



# NOTE ON REVERSION, ROTATION AND EXPONENTIATION IN DIMENSIONS FIVE AND SIX

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**Abstract.** The explicit matrix realizations of reversion and spin groups depend on the set of matrices chosen to represent a basis of one-vectors for a Clifford algebra. On the other hand, there are iterative procedures to obtain bases of one-vectors for higher dimensional Clifford algebras, starting from those for lower dimensional ones. For a basis of one-vectors for  $Cl(0, 5)$ , obtained by applying such procedures to the Pauli basis for  $Cl(3, 0)$  the matrix form of reversion involves neither of the two standard matrices representing the symplectic form. However, by making use of the relation between  $4 \times 4$  real matrices and the quaternion tensor product  $(\mathbb{H} \otimes \mathbb{H})$ , the matrix form of reversion for this basis of one-vectors is identified. The corresponding version of the Lie algebra of the spin group,  $\mathfrak{spin}(5)$ , has useful matrix properties which are explored. Next, the form of reversion for a basis of one-vectors for  $Cl(0, 6)$  obtained iteratively from  $Cl(0, 0)$  is obtained. This is then applied to computing exponentials of  $5 \times 5$  and  $6 \times 6$  real antisymmetric matrices in closed form, by reduction to the simpler task of computing exponentials of certain  $4 \times 4$  matrices. For the latter purpose closed form expressions for the minimal polynomials of these  $4 \times 4$  matrices are obtained, without availing of their eigenstructure. Among the byproducts of this work are natural interpretations for members of an orthogonal basis for  $M(4, \mathbb{R})$  provided by the isomorphism with  $\mathbb{H} \otimes \mathbb{H}$ , and a first principles approach to the spin groups in dimensions five and six.

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## 1. Introduction

The anti-automorphism reversion is central to the theory of Clifford algebras. While it is unambiguously defined at the level of abstract Clifford algebras, its explicit form as an involution of the matrix algebra, to which the Clifford Algebra in question is isomorphic to, very much depends on the specific basis of matrices for one-vectors chosen to make concrete this isomorphism (see Definition 4 in Section 2.2 for the terminology one and two-vectors). Since there are canonical iterations supplying bases of one-vectors for higher dimensional Clifford algebras, starting from well known bases of one-vectors for lower dimensional ones (such as the Pauli matrices for  $\text{Cl}(3, 0)$ ), it is natural to endow these bases with a privileged status. Hence finding the form of reversion and Clifford conjugation with respect to these bases is interesting. For Clifford conjugation it is known [12] that there is (usually more than one) a choice of basis of one-vectors for  $\text{Cl}(0, n)$ , with respect to which Clifford conjugation's matrix form is given by Hermitian conjugation. However, no such easily stated result is available for the *matrix form of reversion* on  $\text{Cl}(0, n)$ .

Explicit expressions for these two anti-automorphisms are important for a variety of applications. An application, motivating this work, is that explicit matrix forms of these 2 involutions are very much needed for the success of a useful technique for computing the exponentials of elements of  $\mathfrak{so}(n, \mathbb{R})$  (the Lie algebra of  $n \times n$  real, antisymmetric matrices). We note that this Lie algebra and its Lie group arise in several applications such as robotics, electrical and energy networks, photonic lattice filters, communication satellites etc., [3–5, 8–10, 24]

Computing the exponential of a matrix is arguably one of the central tasks of applied mathematics. In general, this is quite a thankless job, [19]. However, for

matrices with additional structure certain simplifications may be available. In particular, the theory of Clifford Algebras and spin groups enables the reduction of finding  $e^X$ , with  $X \in \mathfrak{so}(n, \mathbb{R})$ , to the computation of  $e^Y$ , where  $Y$  is the associated element in the Lie algebra of the corresponding spin group. Frequently this means dealing with a matrix of smaller size. In particular, the minimal polynomial of  $Y$  is typically of lower degree than that of  $X$ . This connection, perhaps folklore, seems to have escaped the notice of a variety of practitioners (see, however, [6, 21] for variants on this theme). Let us first illustrate this via the famous Euler-Rodrigues formula for  $\mathfrak{so}(3, \mathbb{R})$ .

**Example 1.** Let  $X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$  be a  $3 \times 3$  antisymmetric real matrix.

As is well known,  $X$  has a cubic minimal polynomial, viz.,  $X^3 + \lambda^2 X = 0$ , with  $\lambda^2 = a^2 + b^2 + c^2$ . Hence  $e^X = I + \frac{\sin \lambda}{\lambda} X + \frac{1 - \cos \lambda}{\lambda^2} X^2$ . This is the famous Euler-Rodrigues formula. We will now show that this formula coincides with the following procedure:

Step 1 Identify  $\mathfrak{su}(2)$  with  $P$ , the purely imaginary quaternions, and  $SU(2)$  with the unit quaternions.

Step 2 Let  $\psi : P \rightarrow \mathfrak{so}(3, \mathbb{R})$  be the map obtained by linearizing the covering map  $\Phi : SU(2) \rightarrow SO(3, \mathbb{R})$ , where  $\Phi$  is the matrix of the map, which sends  $v \in P$  to  $gvg^{-1}$ , with  $g$  a unit quaternion.

Step 3 Find  $\psi^{-1}(X)$ . This is  $\frac{1}{2}(ai + bj + ck)$ .

Step 4 Compute the exponential of  $\psi^{-1}(X)$ . This is the unit quaternion

$$p = \cos\left(\frac{\lambda}{2}\right)1 + \frac{\sin\left(\frac{\lambda}{2}\right)}{\lambda}(ai + bj + ck), \text{ with } \lambda = \sqrt{a^2 + b^2 + c^2}.$$

Step 5 Compute the matrix of the map  $x \in P \rightarrow px\bar{p} \in P$ , with respect to the basis  $\{i, j, k\}$ .

The matrix computed in Step 5 coincides with the matrix provided by the Euler-Rodrigues formula,  $e^X = I + \frac{\sin \lambda}{\lambda} X + \frac{1 - \cos \lambda}{\lambda^2} X^2$ . For instance, the first column of the matrix is Step 5 is found by computing  $pi\bar{p}$  and rewriting this element of  $P$  as a vector in  $\mathbb{R}^3$ . Computing  $pi\bar{p}$  we find, it is

$$pi\bar{p} = \cos^2\left(\frac{\lambda}{2}\right)i + \frac{\cos\left(\frac{\lambda}{2}\right)\sin\left(\frac{\lambda}{2}\right)}{\lambda}(2cj - 2bk) + \frac{\sin^2\left(\frac{\lambda}{2}\right)}{\lambda^2}(a^2i - b^2i - c^2i + 2ack + 2abj)$$

This can be rewritten as

$$\cos^2\left(\frac{\lambda}{2}\right)i + \frac{\sin(\lambda)}{\lambda}(cj - bk) + \frac{\sin^2\left(\frac{\lambda}{2}\right)}{\lambda^2}((a^2 + b^2 + c^2)i - 2(b^2 + c^2)i + 2ack + 2abj).$$

This simplifies to

$$i + \frac{\sin \lambda}{\lambda}(cj - bk) + \frac{1 - \cos \lambda}{\lambda^2}(-(b^2 + c^2)i + ack + abj).$$

Rewritten as a vector in  $\mathbb{R}^3$  it is

$$\begin{pmatrix} 1 - (b^2 + c^2)\frac{1 - \cos \lambda}{\lambda^2} \\ c\frac{\sin \lambda}{\lambda} + ab\frac{1 - \cos \lambda}{\lambda^2} \\ -b\frac{\sin \lambda}{\lambda} + ac\frac{1 - \cos \lambda}{\lambda^2} \end{pmatrix}$$

which is precisely the first column of Euler-Rodrigues formula for  $e^X$ .

