & \eta(s) &= \sqrt{2}s + d, & d &\in \mathbb{R} \\ r(s) &= \sqrt{2}s + d, & \Phi(s) &= -\frac{1}{\sqrt{2}} \ln \left| \sqrt{2}s + d \right|. \end{aligned} \quad (21)$$

Thus, it can be easily obtained that

$$\begin{aligned} x &= (2s + \sqrt{2}d) \sinh \Phi + (\sqrt{2}s + d) \cosh \Phi \\ z &= (2s + \sqrt{2}d) \cosh \Phi + (\sqrt{2}s + d) \sinh \Phi. \end{aligned} \quad (22)$$

iii) Let $\varepsilon = -1$. Then from (15) we get

$$\eta = -r + c, \quad r' = \frac{-\sqrt{c^2 - 2cr}}{r}, \quad c < 0. \quad (23)$$

Exchanging as before the arc-length parameter with t via (17), we obtain

$$\xi(t) = ct, \quad \eta(t) = \frac{c}{2} (1 + t^2), \quad r(t) = \frac{c}{2} (1 - t^2) \quad (24)$$

where $t \in (-\infty, -1) \cup (1, \infty)$.

The integration of (3) leads to

$$\Phi = -t \quad (25)$$

which combined with (19) produces the parametrization

$$\begin{aligned} x &= -ct \sinh t + \frac{c}{2} (1 + t^2) \cosh t \\ z &= ct \cosh t - \frac{c}{2} (1 + t^2) \sinh t. \end{aligned} \quad (26)$$

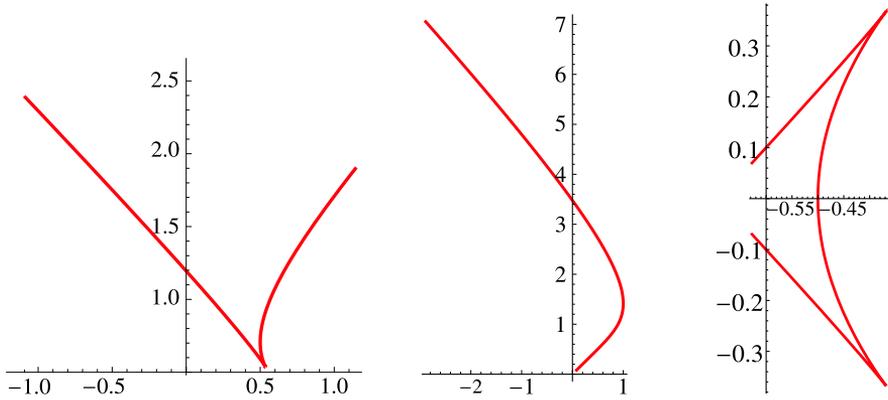


Figure 1. The standard spacelike Sturmian spirals generated by (20) with $c = 1$ (left), by (22) with $d = 0$ (middle) and by (26) with $c = -1$ (right).

3.2. Generalized Spacelike Sturmian Spirals

Forgetting about the restriction on the value of σ in previous section, one can consider as well the other two obvious possibilities, $\sigma > 1$ and $0 < \sigma < 1$ which have to be viewed as a generalization of the spacelike Sturmian spirals.

Case 1. Let $\sigma > 1$. Then from (13) and (14), we get

$$\xi = \sqrt{\sigma_\varepsilon^2 r^2 + 2\varepsilon\sigma cr + c^2}, \quad \eta = \varepsilon\sigma r + c, \quad \sigma_\varepsilon = \sqrt{\sigma^2 + \varepsilon} \quad (27)$$

and

$$r' = \frac{\varepsilon\sqrt{\sigma_\varepsilon^2 r^2 + 2\varepsilon\sigma cr + c^2}}{r}. \quad (28)$$

Now, let us consider again the cases $\varepsilon = 1$ and $\varepsilon = -1$ separately.

i) Let $\varepsilon = 1$. Then we have

$$\eta = \sigma r + c, \quad c \in \mathbb{R}. \quad (29)$$

and

$$r' = \frac{\sqrt{\sigma_+^2 r^2 + 2\sigma cr + c^2}}{r}, \quad \sigma_+ = \sqrt{\sigma^2 + 1}. \quad (30)$$