Strictly speaking, the above calculation is not what stems from considering  $\text{Cl}(0, 3)$ , since the latter is the double ring of the quaternions. However, it is an easy exercise to show that doing all calculations in  $\text{Cl}(0, 3)$  amounts to the same calculation outlined in the five step procedure above.

Though not of immense computational superiority in this simple instance, it worth noting that the exponentiation of a  $3 \times 3$  matrix has been reduced to the exponentiation of a  $2 \times 2$  matrix in  $\mathfrak{su}(2)$ , the Lie algebra of  $2 \times 2$  traceless, anti-Hermitian matrices (equivalently of a purely imaginary quaternion). Such matrices have quadratic minimal polynomials, unlike  $X$  which has a cubic minimal polynomial.

The methodology of the above example extends in general. We will restrict ourselves to  $\text{Cl}(0, n)$  for simplicity. The method proceeds as follows:

### Algorithm 2.

- Step 1 Identify a collection of matrices which serve as a basis of one-vectors for the Clifford Algebra  $\text{Cl}(0, n)$ .
- Step 2 Identify the explicit form of Clifford conjugation ( $\phi^{cc}$ ) and the grade (or so-called main) automorphism on  $\text{Cl}(0, n)$ , with respect to this collection of matrices. Equivalently identify the explicit form of Clifford conjugation and reversion ( $\phi^{rev}$ ) with respect to this collection of matrices.
- Step 3 Steps 1 and 2 help in identifying both the spin group  $\text{Spin}(n)$  and its Lie algebra  $\mathfrak{spin}(n)$ , as sets of matrices, within the same matrix algebra, that the matrices in Step 1 live in. Hence, one finds an matrix form for the double

covering  $\Phi_n : \text{Spin}(n) \rightarrow \text{SO}(n, \mathbb{R})$ . This is given typically as the matrix, with respect to the basis of one-vectors in Step 1, of the linear map  $H \rightarrow ZH\phi^{cc}(Z)$ , with  $H$  a matrix in the collection of one-vectors in Step 1 and  $Z \in \text{Spin}(n)$ . This enables one to express  $\Phi_n(Z)$  as a matrix in  $\text{SO}(n, \mathbb{R})$ .

Step 4 Linearize  $\Phi_n$  to obtain Lie algebra isomorphism  $\Psi_n : \mathfrak{spin}(n) \rightarrow \mathfrak{so}(n, \mathbb{R})$ . This reads as  $W \rightarrow YW - WY$ , with  $W$  once again a one-vector and  $Y \in \mathfrak{spin}(n)$ . Once again this leads to a matrix in  $\mathfrak{so}(n, \mathbb{R})$  which is  $\Psi_n(Y)$ .

Step 5 Given  $X \in \mathfrak{so}(n, \mathbb{R})$  find  $\Psi_n^{-1}(X) = Y \in \mathfrak{spin}(n)$ .

Step 6 Compute the matrix  $e^Y$  and use Step 3 to find the matrix  $\Phi_n(e^Y)$ . This matrix is  $e^X$ .

The key steps for the success of this algorithm are really Steps 1, 2 and 3.

In the literature, the identification of  $\text{Spin}(n)$ , is usually achieved by using the isomorphism between  $\text{Cl}(0, n-1)$  and the even vectors in  $\text{Cl}(0, n)$ , see [16,20]. In other words,  $\text{Spin}(n)$ , is identified as a subset of  $\text{Cl}(0, n-1)$ . However, this does not enable the finding of the matrix form of reversion. Similarly, to use Algorithm 2 above, one needs the one-vectors, the two-vectors (since they intervene in the Lie algebra of the spin group) and  $\text{Spin}(n)$  to be identified as explicit subcollections of matrices within the same matrix algebra that  $\text{Cl}(0, n)$  is isomorphic to. Therefore, once a basis of one-vectors as a specific collection of matrices has been found, one needs to find what forms Clifford conjugation and reversion take with respect to this collection for the successful realization of the applications above. Even if a realization of one-vectors of  $\text{Cl}(0, n)$  as a subset of  $\text{Cl}(0, n-1)$ , is specified, one still needs a prescription of how both  $\text{Spin}(n)$  and  $\mathfrak{spin}(n)$  act on this set of one-vectors. Furthermore, the latter action should be the linearization of the former action for applicability to the problem of finding exponentials of matrices in  $\mathfrak{so}(n, \mathbb{R})$ . See Remark 3 below for more on this issue.

In this note, therefore, we prefer to do all calculations within  $\text{Cl}(0, n)$ . One virtue of this is that it is a first principles approach to the problem of identifying the spin group and thus has some *didactical* advantages also.

As mentioned above, there are iterative constructions enabling one to find a basis of one-vectors for  $\text{Cl}(0, n)$ , starting from certain obvious bases of one-vectors for lower-dimensional Clifford algebras (the iterative constructions, pertinent to this work, are summarized in Section 2.3). Hence, it seems natural to use these for Step 1 of the last algorithm.

We found, to our initial chagrin, that for a basis of one-vectors for  $\text{Cl}(0, 5)$ , obtained from the Pauli basis  $\{\sigma_j; j = 1, 2, 3\}$  for  $\text{Cl}(3, 0)$ , reversion is not given by  $X \rightarrow M^{-1}X^T M$  for  $M = J_4$  or  $M = \tilde{J}_4$ , as one might expect from the circumstance that  $\text{Spin}(5)$  is isomorphic to  $\text{Sp}(4)$  (the group of  $4 \times 4$  matrices which are both unitary and symplectic). The matrices  $M = J_{2n}$  and  $M = \tilde{J}_{2n}$  are, of course, the two standard matrices representing the symplectic bilinear form - see Section 2.1 for the precise definitions of these matrices.

To circumvent this difficulty, we use the isomorphism between  $\mathbb{H} \otimes \mathbb{H}$  and  $\text{M}(4, \mathbb{R})$  to find a skew-symmetric and orthogonal  $M$ , for which reversion is indeed described by  $X \rightarrow M^{-1}X^T M$ . Furthermore, this isomorphism also enables us to find a conjugation between this  $M$  and  $J_4$ , and thus produce a basis of one-vectors for  $\text{Cl}(0, 5) = \text{M}(4, \mathbb{C})$ , with respect to which  $\text{Spin}(5)$  is indeed the standard representation of  $\text{Sp}(4)$ . It is emphasized, however, that it is not obvious how to obtain this latter basis from first principles, and hence the detour through  $\mathbb{H} \otimes \mathbb{H}$  is really useful, apart from being of independent interest. See, Remark26, for instance, for another illustration of this utility.

It turns out that one obstacle to reversion not involving either  $J_4$  nor  $\tilde{J}_4$  is the presence of either of these matrices themselves in the basis of one-vectors for  $\text{Cl}(0, 5)$ . Not having a tool such as the  $\mathbb{H} \otimes \mathbb{H}$  isomorphism in higher dimensions, we work very carefully to arrive at a basis of one-vectors for  $\text{Cl}(0, 6)$  which contains neither  $J_8$  nor  $\tilde{J}_8$ . For this we start with the sole possible basis for  $\text{Cl}(0, 0)$  and apply a judicious combination of the iterative procedures in Section 2.3, to find a desirable basis of one-vectors for  $\text{Cl}(0, 6)$ . This then very naturally leads to  $\text{SU}(4)$  being the covering group in dimension six.

**Remark 3.** In [20] the derivation of  $\text{SU}(4)$  as the spin group in dimension six, is carried out in pp 80, 151 and 264 – 265. As mentioned before, the Clifford algebra that [20] works with for this purpose is actually  $\text{Cl}(0, 5)$ . In particular, on pp 264 – 265, an embedding of  $\mathbb{R}^6$ , - the one-vectors for  $\text{Cl}(0, 6)$ , in  $\text{Cl}(0, 5) = \text{M}(4, \mathbb{C})$  is used. Specifically,  $\mathbb{R}^6$  is identified with  $\mathbb{C}^3$  and then  $(z_0, z_1, z_2) \in \mathbb{C}^3$  is identified with the following matrix in  $\text{M}(4, \mathbb{C})$

$$X(z_0, z_1, z_2) = \begin{pmatrix} \bar{z}_2 & 0 & z_0 & \bar{z}_1 \\ 0 & \bar{z}_2 & z_1 & -\bar{z}_0 \\ -\bar{z}_0 & -\bar{z}_1 & z_2 & 0 \\ -z_1 & z_0 & 0 & z_2 \end{pmatrix}.$$

But then the action of  $\mathfrak{spin}(6) = \mathfrak{su}(4)$  cannot be the usual one, viz.,  $A \in \mathfrak{su}(4)$  sending the one vector  $X(z_0, z_1, z_2)$  to the matrix  $AX(z_0, z_1, z_2) - X(z_0, z_1, z_2)A$ , since the latter is not of the form  $X(w_0, w_1, w_2)$  for some triple  $(w_0, w_1, w_2) \in$

$\mathbb{C}^3$ . Indeed, the  $(1, 2)$  entry of  $AX(z_0, z_1, z_2) - X(z_0, z_1, z_2)A$  is non-zero typically.

*It is emphasized that [20] does not make the claim in the above paragraph, and the matrix  $X(z_0, z_1, z_2)$  is used therein for an entirely different reason, viz., to avail of the fact that every element of  $\text{Spin}(n)$  can be factorized as a product of an element in  $S^{n-1}$  (the unit sphere in  $\mathbb{R}^n$ ) and an element in  $\text{Spin}(n-1)$ . The association of the matrix  $X(z_0, z_1, z_2)$  to the triple  $(z_0, z_1, z_2)$  is indeed elegant and the associated factorization is quite useful. However, for the purposes of this note it is necessary to proceed from first principles and work directly with  $\text{Cl}(0, 6) = \text{M}(8, \mathbb{R})$ . It seems that this is also didactically simpler for these purposes.*

There is also an unexpected benefit from working in  $\text{Cl}(0, 6)$ . Specifically, by starting with the obvious basis for  $\text{Cl}(0, 1)$  and mimicking for  $\text{Cl}(0, 5)$ , the iterative constructions for  $\text{Cl}(0, 6)$ , alluded to above, we arrive at a basis of one-vectors for  $\text{Cl}(0, 5)$  which sheds some light on the matrix  $X(z_0, z_1, z_2)$  - see Remark (33), part b). Further, by slightly modifying this construction we find a natural interpretation of yet another member of the  $\mathbb{H} \otimes \mathbb{H}$  basis for  $\text{M}(4, \mathbb{R})$ .

Thus, one by-product of this note is useful interpretations for at least three elements of a basis of orthogonal matrices for  $\text{M}(4, \mathbb{R})$ , yielded by its isomorphism to  $\mathbb{H} \otimes \mathbb{H}$  are provided. More generally, our work can be seen as showing the utility of Clifford Algebras for questions in algorithmic/computational linear algebra. Thus this note is in the spirit of [1, 2, 7, 11, 17, 18, 21–23].

The other component of this work is an explicit characterization of the minimal polynomials of matrices in the Lie algebra of the spin groups of dimensions 5 and 6. These expressions are constructive and do not require any knowledge of the eigenvalues/eigenvectors of these matrices. Once one has access to these minimal polynomials computing the exponentials of matrices in these Lie algebras is facile. One can either use recursions for the coefficients of the exponential or use simple Lagrange interpolation (since the matrices in question are all evidently diagonalizable and thus their minimal polynomials have distinct roots). As mentioned before it is often the case that the minimal polynomials of matrices in the Lie algebra of the spin group is far lower than that of the corresponding element in  $\mathfrak{so}(n, \mathbb{R})$ . Example 30 provides a striking illustration of this circumstance. Of course, a natural question that could be asked is whether one could not directly compute exponentials of elements of  $\mathfrak{spin}(n)$ , without passing to a matrix algebra representation of them, e.g., without using the fact that  $\mathfrak{spin}(6) \cong \mathfrak{su}(4)$ , for instance. Computing exponentials of matrices by computing exponentials directly within Clifford algebras has indeed been proposed in [1]. However, it has been our experience that it is only by passing to the matrix representation that we are able to avail of certain

simplifications. For example, the fact that only certain types of polynomials can arise as the minimal polynomials of matrices in  $\mathfrak{su}(4)$  is not evident from the fact that it is isomorphic to  $\mathfrak{spin}(6)$ . A full analysis of the advantages/disadvantages of passing to the matrix representation is beyond the scope of this paper, though it certainly is an interesting question to investigate.

The balance of this note is organized as follows. In the next section basic notation and preliminary facts are presented. Section 3 derives the explicit form of the reversion map for  $\text{Cl}(0, 5)$  with respect to a basis of one-vectors obtained iteratively from the Pauli matrices. An algorithm is then presented, which uses the derived form of reversion on  $\text{Cl}(0, 5)$  to exponentiate in closed form a matrix in  $\mathfrak{so}(5, \mathbb{R})$  by reducing this to the exponentiation of a  $4 \times 4$  matrix in a Lie algebra, denoted  $\widehat{\mathfrak{sp}}(4)$ . Section 4 derives explicit forms for minimal polynomials of matrices in  $\widehat{\mathfrak{sp}}(4)$ , thereby providing a complete solution to the problem of exponentiation of matrices in  $\mathfrak{so}(5, \mathbb{R})$ . The block structure of elements of  $\widehat{\mathfrak{sp}}(4)$  is shown to be amenable for calculation of the quantities intervening in the expressions for these minimal polynomials. Section 5 obtains the form of reversion on  $\text{Cl}(0, 6)$  with respect to a basis of one-vectors obtained iteratively from the sole possible basis for  $\text{Cl}(0, 0)$ . This is then applied to provide an algorithm for exponentiating a matrix in  $\mathfrak{so}(6, \mathbb{R})$  by reducing it to the corresponding problem in  $\mathfrak{su}(4)$ . The next section then provides a complete list of closed form expressions for minimal polynomials of matrices in  $\mathfrak{su}(4)$ . Remark 33 in this section revisits reversion on  $\text{Cl}(0, 5)$  and sheds light on the matrix  $X(z_0, z_1, z_2)$  in Remark 3 and also finds an interpretation for yet another element of the  $\mathbb{H} \otimes \mathbb{H}$  basis. The final section offers conclusions.

## 2. Notation and Preliminary Observations

### 2.1. Notation

We use the following notation throughout

N1  $\mathbb{H}$  is the set of quaternions, while  $\mathbb{P}$  is the set of purely imaginary quaternions. Let  $K$  be an associative algebra. Then  $M(n, K)$  is just the set of  $n \times n$  matrices with entries in  $K$ . For  $K = \mathbb{C}, \mathbb{H}$  we define  $X^*$  as the matrix obtained by performing entrywise complex (resp. quaternionic) conjugation first, and then transposition. For  $K = \mathbb{C}$ ,  $\bar{X}$  is the matrix obtained by performing entrywise complex conjugation.

N2  $J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ . Associated to  $J_{2n}$  are

i)  $\text{Sp}(2n) = \{X \in M(2n, \mathbb{C}) ; X^*X = I_n, J_{2n}^{-1}X^T J_{2n} = J_{2n}\}$ .



$\text{Sp}(2n)$  is a Lie group, and

ii)  $\mathfrak{sp}(2n) = \{X \in \text{M}(2n, \mathbb{C}) ; X^* = -X, X^T J_{2n} = -J_{2n} X\}$ .

$\mathfrak{sp}(2n)$  is the Lie algebra of  $\text{Sp}(2n)$ . Note many authors write  $\text{Sp}(n)$  instead of our  $\text{Sp}(2n)$ .