Changing the parameter as in (17), we get

$$\begin{aligned} \xi(t) &= \frac{c}{\sigma_+} \cosh(\sigma_+ t), & \eta(t) &= \frac{c}{\sigma_+^2} (\sigma \sinh(\sigma_+ t) + 1) \\ r(t) &= \frac{c}{\sigma_+^2} (\sinh(\sigma_+ t) - \sigma), & \Phi &= -\sigma t. \end{aligned} \quad (31)$$

Further on, by making use of formula (10) one finds easily the parametrization of the generalized Sturmian spirals

$$\begin{aligned} x &= \left[-\frac{c}{\sigma_+} \cosh(\sigma_+ t) \right] \sinh(\sigma t) + \left[\frac{c}{\sigma_+^2} (\sigma \sinh(\sigma_+ t) + 1) \right] \cosh(\sigma t) \\ z &= \left[\frac{c}{\sigma_+} \cosh(\sigma_+ t) \right] \cosh(\sigma t) - \left[\frac{c}{\sigma_+^2} (\sigma \sinh(\sigma_+ t) + 1) \right] \sinh(\sigma t). \end{aligned} \quad (32)$$

ii) Let $\varepsilon = -1$. Then from (27) and (28), we get

$$\xi = -\sqrt{\sigma_-^2 r^2 - 2\sigma cr + c^2}, \quad \eta = -\sigma r + c, \quad \sigma_- = \sqrt{\sigma^2 - 1}, \quad c > 0 \quad (33)$$

and

$$r' = -\frac{\sqrt{\sigma_-^2 r^2 - 2\sigma cr + c^2}}{r}. \quad (34)$$

One easily concludes that the expression under the radical on the right-hand side is positive provided that $c > 0$ and if r lies in the interval

$$\frac{c}{\sigma + 1} < r < \frac{c}{\sigma - 1}. \quad (35)$$

Passing to the t -parametrization (17), we find

$$\begin{aligned} \xi(t) &= -\frac{c}{\sigma_-} \sinh(\sigma_- t), & \eta(t) &= -\frac{c}{\sigma_-^2} (\sigma \cosh(\sigma_- t) + 1) \\ r(t) &= \frac{c}{\sigma_-^2} (\cosh(\sigma_- t) + \sigma), & \Phi &= -\sigma t. \end{aligned} \quad (36)$$

By using (36) in (10), we get the components of the Sturmian spirals

$$\begin{aligned} x &= \left[\frac{c}{\sigma_-} \sinh(\sigma_- t) \right] \sinh(\sigma t) - \left[\frac{c}{\sigma_-^2} (\sigma \cosh(\sigma_- t) + 1) \right] \cosh(\sigma t) \\ z &= \left[-\frac{c}{\sigma_-} \sinh(\sigma_- t) \right] \cosh(\sigma t) + \left[\frac{c}{\sigma_-^2} (\sigma \cosh(\sigma_- t) + 1) \right] \sinh(\sigma t). \end{aligned} \quad (37)$$

Case 2. Let $\sigma > 1$ and $c = 0$. Then from (27) and (28), we obtain

$$\xi = \sigma_\varepsilon r, \quad \eta = \varepsilon \sigma r, \quad r' = \varepsilon \sigma_\varepsilon. \quad (38)$$

These lead further to the useful formulas

$$\begin{aligned} \xi(s) &= \varepsilon \sigma_\varepsilon^2 s + d \sigma_\varepsilon, & \eta(s) &= \sigma \sigma_\varepsilon s + d \varepsilon \sigma, & d &\in \mathbb{R} \\ r(s) &= \varepsilon \sigma_\varepsilon s + d, & \Phi(s) &= -\frac{\varepsilon \sigma}{\sigma_\varepsilon} \ln |\varepsilon \sigma_\varepsilon s + d|. \end{aligned} \quad (39)$$

Inserting these results in (10) one finds the parametrization

$$\begin{aligned} x &= [\varepsilon \sigma_\varepsilon^2 s + d \sigma_\varepsilon] \sinh \Phi + [\sigma \sigma_\varepsilon s + d \varepsilon \sigma] \cosh \Phi \\ z &= [\varepsilon \sigma_\varepsilon^2 s + d \sigma_\varepsilon] \cosh \Phi + [\sigma \sigma_\varepsilon s + d \varepsilon \sigma] \sinh \Phi. \end{aligned} \quad (40)$$

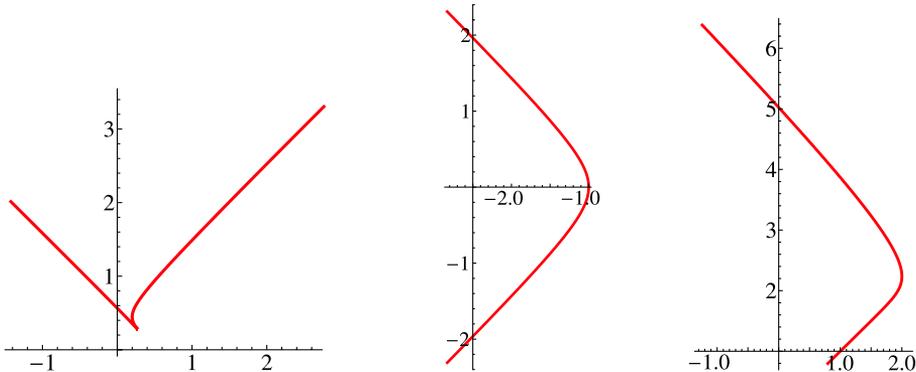


Figure 2. The generalized spacelike Sturmian spirals generated by (32) with $c = 1$ and $\sigma = 2$ (left), by (37) with $c = 1$ and $\sigma = 2$ (middle), by (40) with $d = 0$ and $\varepsilon = 1$ (right).

Case 3. Let $0 < \sigma < 1$. This time via (27) and (28), we get

$$\xi = \sqrt{\sigma_\varepsilon^2 r^2 + 2\varepsilon \sigma c r + c^2}, \quad \eta = \varepsilon \sigma r + c, \quad c \in \mathbb{R} \quad (41)$$

and

$$r' = \frac{\varepsilon \sqrt{\sigma_\varepsilon^2 r^2 + 2\varepsilon \sigma c r + c^2}}{r}. \quad (42)$$

As before we consider $\varepsilon = 1$ and $\varepsilon = -1$ cases separately.

i) Let $\varepsilon = 1$. Then we have

$$\eta = \sigma r + c, \quad r' = \frac{\sqrt{\sigma_+^2 r^2 + 2\sigma c r + c^2}}{r}. \quad (43)$$

By going to the new independent variable t as in (17), we find

$$\begin{aligned} \xi(t) &= \frac{c}{\sigma_+} \cosh(\sigma_+ t), & \eta(t) &= \frac{c}{\sigma_+^2} (\sigma \sinh(\sigma_+ t) + 1) \\ r(t) &= \frac{c}{\sigma_+^2} (\sinh(\sigma_+ t) - \sigma), & \Phi &= -\sigma t. \end{aligned} \quad (44)$$

Rewriting (10) in its components, we obtain the parametrization

$$\begin{aligned} x &= \left[-\frac{c}{\sigma_+} \cosh(\sigma_+ t) \right] \sinh(\sigma t) + \left[\frac{c}{\sigma_+^2} (\sigma \sinh(\sigma_+ t) + 1) \right] \cosh(\sigma t) \\ z &= \left[\frac{c}{\sigma_+} \cosh(\sigma_+ t) \right] \cosh(\sigma t) - \left[\frac{c}{\sigma_+^2} (\sigma \sinh(\sigma_+ t) + 1) \right] \sinh(\sigma t). \end{aligned} \quad (45)$$

ii) Let $\varepsilon = -1$. Then from (27) and (28), we get

$$\eta = -\sigma r + c, \quad r' = -\frac{\sqrt{\sigma_-^2 r^2 - 2\sigma c r + c^2}}{r}. \quad (46)$$

One easily concludes that the expression under the radical on the right-hand side is positive provided that

$$c > 0 \quad \text{and} \quad 0 < r < \frac{c}{\sigma + 1}. \quad (47)$$

By the similar techniques used till now we obtain

$$\begin{aligned} \xi(t) &= \frac{c}{\omega} \cos(\omega t), & \eta(t) &= \frac{c}{\omega^2} (1 + \sigma \sin(\omega t)), & \omega &= \sqrt{1 - \sigma^2} \\ r(t) &= \frac{c}{\omega^2} (-\sin(\omega t) + \sigma), & t &\in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega} \right], & \Phi &= -\sigma t. \end{aligned} \quad (48)$$