N3  $\tilde{J}_{2n} = J_2 \oplus J_2 \oplus \dots \oplus J_2$ . Thus  $\tilde{J}_{2n}$  is the  $n$ -fold direct sum of  $J_2$ .  $\tilde{J}_{2n}$ , is of course, explicitly permutation similar to  $J_{2n}$ , but it is important for our purposes to maintain the distinction. Accordingly

i)  $\widetilde{\text{Sp}}(2n) = \{X \in \text{M}(2n, \mathbb{C}) ; X^* X = I_n, \tilde{J}_{2n}^{-1} X^T \tilde{J}_{2n} = \tilde{J}_{2n}\}$ .

$\widetilde{\text{Sp}}(2n)$  is a Lie group, and

ii)  $\widetilde{\mathfrak{sp}}(2n) = \{X \in \text{M}(2n, \mathbb{C}) ; X^* = -X, X^T \tilde{J}_{2n} = -\tilde{J}_{2n} X\}$ .

$\widetilde{\mathfrak{sp}}(2n)$  is the Lie algebra of  $\widetilde{\text{Sp}}(2n)$ .

Other variants of  $J_4$  are of importance to this paper, and they will be introduced later at appropriate points (see Remark 17 below).

N4 The Pauli Matrices are

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

N5  $\text{SO}(n, \mathbb{R})$  stands for the  $n \times n$  real orthogonal matrices with determinant one.  $\mathfrak{so}(n, \mathbb{R})$  is its Lie algebra - the set of  $n \times n$  real antisymmetric matrices.

N6  $\text{SU}(n)$  is the Lie group of unitary matrices with unit determinant, and  $\mathfrak{su}(n)$  is its Lie algebra - the set of anti-Hermitian matrices with zero trace.

N7  $A \otimes B$  stands for the Kronecker product of  $A$  and  $B$ .

N8  $\|X\|_F$ , for a matrix  $X$ , is  $\sqrt{\text{Tr}(X^* X)} = \sqrt{\sum \sum_{i,j} |x_{ij}|^2}$ .

## 2.2. Reversion and Clifford Conjugation

We will begin with informal definitions of the notions of one and two-vectors for a Clifford algebra, which is sufficient for the purpose of this work. The texts [16,20] are excellent sources for more precise definitions in the theory of Clifford algebras.

**Definition 4.** Let  $p, q$  be non-negative integers with  $p + q = n$ . A collection of matrices  $\{X_1, \dots, X_p, X_{p+1}, \dots, X_{p+q}\}$ , with entries in  $\mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$  is a basis of one-vectors for the Clifford algebra  $\text{Cl}(p, q)$  if

1.  $X_i^2 = \text{Id}$ , for  $i = 1, 2, \dots, p$ , where  $\text{Id}$  is the identity matrix of the appropriate size (this size is typically different from  $n$ ).

2.  $X_i^2 = -\text{Id}$ , for  $i = p + 1, p + 2, \dots, p + q$ .
3.  $X_i X_j = -X_j X_i$ , for  $i \neq j; i, j = 1, 2, \dots, n$ .

A one-vector is just a real linear combination of the  $X_i$ 's,  $i = 1, 2, \dots, n$ . Similarly, a two-vector is a real linear combination of the matrices  $X_i X_j$ ,  $i < j; i, j = 1, 2, \dots, n$ . Analogously, we can define three, four, ...  $n$ -vectors, etc.  $\text{Cl}(p, q)$  is just a real linear combination of  $\text{Id}$ , one-vectors, ...,  $n$ -vectors.

- Definition 5.** I) The reversion anti-automorphism on a Clifford algebra,  $\phi^{rev}$ , is the linear map defined by requiring that i)  $\phi^{rev}(ab) = \phi^{rev}(b)\phi^{rev}(a)$ ; ii)  $\phi^{rev}(v) = v$ , for all one-vectors  $v$ ; and iii)  $\phi^{rev}(1) = 1$ . For brevity we will write  $X^{rev}$  instead of  $\phi^{rev}(X)$ .
- II) The Clifford conjugation anti-automorphism on a Clifford algebra,  $\phi^{cc}$ , is the linear map defined by a requiring that i)  $\phi^{cc}(ab) = \phi^{cc}(b)\phi^{cc}(a)$ ; ii)  $\phi^{cc}(v) = -v$ , for all one-vectors  $v$ ; and iii)  $\phi^{cc}(1) = 1$ . For brevity  $\phi^{cc}(X)$  will be written in the form  $X^{cc}$ .
- III) The grade automorphism on a Clifford algebra,  $\phi^{gr}$  is  $\phi^{rev} \circ \phi^{cc}$ . As is well known it is also true that  $\phi^{gr} = \phi^{cc} \circ \phi^{rev}$ . Once again we write  $X^{gr}$  for  $\phi^{gr}(X)$ .
- IV) Spin ( $n$ ) is the collection of elements  $x$  in  $\text{Cl}(0, n)$  satisfying the following requirements: i)  $x^{gr} = x$ , i.e.,  $x$  is even, ii)  $xx^{cc} = 1$ , and iii) For all one-vectors  $v$  in  $\text{Cl}(0, n)$ ,  $xvx^{cc}$  is also a one-vector. The last condition, in the presence of the first two conditions, is known to be superfluous for  $n \leq 5$ , [16, 20].

### 2.3. Iterative Constructions in Clifford Algebras

Here will outline three iterative constructions of one-vectors for certain Clifford Algebras, given a choice of one vectors for another Clifford Algebra, [16, 20]:

IC1  $\text{Cl}(p + 1, q + 1)$  as  $M(2, \text{Cl}(p, q))$ , where  $M(2, \mathfrak{A})$  stands for the set of  $2 \times 2$  matrices with entries in an associative algebra  $\mathfrak{A}$ :

Suppose  $\{e_1, \dots, e_p, f_1, \dots, f_q\}$  is a basis of one-vectors for  $\text{Cl}(p, q)$ . So, in particular,  $e_k^2 = +1$ ,  $k = 1, \dots, p$  and  $f_l^2 = -1$ ,  $l = 1, \dots, q$ . Then a basis of one-vectors for  $\text{Cl}(p + 1, q + 1)$  is given by the following collection

of elements in  $M(2, Cl(p, q))$

$$\begin{aligned} \begin{pmatrix} e_k & 0 \\ 0 & -e_k \end{pmatrix}, & \quad k = 1, \dots, p, & \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} f_l & 0 \\ 0 & -f_l \end{pmatrix}, & \quad l = 1, \dots, q, & \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

The 1 and the 0 in the matrices above are the identity and zero elements of  $Cl(p, q)$  respectively.

IC2 From  $Cl(p, q)$  to  $Cl(p - 4, q + 4)$ , for  $p \geq 4$ :

Suppose  $\{e_1, \dots, e_p, f_1, \dots, f_q\}$  is a basis of one-vectors for  $Cl(p, q)$ . Let us label this basis as  $\{g_i; i = 1, \dots, n\}$ . Thus,  $g_i = e_i, i = 1, \dots, p$  and  $g_{p+j} = f_j, j = 1, \dots, q$ . Then, to obtain a basis of one-vectors for  $Cl(p - 4, q + 4)$ , we first compute

$$g = e_1 e_2 e_3 e_4.$$

Then a basis  $\{h_i; i = 1, \dots, p + q\}$  of one-vectors for  $Cl(p - 4, q + 4)$  is obtained by setting

$$h_i = g_i g, \quad i = 1, \dots, 4, \quad h_i = g_i, \quad i > 4.$$

IC3 From  $Cl(p, q)$  to  $Cl(q + 1, p - 1)$  if  $p \geq 1$ .