Substituting (44) in (10), we have

$$\begin{aligned} x &= \left[-\frac{c}{\omega} \cos(\omega t)\right] \sinh(\sigma t) + \left[\frac{c}{\omega^2} (1 + \sigma \sin(\omega t))\right] \cosh(\sigma t) \\ z &= \left[\frac{c}{\omega} \cos(\omega t)\right] \cosh(\sigma t) - \left[\frac{c}{\omega^2} (1 + \sigma \sin(\omega t))\right] \sinh(\sigma t). \end{aligned} \tag{49}$$

Case 4. Let $0 < \sigma < 1$, $c = 0$ and $\varepsilon = 1$. Then we have

$$\begin{aligned} \xi(s) &= \sigma_+^2 s + d\sigma_+, & \eta(s) &= \sigma\sigma_+ s + \sigma d \\ r(s) &= \sigma_+ s + d, & \Phi(s) &= -\frac{\sigma}{\sigma_+} \ln|\sigma_+ s + d|. \end{aligned}$$

Considering the equations (10) and (48) together, we find the following

$$\begin{aligned} x &= [\sigma_+^2 s + d\sigma_+] \sinh \Phi + [\sigma\sigma_+ s + \sigma d] \cosh \Phi \\ z &= [\sigma_+^2 s + d\sigma_+] \cosh \Phi + [\sigma\sigma_+ s + \sigma d] \sinh \Phi. \end{aligned} \tag{50}$$

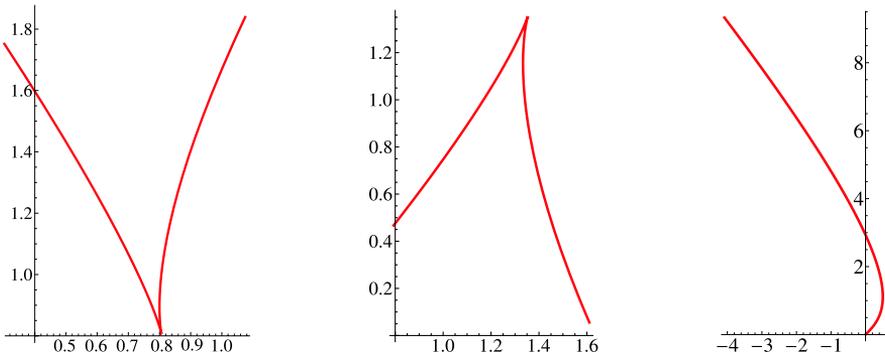


Figure 3. The generalized spacelike Sturmian spirals generated by (45) with $c = 1$, $\sigma = \frac{1}{2}$ (left) by (49) with $c = 1$, $\sigma = \frac{1}{2}$ (middle) and by (50) with $d = \frac{1}{2}$ and $\sigma = \frac{1}{2}$ (right).

4. Timelike Sturmian Spirals in Lorentz-Minkowski Plane

In this section, we study the timelike Sturm spirals and the generalized timelike Sturmian spirals by considering relations (9).

Let \mathbf{x} be a timelike curve and the position vector \mathbf{x} is presented in the form

$$\mathbf{x}(s) = \xi(s)\mathbf{T}(s) + \eta(s)\mathbf{N}(s) \tag{51}$$

where ξ and η are differentiable functions of s .

Differentiating (51) with respect to s and using (8), we get

$$\mathbf{T} = (\xi' + \eta\kappa)\mathbf{T} + (\eta' + \xi\kappa)\mathbf{N}. \quad (52)$$

From (51) and (52), we have

$$\xi' = 1 - \eta\kappa, \quad \eta' = -\xi\kappa, \quad \kappa = \frac{\sigma}{r} = \frac{\sigma}{\sqrt{\varepsilon(\eta^2 - \xi^2)}} \quad (53)$$

where $\varepsilon = \pm 1$ and such that $\varepsilon(\eta^2 - \xi^2) \geq 0$.

Multiplying the first equation above by ξ , the second one by η and summing the so obtained expressions we end up with the equation

$$\xi = -\varepsilon r r'. \quad (54)$$

Substituting back the relation (54) into the second equation of (53) and integrating we have finally

$$\eta = \varepsilon \int r \kappa(r) dr + c \quad (55)$$

where c is the integration constant.

4.1. Timelike Sturmian Spirals

From [15] these curves have the property that at each point their curvature radius \mathfrak{R} is equal to the distance r from the origin. Thus, κ is given by the third formula in (53) in which $\sigma = 1$. From (54) and (55) we have

$$\eta = \varepsilon r + c, \quad r' = \frac{-\varepsilon \sqrt{(1 - \varepsilon)r^2 + 2\varepsilon cr + c^2}}{r}, \quad c > 0 \quad (56)$$

and let us consider again the cases $\varepsilon = 1$ and $\varepsilon = -1$ separately.

i) Let $\varepsilon = 1$. Then we have

$$\eta = r + c, \quad r' = \frac{-\sqrt{2cr + c^2}}{r}. \quad (57)$$

Using (17), we get the followings

$$\xi(t) = ct, \quad \eta(t) = \frac{c}{2}(t^2 + 1), \quad r(t) = \frac{c}{2}(t^2 - 1), \quad \Phi = t. \quad (58)$$

By rewriting (51) in its components, we obtain the relations

$$x = \xi \cosh \Phi + \eta \sinh \Phi, \quad z = \xi \sinh \Phi + \eta \cosh \Phi \quad (59)$$

which combined with the above findings leads to the parametrization of the Sturm spirals

$$x = ct \cosh t + \frac{c}{2} (t^2 + 1) \sinh t, \quad z = ct \sinh t + \frac{c}{2} (t^2 + 1) \cosh t. \quad (60)$$

ii) Let $\varepsilon = -1$. Then from (56) we have directly

$$\eta = -r + c, \quad r' = \frac{\sqrt{2r^2 - 2cr + c^2}}{r}, \quad c > 0. \quad (61)$$

Changing the parameter like in (17), leads to

$$\xi(t) = \frac{c}{\sqrt{2}} \cosh(\sqrt{2}t), \quad \eta(t) = \frac{c}{2} (1 - \sinh(\sqrt{2}t)) \quad (62)$$

$$r(t) = \frac{c}{2} (1 + \sinh(\sqrt{2}t)), \quad \Phi = t. \quad (63)$$

By using (51), we get finally

$$\begin{aligned} x &= \frac{c}{\sqrt{2}} \cosh(\sqrt{2}t) \cosh t + \frac{c}{2} (1 - \sinh(\sqrt{2}t)) \sinh t \\ z &= \frac{c}{\sqrt{2}} \cosh(\sqrt{2}t) \sinh t + \frac{c}{2} (1 - \sinh(\sqrt{2}t)) \cosh t. \end{aligned} \quad (64)$$

iii) Let $\varepsilon = -1$ and $c = 0$. Then we have

$$\begin{aligned} \xi(s) &= 2s + \sqrt{2}d, & \eta(s) &= -\sqrt{2}s - d, & d &\in \mathbb{R} \\ r(s) &= \sqrt{2}s + d, & \Phi(s) &= \frac{1}{\sqrt{2}} \ln |\sqrt{2}s + d|. \end{aligned} \quad (65)$$

Thus, it can be easily obtained that

$$\begin{aligned} x &= (2s + \sqrt{2}d) \cosh \Phi + (-\sqrt{2}s - d) \sinh \Phi \\ z &= (2s + \sqrt{2}d) \sinh \Phi + (-\sqrt{2}s - d) \cosh \Phi. \end{aligned} \quad (66)$$

4.2. Generalized Timelike Sturmian Spirals

Disregarding the restriction on the value of σ , one can consider as well the other two obvious possibilities, $\sigma > 1$ and $0 < \sigma < 1$ which have to be viewed as generalizations of the timelike Sturmian spirals.

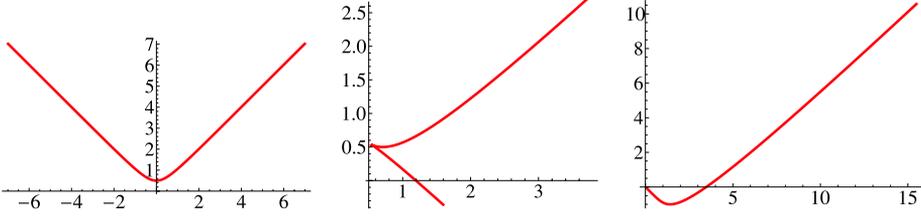


Figure 4. The standard timelike Sturmian spirals generated by (60) with $c = 1$ (left), (64) with $c = 1$ (middle) and (66) with $d = 0$ (right).