Suppose  $\{e_1, \dots, e_p, f_1, \dots, f_q\}$  is a basis of one-vectors for  $Cl(p, q)$ . Then a basis  $\{\epsilon_1, \dots, \epsilon_{q+1}, \mu_1, \dots, \mu_{p-1}\}$  is obtained by defining

$$\epsilon_1 = e_1, \quad \epsilon_{k+1} = f_k e_1, \quad k = 1, \dots, q$$

and

$$\mu_k = e_{k+1} e_1, \quad k = 1, \dots, p - 1.$$

In this last basis, the  $\epsilon$ 's square to  $+1$ , while the  $\mu$ 's square to  $-1$ .

**Remark 6.** *In the last construction IC3 above, the special role played by  $e_1$  could have been played by any one of the  $e_k, k = 1, \dots, p$ . This would yield different sets of bases of one-vectors for  $Cl(q + 1, p - 1)$ , starting from a basis of one-vectors for  $Cl(p, q)$ . We will make use of this observation in Remark 33 part c).*

## 2.4. $\theta_{\mathbb{C}}$ and $\theta_{\mathbb{H}}$ Matrices

Some of the material here is to be found in [13], for instance.

**Definition 7.** Given a matrix  $M \in M(n, \mathbb{C})$ , define a matrix  $\theta_{\mathbb{C}}(M) \in M(2n, \mathbb{R})$  by first setting  $\theta_{\mathbb{C}}(z) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$  for a complex scalar  $z = x + iy$ . We then define  $\theta_{\mathbb{C}}(M) = (\theta_{\mathbb{C}}(m_{ij}))$ , i.e.,  $\theta_{\mathbb{C}}(M)$  is a  $n \times n$  block matrix, with the  $(i, j)$ th block equal to the  $2 \times 2$  real matrix  $\theta_{\mathbb{C}}(m_{ij})$ .

**Remark 8.** Properties of  $\theta_{\mathbb{C}}$

- i)  $\theta_{\mathbb{C}}$  is an  $\mathbb{R}$ -linear map.
- ii)  $\theta_{\mathbb{C}}(MN) = \theta_{\mathbb{C}}(M)\theta_{\mathbb{C}}(N)$ .
- iii)  $\theta_{\mathbb{C}}(M^*) = [\theta_{\mathbb{C}}(M)]^T$ .
- iv)  $\theta_{\mathbb{C}}(I_n) = I_{2n}$ .
- v) A useful property is the following:  $X \in M(2n, \mathbb{R})$  is in the image of  $\theta_{\mathbb{C}}$  iff  $X^T = \tilde{J}_{2n}^{-1} X^T \tilde{J}_{2n}$ .

**Remark 9.** We call an  $X \in \text{im}(\theta_{\mathbb{C}})$ , a  $\theta_{\mathbb{C}}$  matrix. It is tempting, but confusing, to call such matrices complex matrices. Similarly, if  $X \in M(2n, \mathbb{R})$  satisfies  $X^T = -\tilde{J}_{2n}^{-1} X^T \tilde{J}_{2n}$ , it will be called an anti- $\theta_{\mathbb{C}}$  matrix. These are precisely the linear anti-holomorphic maps on  $\mathbb{R}^{2n}$ .

Next, to a matrix with quaternion entries will be associated a complex matrix. First, if  $q \in \mathbb{H}$  is a quaternion, it can be written uniquely in the form  $q = z + wj$ , for some  $z, w \in \mathbb{C}$ . Note that  $j\eta = \bar{\eta}j$ , for any  $\eta \in \mathbb{C}$ . With this at hand, the following construction associating complex matrices to matrices with quaternionic entries (see [13] for instance) is useful

**Definition 10.** Let  $X \in M(n, \mathbb{H})$ . By writing each entry  $x_{pq}$  of  $X$  as

$$x_{pq} = z_{pq} + w_{pq}j, \quad z_{pq}, w_{pq} \in \mathbb{C}$$

we can write  $X$  uniquely as  $X = Z + Wj$  with  $Z, W \in M(n, \mathbb{C})$ . Associate to  $X$  the following matrix  $\theta_{\mathbb{H}}(X) \in M(2n, \mathbb{C})$

$$\theta_{\mathbb{H}}(X) = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix}$$

**Remark 11.** Viewing an  $X \in M(n, \mathbb{C})$  as an element of  $M(n, \mathbb{H})$  it is immediate that  $jX = \bar{X}j$ , where  $\bar{X}$  is entrywise complex conjugation of  $X$ .

Next some useful properties of the map  $\theta_{\mathbb{H}} : M(n, \mathbb{H}) \rightarrow M(2n, \mathbb{C})$  are collected.

**Remark 12.** *Properties of  $\theta_{\mathbb{H}}$*

- i)  $\theta_{\mathbb{H}}$  is an  $\mathbb{R}$ -linear map.
- ii)  $\theta_{\mathbb{H}}(XY) = \theta_{\mathbb{H}}(X)\theta_{\mathbb{H}}(Y)$ .
- iii)  $\theta_{\mathbb{H}}(X^*) = [\theta_{\mathbb{H}}(X)]^*$ . Here the  $*$  on the left is quaternionic Hermitian conjugation, while that on the right is complex Hermitian conjugation.
- iv)  $\theta_{\mathbb{H}}(I_n) = I_{2n}$ .
- v) A less known property is the following:  $\Lambda \in M(2n, \mathbb{C})$  is in the image of  $\theta_{\mathbb{H}}$  iff  $\Lambda^* = J_{2n}^{-1} X^T J_{2n}$ .

**Remark 13.** We call an  $\Lambda \in \text{im}(\theta_{\mathbb{H}})$ , a  $\theta_{\mathbb{H}}$  matrix. In [13] such matrices are called matrices of the quaternion type. But we eschew this nomenclature for the same reason as for avoiding the terminology complex matrices.

Similarly, if  $\Lambda \in M(2n, \mathbb{C})$  satisfies  $\Lambda^* = -J_{2n}^{-1} X^T J_{2n}$ , we say  $\Lambda$  is an anti- $\theta_{\mathbb{H}}$  matrix.

**2.5. Minimal Polynomials and Exponential Formulae**

The minimal polynomial of a matrix  $X \in M(n, \mathbb{C})$  is the unique monic polynomial,  $m_X(x)$ , of minimal degree which annihilates  $X$ . Minimal polynomials can, just as any other annihilating polynomial, be used to compute functions of  $X$ . One typical mode to do so is to use the annihilating polynomial to establish recurrences for higher powers of  $X$ , and in turn for any analytic function of  $X$ . Naturally the recurrences are simpler on the eye, when the minimal polynomial is used. An alternative method is to use such polynomials and interpolation techniques for constructing functions of  $X$ , [14]. This method is particularly useful when it is known in advance that  $X$  is diagonalizable (the only case of pertinence to this paper). In this case the roots of the minimal polynomial are distinct and the venerable Lagrange interpolation technique yields the desired function. We will confine ourselves to giving explicit formulae for  $e^X$  when  $m_X$  is one of the four following polynomials. Both the recurrence method and the interpolation method lead to the same representation for  $e^X$  as one may confirm.

**Theorem 14.** *Let  $X \in M(n, \mathbb{C})$  be non-zero. Then we have*

I) If  $m_X(x) = x^2 + \lambda^2$ , with  $0 \neq \lambda \in \mathbb{R}$ , then  $e^X = \cos(\lambda)I + \frac{\sin(\lambda)}{\lambda} X$ .