Case 5. Let $\sigma > 1$. Then from (54) and (55), we get

$$\xi = \sqrt{(\sigma^2 - \varepsilon) r^2 + 2\varepsilon\sigma cr + c^2}, \quad \eta = \varepsilon\sigma r + c \quad (67)$$

and

$$r' = -\frac{\varepsilon\sqrt{\sigma_-^2 r^2 + 2\varepsilon\sigma cr + c^2}}{r}. \quad (68)$$

Let us consider the cases $\varepsilon = 1$ and $\varepsilon = -1$ separately.

i) Let $\varepsilon = 1$. Then we have

$$\eta = \sigma r + c, \quad r' = -\frac{\sqrt{\sigma_-^2 r^2 + 2\sigma cr + c^2}}{r}, \quad c > 0. \quad (69)$$

Using as independent variable t and (51), we find

$$\begin{aligned} \xi(t) &= -\frac{c}{\sigma_-} \sinh(\sigma_- t), & \eta(t) &= \frac{c}{\sigma_-} (\sigma \cosh(\sigma_- t) - 1) \\ r(t) &= \frac{c}{\sigma_-^2} (\cosh(\sigma_- t) - \sigma), & \Phi &= \sigma t. \end{aligned} \quad (70)$$

If we rewrite (51) in its components, we get the parametrization

$$\begin{aligned} x &= \left[-\frac{c}{\sigma_-} \sinh(\sigma_- t) \right] \cosh \sigma t + \left[\frac{c}{\sigma_-^2} (\sigma \cosh(\sigma_- t) - 1) \right] \sinh \sigma t \\ z &= \left[-\frac{c}{\sigma_-} \sinh(\sigma_- t) \right] \sinh \sigma t + \left[\frac{c}{\sigma_-^2} (\sigma \cosh(\sigma_- t) - 1) \right] \cosh \sigma t. \end{aligned} \quad (71)$$

ii) Let $\varepsilon = -1$. Then from (67) and (68), we get

$$\xi = \sqrt{\sigma_+^2 r^2 - 2\sigma cr + c^2}, \quad \eta = -\sigma r + c, \quad c > 0 \quad (72)$$

and

$$r' = \frac{\sqrt{\sigma_+^2 r^2 - 2\sigma cr + c^2}}{r}. \quad (73)$$

By a similar change of the parameter as in (17), we have

$$\begin{aligned} \xi(t) &= \frac{c}{\sigma_+} \cosh(\sigma_+ t), & \eta(t) &= \frac{c}{\sigma_+^2} (1 - \sigma \sinh(\sigma_+ t)) \\ r(t) &= \frac{c}{\sigma_+^2} (\sinh(\sigma_+ t) + \sigma), & \Phi &= \sigma t. \end{aligned} \quad (74)$$

Relying on the equations (51) and (74) one gets at the end

$$\begin{aligned} x &= \left[\frac{c}{\sigma_+} \cosh(\sigma_+ t) \right] \cosh \sigma t + \left[\frac{c}{\sigma_+^2} (1 - \sigma \sinh(\sigma_+ t)) \right] \sinh \sigma t \\ z &= \left[\frac{c}{\sigma_+} \cosh(\sigma_+ t) \right] \sinh \sigma t + \left[\frac{c}{\sigma_+^2} (1 - \sigma \sinh(\sigma_+ t)) \right] \cosh \sigma t. \end{aligned} \quad (75)$$

Case 6. Let $\sigma > 1$ and $c = 0$. Then from (67) and (68), we obtain

$$\xi = \sqrt{\sigma^2 - \varepsilon} r, \quad \eta = \varepsilon \sigma r, \quad r' = -\varepsilon \sqrt{\sigma^2 - \varepsilon}. \quad (76)$$

It can be easily obtained that

$$\begin{aligned} \xi(s) &= -\varepsilon(\sigma^2 - \varepsilon)s + d\sqrt{\sigma^2 - \varepsilon}, & \eta(s) &= -\sigma\sqrt{\sigma^2 - \varepsilon}s + d\varepsilon\sigma \\ r(s) &= -\varepsilon\sqrt{\sigma^2 - \varepsilon}s + d, & \Phi(s) &= \frac{-\varepsilon\sigma}{\sqrt{\sigma^2 - \varepsilon}} \ln \left| -\varepsilon\sqrt{\sigma^2 - \varepsilon}s + d \right|. \end{aligned}$$