- II) If  $m_X = x^2 + 2i\gamma x + \lambda^2$ , with  $\gamma, \lambda \in \mathbb{R}$ , both non-zero, then  $e^X = e^{-i\gamma}[(\cos(\sigma) + \frac{i\gamma}{\sigma} \sin(\sigma))I + \frac{\sin(\sigma)}{\sigma} X]$ , where  $\sigma$  is the positive square root of  $\lambda^2 + \gamma^2$ .
- III) If  $m_X = x^3 + c^2 x$ , with  $0 \neq c \in \mathbb{R}$ , then  $e^X = I + \frac{\sin c}{c} X + \frac{1 - \cos c}{c^2} X^2$ .
- IV) If  $m_X(x) = x^4 + \theta^2 x^2 + \lambda^2$ , with  $\theta, \lambda \in \mathbb{R}$ , both non-zero, and satisfying  $\theta^4 > 4\lambda^2$ , then

$$e^X = \frac{1}{b^2 - a^2} \left[ \left( \frac{b \sin a - a \sin b}{ab} \right) X^3 + (\cos a - \cos b) X^2 + \left( \frac{b^3 \sin a - a^3 \sin b}{ab} \right) X + (b^2 \cos a - a^2 \cos b) I \right]$$

Here  $a$  and  $b$  are positive square roots of positive numbers  $a^2$  and  $b^2$ , which in turn are defined to be the unique positive solutions to  $a^2 + b^2 = \theta^2$ ,  $a^2 b^2 = \lambda^2$ .

**Remark 15.** It is possible that a matrix may be the sum of commuting summands, each of which has a low degree minimal polynomial, even though the original matrix has a high degree minimal polynomial. Thus, the exponential of such matrices can be quite easily found. Some instances of this phenomenon are to be found in [22].

## 2.6. $\mathbb{H} \otimes \mathbb{H}$ and $M(4, \mathbb{R})$

The algebra isomorphism between  $\mathbb{H} \otimes \mathbb{H}$  and  $M(4, \mathbb{R})$  (also denoted by  $\mathfrak{gl}(4, \mathbb{R})$ ) may be summarized as follows

- Associate to each product tensor  $p \otimes q \in \mathbb{H} \otimes \mathbb{H}$ , the matrix,  $M_{p \otimes q}$ , of the map which sends  $x \in \mathbb{H}$  to  $px\bar{q}$ , identifying  $\mathbb{R}^4$  with  $\mathbb{H}$  via the basis  $\{1, i, j, k\}$ . Here,  $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$ .
- Extend this to the full tensor product by linearity. This yields an associative algebra isomorphism between  $\mathbb{H} \otimes \mathbb{H}$  and  $M(4, \mathbb{R})$ . Furthermore, a basis for  $\mathfrak{gl}(4, \mathbb{R})$  is provided by the sixteen matrices  $M_{e_x \otimes e_y}$  as  $e_x, e_y$  run through 1,  $i, j, k$ .
- We define conjugation on  $\mathbb{H} \otimes \mathbb{H}$  by setting  $p \bar{\otimes} q = \bar{p} \otimes \bar{q}$  and then extending by linearity. Conjugation in  $\mathbb{H} \otimes \mathbb{H}$  corresponds to matrix transposition, i.e.,  $M_{\bar{p} \bar{\otimes} \bar{q}} = (M_{p \otimes q})^T$ . A consequence of this is that any matrix of the form  $M_{1 \otimes p}$  or  $M_{q \otimes 1}$ , with  $p, q \in \mathbb{P}$  is a real antisymmetric matrix. Similarly, the

most general special orthogonal matrix in  $M(4, \mathbb{R})$  admits an expression of the form  $M_{p \otimes q}$ , with  $p$  and  $q$  both unit quaternions.

**Remark 16.**  $M(4, \mathbb{C})$  : Since any complex matrix can be written  $Y + iZ$ , with  $Y, Z$  in  $M(n, \mathbb{R})$ , it follows that matrices in  $M(4, \mathbb{C})$  also possess quaternionic representations. In particular a complex symmetric matrix can be written as  $M_{p \otimes i + q \otimes j + r \otimes k}$ , with  $p, q, r \in \mathbb{C}^3$ . It should be clear from the context whether  $i$  is a complex number or a quaternion, in this regard. For instance  $iM_{i \otimes j}$  [or just  $i(i \otimes j)$ ] is the complex matrix equalling the complex number  $i$  times the real matrix  $M_{i \otimes j}$ .

**Remark 17.** Three matrices from this basis for  $M(4, \mathbb{R})$  provided by  $\mathbb{H} \otimes \mathbb{H}$  are important for us. They are

- $M_{1 \otimes j}$  is precisely  $J_4$ .
- The matrix  $M_{1 \otimes i}$ , which we denote by  $\widehat{J}_4$ .
- The matrix  $M_{j \otimes 1}$ , which we denote by  $\check{J}_4$ .

### 2.7. Other Matrix Theoretic Facts

Throughout this note many important matrices are expressible as Kronecker products  $A \otimes B$  and so, the following properties of Kronecker products will be freely used

- $(A \otimes B)(C \otimes D) = AC \otimes BD$ ,  $(A \otimes B)^T = A^T \otimes B^T$ .
- If  $A$  and  $B$  are square then  $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$ .

*Schur's Determinantal Formulae:* We will use the following special case of Schur's Determinantal Formulae, [13]: Suppose  $X_{2n \times 2n}$  is

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with  $A, B, C, D$  all  $n \times n$ . Then if  $B$  is invertible,

$$\det(X) = (-1)^{n^2} \det(B)\det(C - DB^{-1}A).$$

### 3. Reversion and Rotation in Dimension Five

First a basis of one-vectors for  $\text{Cl}(0, 5)$  will be constructed by starting with the Pauli basis for  $\text{Cl}(3, 0)$  and applying the iterative constructions IC1 and IC2 of Section 2.3. Thus, let  $\{Z_1 = \sigma_x, Z_2 = \sigma_y, Z_3 = \sigma_z\}$  be a basis of one-vectors for  $\text{Cl}(3, 0)$ . Applying IC1 to this yields the following basis for  $\text{Cl}(4, 1)$

$$\begin{aligned} \epsilon_1 &= \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, & \epsilon_2 &= \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, & \epsilon_3 &= \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} \\ \epsilon_4 &= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, & e_1 &= \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}. \end{aligned}$$

Next let us apply IC2 of Section 2.3 to this last basis to arrive at a basis for  $\text{Cl}(0, 5)$ . To that end we first need the product  $\epsilon_1\epsilon_2\epsilon_3\epsilon_4$ . A quick calculation shows

$$\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = i \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} = iJ_4$$

**Table 1.** One-vectors for  $\text{Cl}(0, 5)$ .

$$\begin{aligned} F_1 &= (\epsilon_1\epsilon_2\epsilon_3\epsilon_4)\epsilon_1 i \begin{pmatrix} 0 & -\sigma_x \\ -\sigma_x & 0 \end{pmatrix} = \sigma_x \otimes (-i\sigma_x) \\ F_2 &= (\epsilon_1\epsilon_2\epsilon_3\epsilon_4)\epsilon_2 \begin{pmatrix} 0 & -i\sigma_y \\ -i\sigma_y & 0 \end{pmatrix} = \sigma_x \otimes (-i\sigma_y) \\ F_3 &= (\epsilon_1\epsilon_2\epsilon_3\epsilon_4)\epsilon_3 \begin{pmatrix} 0 & -i\sigma_z \\ -i\sigma_z & 0 \end{pmatrix} = \sigma_x \otimes (-i\sigma_z) \\ F_4 &= (\epsilon_1\epsilon_2\epsilon_3\epsilon_4)\epsilon_2 \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix} = i\sigma_z \otimes I_2 \\ F_5 &= e_1 = J_4 \end{aligned}$$

Then IC2 says that a basis of one-vectors for  $\text{Cl}(0, 5)$  is  $\{F_i; i = 1, \dots, 5\}$ , where  $F_i$ 's are given in Table 1. Note that the presence of  $J_4$  in the basis is *unavoidable*, by construction, since the presence of  $e_1 = J_4$  in a basis of one-vectors for  $\text{Cl}(4, 1)$  and hence in that for  $\text{Cl}(0, 5)$  is required by construction. Inspired by the expected role of  $J_4$ , we now seek an expression for reversion on  $\text{Cl}(0, 5)$  of the form

$$\Phi^{rev}(X) = M^{-1}X^T M$$



where  $M$  is a real orthogonal antisymmetric matrix. The unavoidable presence of  $J_4$  in the basis of one-vectors, immediately implies that  $M \neq J_4$  and  $M \neq \tilde{J}_4$ . Indeed, for these two choices of  $M$ , we find that

$$M^{-1}F_5^T M = M^{-1}J_4^T M = -F_5 \neq F_5.$$

So an alternative choice for  $M$  is needed.

Given that we are working  $4 \times 4$  matrices, we are lead inexorably to the  $\mathbb{H} \otimes \mathbb{H}$  basis for  $M(4, \mathbb{R})$ . Slight experimentation reveals that

$$M = M_{1 \otimes i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

does the job, i.e.,  $M_{1 \otimes i}^{-1}F_i^T M_{1 \otimes i} = F_i$ , for all  $i = 1, \dots, 5$ .

It is useful to note that  $M_{1 \otimes i}$  also equals the following two matrices:

i)  $M_{1 \otimes i} = J_2 \oplus (-J_2)$ .

ii)  $M_{1 \otimes i} = \sigma_z \otimes (i\sigma_y)$ , and thus,  $M_{1 \otimes i}^{-1} = \sigma_z \otimes (-i\sigma_y)$ . This representation is pertinent since the  $F_i$  all have the form of Kronecker products of  $2 \times 2$  matrices and thus we will be able to use the properties of the Kronecker product (see Section 2.7) to facilitate calculation of  $M_{1 \otimes i}^{-1}F_i^T M_{1 \otimes i}$ .

The second of these two representations confirms that  $\phi^{rev}(X) = M_{1 \otimes i}^{-1}X^T M_{1 \otimes i}$ . For future convenience we denote  $M_{1 \otimes i}$  as  $\hat{J}_4$ , and correspondingly denote

$$\widehat{\text{Sp}}(4) = \left\{ X \in M(4, \mathbb{C}) ; X \in U(4), X^T \hat{J}_4 X = \hat{J}_4 \right\}.$$

It is well-known, and confirmed also by the above basis  $\{F_i\}$ , that Clifford conjugation on  $\text{Cl}(0, 5)$  is

$$\phi^{cc}(X) = X^*.$$

Hence the grade automorphism becomes

$$\phi^{gr}(X) = \hat{J}_4^{-1} \bar{X} \hat{J}_4.$$

Thus, with respect to this choice of a basis of one-vectors, it is seen that

$$\text{Spin}(5) = \left\{ X \in M(4, \mathbb{C}) ; X^* X = I_4, M_{1 \otimes i} X = \bar{X} M_{1 \otimes i} \right\} = \widehat{\text{Sp}}(4).$$

In summary, we have shown the following

**Proposition 18.** *Let  $B$  be the following set*

$$\{F_1 = \sigma_x \otimes (-i\sigma_x), F_2 = \sigma_x \otimes (-i\sigma_y), F_3 = \sigma_x \otimes (-i\sigma_z), F_4 = i\sigma_z \otimes I_2, F_5 = J_4\}.$$

*Then  $B$  is a basis for  $V$ , the space of one-vectors for  $\text{Cl}(0, 5)$ . With respect to  $B$  we have the following*

i) *The reversion anti-automorphism on  $\text{Cl}(0, 5)$  is given by*

$$\phi^{\text{rev}}(X) = M_{1 \otimes i}^{-1} X^T M_{1 \otimes i}.$$

ii) *Clifford conjugation is given by  $\phi^{\text{cc}}(X) = X^*$ .*

iii)  $\text{Spin}(5) = \widehat{\text{Sp}}(4) = \left\{ X \in \text{M}(4, \mathbb{C}) ; X^* X = I_4, X^T \widehat{J}_4 X = \widehat{J}_4 \right\}$ , where  $\widehat{J}_4 = M_{1 \otimes i}$ .

iv) *The standard covering map  $\Phi_5 : \text{Spin}(5) \rightarrow \text{SO}(5, \mathbb{R})$  is given by sending  $G \in \widehat{\text{Sp}}(4)$  to the matrix of the linear map, with respect to the basis  $B$ ,  $\Psi_G : V \rightarrow V$ , where*

$$\Phi_G(Y) = G Y G^*$$

v) *The Lie algebra isomorphism  $\Psi_5 : \widehat{\mathfrak{sp}}(4) \rightarrow \mathfrak{so}(5, \mathbb{R})$ , where  $\widehat{\mathfrak{sp}}(4)$  is the Lie algebra of the group  $\widehat{\text{Sp}}(4)$ , is obtained by linearizing  $\Phi_5 : \text{Spin}(5) \rightarrow \text{SO}(5, \mathbb{R})$ . Thus it is the map which sends  $A \in \widehat{\mathfrak{sp}}(4)$  to the matrix, with respect to  $B$ , of the linear map  $\psi_A : V \rightarrow V$ , where*

$$\psi_A(Z) = AZ - ZA.$$

### 3.1. Computing the Lie Algebra Isomorphism $\psi : \widehat{\mathfrak{sp}}(4) \rightarrow \mathfrak{so}(5, \mathbb{R})$

The Lie algebra of the  $\widehat{\text{Sp}}(4)$  is given by

$$\widehat{\mathfrak{sp}}(4) = \left\{ X \in \text{M}(4, \mathbb{C}) ; X^* = -X, X^T \widehat{J}_4 = -\widehat{J}_4 X \right\}$$

The second condition is equivalent to saying that the  $X \in \widehat{\mathfrak{sp}}(4)$  can be expressed as  $\widehat{J}_4 S$ , where  $S$  is a complex symmetric matrix. In view of Remark 16, this condition alone says that such an  $X$ 's  $\mathbb{H} \otimes \mathbb{H}$  representation must be of the form

$$X = (1 \otimes i)(p \otimes i + q \otimes j + r \otimes k + a 1 \otimes 1)$$

with  $p, q, r \in \mathbb{C}^3$  and  $a \in \mathbb{C}$ . However, the other condition,  $X^* = -X$ , forces  $p \in \mathbb{R}^3$ ,  $a \in \mathbb{R}$  and  $q, r \in (i\mathbb{R})^3$  (that is the components of  $q, r$  are purely imaginary).

**Table 2.** Basis for  $\widehat{\mathfrak{sp}}(4)$ .

$X_1 = i(j \otimes j)$	$X_6 = i(i \otimes j)$
$X_2 = i \otimes 1$	$X_7 = 1 \otimes i$
$X_3 = k \otimes 1$	$X_8 = j \otimes 1$
$X_4 = i(j \otimes k)$	$X_9 = i(k \otimes k)$
$X_5 = i(k \otimes j)$	$X_{10} = i(i \otimes k)$

Thus the most general such  $X$  has an  $\mathbb{H} \otimes \mathbb{H}$  representation of the form

$$X = -p \otimes 1 + a1 \otimes i + q \otimes k - r \otimes j$$

with  $p \in \mathbb{R}^3$ ,  $a \in \mathbb{R}$  and  $q, r \in (i\mathbb{R})^3$ . The negative signs are inessential and so a basis of  $\widehat{\mathfrak{sp}}(4)$  can be written in  $\mathbb{H} \otimes \mathbb{H}$  form, keeping in mind the remark on notation in Remark 16, as in Table 2.