Thus, we have

$$\begin{aligned} x &= \left(-\varepsilon(\sigma^2 - \varepsilon)s + d\sqrt{\sigma^2 - \varepsilon} \right) \cosh \Phi + \left(-\sigma\sqrt{\sigma^2 - \varepsilon}s + d\varepsilon\sigma \right) \sinh \Phi \\ z &= \left(-\varepsilon(\sigma^2 - \varepsilon)s + d\sqrt{\sigma^2 - \varepsilon} \right) \sinh \Phi + \left(-\sigma\sqrt{\sigma^2 - \varepsilon}s + d\varepsilon\sigma \right) \cosh \Phi. \end{aligned} \quad (77)$$

Case 7. Let $0 < \sigma < 1$. Then via (67) and (68), we get

$$\xi = \sqrt{(\sigma^2 - \varepsilon)r^2 + 2\varepsilon\sigma cr + c^2}, \quad \eta = \varepsilon\sigma r + c \quad (78)$$

and

$$r' = \frac{-\varepsilon\sqrt{(\sigma^2 - \varepsilon)r^2 + 2\varepsilon\sigma cr + c^2}}{r}. \quad (79)$$

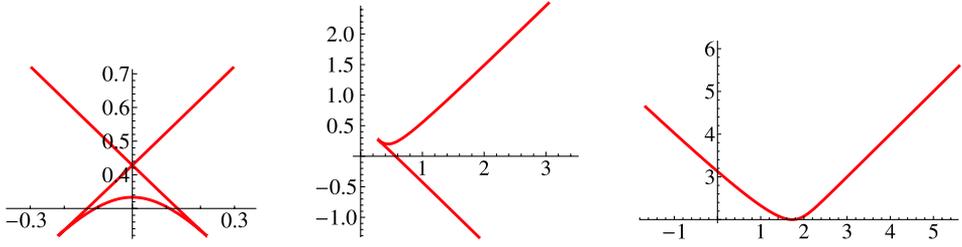


Figure 5. The generalized timelike Sturmiian spirals generated by (71) with $c = 1$, $\sigma = 2$ (left), by (75) with $c = 1$, $\sigma = 2$ (middle), and via (77) with $d = 0$, $\varepsilon = 1$ and $\sigma = 2$ (right).

Now, we consider $\varepsilon = 1$ or $\varepsilon = -1$ separately.

i) Let $\varepsilon = 1$. Then we have

$$\eta = \sigma r + c, \quad r' = \frac{-\sqrt{(\sigma^2 - 1)r^2 + 2\sigma cr + c^2}}{r}. \quad (80)$$

One easily concludes that the expression under the radical on the right-hand side is positive provided that

$$c > 0 \quad \text{and} \quad 0 < r < -\frac{c}{\sigma + 1}.$$

Using new independent variable, we find

$$\begin{aligned} \xi(t) &= \frac{c}{\omega} \cosh(\omega t), & \eta(t) &= \frac{c}{\omega^2} (-\sigma \sinh(\omega t) + 1) \\ r(t) &= \frac{c}{\omega^2} (-\sinh(\omega t) + \sigma), & \Phi &= \sigma t \end{aligned} \quad (81)$$

This time equations (51) and (81) give rise to

$$\begin{aligned} x &= \left[\frac{c}{\omega} \cosh(\omega t) \right] \cosh \sigma t + \left[\frac{c}{\omega^2} (-\sigma \sinh(\omega t) + 1) \right] \sinh \sigma t \\ z &= \left[\frac{c}{\omega} \cosh(\omega t) \right] \sinh \sigma t + \left[\frac{c}{\omega^2} (-\sigma \sinh(\omega t) + 1) \right] \cosh \sigma t. \end{aligned} \quad (82)$$

ii) Let $\varepsilon = -1$. Then from (67) and (68), we get

$$\eta = -\sigma r + c, \quad r' = \frac{\sqrt{\sigma_+^2 r^2 - 2\sigma cr + c^2}}{r} \quad c > 0. \quad (83)$$

By using as independent variable t defined in (17), we find immediately

$$\begin{aligned} \xi(t) &= \frac{c}{\sigma_+} \cos(\sigma_+ t), & \eta(t) &= \frac{c}{\sigma_+^2} (1 - \sigma \sin(\sigma_+ t)) \\ r(t) &= \frac{c}{\sigma_+^2} (\sin(\sigma_+ t) + \sigma), & \Phi &= \sigma t, & t &\in \left[-\frac{\pi}{2\sigma_+}, \frac{\pi}{2\sigma_+}\right]. \end{aligned} \quad (84)$$

From (51) and (84), we reach the equations below

$$\begin{aligned} x &= \left[\frac{c}{\sigma_+} \cos(\sigma_+ t) \right] \cosh \sigma t + \left[\frac{c}{\sigma_+^2} (1 - \sigma \sin(\sigma_+ t)) \right] \sinh \sigma t \\ z &= \left[\frac{c}{\sigma_+} \cos(\sigma_+ t) \right] \sinh \sigma t + \left[\frac{c}{\sigma_+^2} (1 - \sigma \sin(\sigma_+ t)) \right] \cosh \sigma t. \end{aligned} \quad (85)$$

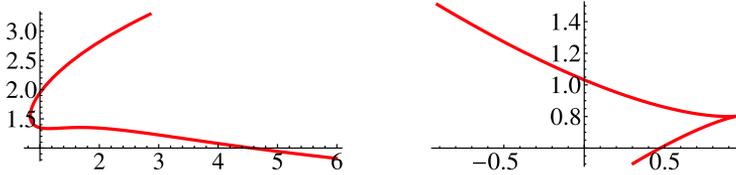


Figure 6. The generalized timelike Sturmian spirals generated by (82) with $c = 1$, $\sigma = \frac{1}{2}$ (left), and (85) with $c = 1$, $\sigma = \frac{1}{2}$ (right).

Case 8. Let $0 < \sigma < 1$, $c = 0$ and $\varepsilon = -1$. Then we get

$$\begin{aligned} \xi(s) &= \sigma_+^2 s + d\sigma_+, & \eta(s) &= -\sigma\sigma_+ s - \sigma d \\ r(s) &= \sigma_+ s + d, & \Phi(s) &= \frac{\sigma}{\sigma_+} \ln |\sigma_+ s + d|. \end{aligned}$$

By rewriting (51) in its components, we get

$$\begin{aligned} x &= [(\sigma_+^2) s + d\sigma_+] \cosh \Phi + [-\sigma\sigma_+ s - \sigma d] \sinh \Phi \\ z &= [(\sigma_+^2) s + d\sigma_+] \sinh \Phi + [-\sigma\sigma_+ s - \sigma d] \cosh \Phi. \end{aligned} \quad (86)$$

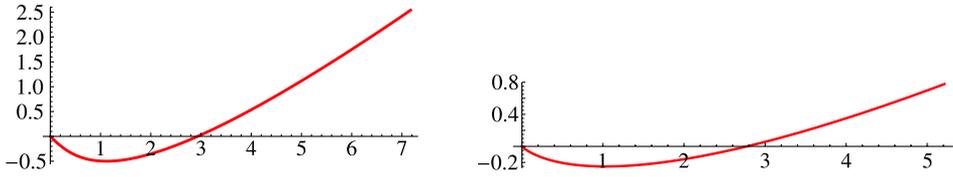


Figure 7. The generalized timelike Sturmian spirals generated by (86) with $d = 2$ and $\sigma = \frac{1}{2}$ (left), with $d = 0$ and $\sigma = \frac{1}{4}$ (right).

5. Conclusion

In [3], Mladenov *et al* have studied Sturmian spirals, planar curves whose curvature functions obey to the relation $\kappa(s) = \frac{\sigma}{r}$, where $r \in \mathbb{R}^+$ is the distance from the origin in the Euclidean plane \mathbb{E}^2 . By drawing inspiration from this work, here we consider spacelike and timelike Sturmian spirals in the Lorentz-Minkowski plane \mathbb{E}_1^2 . More precisely we obtain the parametric equations of spacelike and timelike Sturmian spirals, generalized spacelike and timelike Sturmian spirals in the Lorentz-Minkowski plane. We present many figures of these curves and the graphics are realized using *Mathematica*. Further, we believe that the results in this paper suggest that other curves like Serret's curves, Bernoulli's Lemniscate and the plane curves whose curvatures depend on the distance from some axis in the Lorentz-Minkowski plane \mathbb{E}_1^2 deserve to be studied as well. Work in this direction is in progress and will be reported elsewhere.

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