Now to compute the image under  $\Psi_5$  of such a basis element of  $\widehat{\mathfrak{sp}}(4)$ , call it  $X$ , we have to compute  $XF_i - F_iX$ ,  $i = 1, \dots, 5$  where  $\{F_i\}$  is the basis of one-vectors in Proposition 18 and express the result as a real linear combination of the  $F_i$ . We will content ourselves with an illustration of the calculation for  $X_7 = 1 \otimes i$ . We find

- $X_7F_1 - F_1X_7 = (\sigma_z \otimes i\sigma_y)(\sigma_x \otimes (-i\sigma_x)) - (\sigma_x \otimes (-i\sigma_x))(\sigma_z \otimes i\sigma_y) = 0$ .  
Here, the fact that  $X_7$  can also be written as  $(\sigma_z \otimes i\sigma_y)$  and that  $F_1$  can also be written in the form  $\sigma_x \otimes (-i\sigma_x)$  was employed.
- $X_7F_2 - F_2X_7 = (\sigma_z \otimes i\sigma_y)((\sigma_x \otimes (-i\sigma_y)) - (\sigma_x \otimes (-i\sigma_y))(\sigma_z \otimes i\sigma_y)) = 2\sigma_z\sigma_x \otimes I_2 = 2i\sigma_y \otimes I_2 = 2F_5$ .
- $X_7F_3 - F_3X_7 = (\sigma_z \otimes i\sigma_y)(\sigma_x \otimes (-i\sigma_z)) - (\sigma_x \otimes (-i\sigma_z))(\sigma_z \otimes i\sigma_y) = 0$ .
- $X_7F_4 - F_4X_7 = (\sigma_z \otimes i\sigma_y)(i\sigma_z \otimes I_2 - (i\sigma_z \otimes I_2)(\sigma_z \otimes i\sigma_y)) = 0$ .
- $X_7F_5 - F_5X_7 = (\sigma_z \otimes i\sigma_y)i\sigma_y \otimes I_2 - i\sigma_y \otimes I_2(\sigma_z \otimes i\sigma_y) = 2\sigma_x \otimes (i\sigma_y) = -2F_5$ .

Hence  $\Psi_5(X_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}$ . More compactly,

$$\Psi_5(X_7) = 2(e_5e_2^T - e_2e_5^T) \text{ (here, of course } e_i \text{ is the } i\text{th standard unit vector)}$$

**Table 3.** Lie algebra isomorphism between  $\widehat{\mathfrak{sp}}(4)$  and  $\mathfrak{so}(5, \mathbb{R})$ .

$\widehat{\mathfrak{sp}}(4)$	$\mathfrak{so}(5, \mathbb{R})$	$\widehat{\mathfrak{sp}}(4)$	$\mathfrak{so}(5, \mathbb{R})$
$iM_{j \otimes j}$	$2(e_1 e_2^T - e_2 e_1^T)$	$iM_{i \otimes j}$	$2(e_2 e_4^T - e_4 e_2^T)$
$M_{i \otimes 1}$	$2(e_3 e_1^T - e_1 e_3^T)$	$M_{1 \otimes i}$	$2(e_5 e_2^T - e_2 e_5^T)$
$M_{k \otimes 1}$	$2(e_1 e_4^T - e_4 e_1^T)$	$M_{j \otimes 1}$	$2(e_4 e_3^T - e_3 e_4^T)$
$iM_{j \otimes k}$	$2(e_1 e_5^T - e_5 e_1^T)$	$iM_{k \otimes k}$	$2(e_5 e_3^T - e_3 e_5^T)$
$iM_{k \otimes j}$	$2(e_2 e_3^T - e_3 e_2^T)$	$iM_{i \otimes k}$	$2(e_5 e_4^T - e_4 e_5^T)$

In summary, the following holds

**Theorem 19.** *The Lie algebra isomorphism  $\Psi_5 : \widehat{\mathfrak{sp}}(4) \rightarrow \mathfrak{so}(5, \mathbb{R})$  is described by Table 3.*

**Remark 20.** *We have  $\widehat{J}_4 = M_{1 \otimes i}$ , while the standard representation of the symplectic form,  $J_4$  is  $J_4 = M_{1 \otimes j}$ . This makes it extremely easy to find a special orthogonal conjugation between the two. Since every element of  $\text{SO}(4, \mathbb{R})$  has a  $\mathbb{H} \otimes \mathbb{H}$  representation of the form  $M_{p \otimes q}$ , for unit quaternions, we let  $U^T = M_{p \otimes q}$  and seek  $U$  so that*

$$U^T \widehat{J}_4 U = J_4.$$

Using results of Section 2.6, it is obvious that we can let  $p = 1$  and seek  $q$  to be a unit quaternion satisfying

$$qi\bar{q} = j.$$

Of the infinite choices possible, let us pick  $q = \frac{1}{\sqrt{2}}(1 + k)$  for concreteness.

With this explicit conjugation available, the following are immediate:

I)  $U[\text{Sp}(4)]U^T = \widehat{\text{Sp}}(4)$ , and  $U[\mathfrak{sp}(4)]U^T = \widehat{\mathfrak{sp}}(4)$ .

II) One can use this conjugation to find yet another basis of one-vectors for  $\text{Cl}(0, 5)$ , viz.,

$$\{I_2 \otimes (i\sigma_z), \sigma_x \otimes (i\sigma_y), I_2 \otimes (i\sigma_x), i\sigma_y \otimes \sigma_y, \sigma_z \otimes (i\sigma_y)\}.$$

With respect to this basis Clifford conjugation is once again Hermitian conjugation, but reversion is  $Y \rightarrow J_4^{-1} Y^T J_4$ . Thus,  $\text{Spin}(5)$  is, with respect to this basis, the standard representation of  $\text{Sp}(4)$ .

We emphasize however, that this basis was arrived at only by going through  $\widehat{J}_4$  first. In other words, this basis, to the best of our knowledge, does not naturally arise from first principles as does the basis  $\{F_i; i = 1, \dots, 5\}$  in Proposition 18.

### Computing Exponentials in $\mathfrak{so}(5, \mathbb{R})$

Specializing Algorithm 2 yields the following method for computing the exponential of a matrix in  $\mathfrak{so}(5, \mathbb{R})$ :

- If  $X \in \mathfrak{so}(5, \mathbb{R})$ , find  $Y = \Psi_5^{-1}(X) \in \widehat{\mathfrak{sp}}(4)$  using the table in Theorem 19.
- Compute  $e^Y$ .
- Find  $e^Y F_j e^{-Y}, \forall j = 1, \dots, 5$ . Express  $e^Y F_j e^{-Y} = \sum_{i=1}^5 c_{ij} F_i$ .
- Then  $e^X$  is the matrix whose  $i$ th column is  $\begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{i5} \end{pmatrix}$ .

Thus, the problem of computing  $e^X$  is reduced to the problem of computing the exponential of a  $4 \times 4$  matrix,  $Y$ , which furthermore has additional structure, thereby rendering the computation of  $e^Y$  in closed form very easy.

### 4. Minimal Polynomials of Matrices in $\widehat{\mathfrak{sp}}(4)$

In this section we show that the minimal polynomials of matrices in  $Y \in \widehat{\mathfrak{sp}}(4)$  can be computed explicitly, and that these explicit forms lead correspondingly to explicit formulae for  $e^Y$ . Indeed, as will be seen below, the minimal polynomials that arise are each one of the four types in Theorem 14.

To this end, it is easier to work with matrices in the standard representation, viz.,  $\mathfrak{sp}(4)$ , and use the connection of such matrices to  $M(2, \mathbb{H})$ . It should be pointed that the results obtained below are invariant under conjugation by a special orthogonal matrix, and hence extend verbatim to matrices in  $\widehat{\mathfrak{sp}}(4)$  and thus there is no need to find the element in  $\mathfrak{sp}(4)$  conjugate to the matrix  $Y \in \widehat{\mathfrak{sp}}(4)$  (See Remark 25). In fact, it will be seen in Remark 26 that the quantities intervening in the result about the minimal polynomials are easier to calculate for  $\widehat{\mathfrak{sp}}(4)$ .

Recall that if  $Z \in M(2, \mathbb{H})$ , then  $Z = A + Bj$ , with  $A, B \in M(2, \mathbb{C})$ . Denote

$$Y = \theta_{\mathbb{H}}(Z) = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

Hence by  $v)$  of Remark 12 of Section 2.4

$$Y^* = Y^\dagger$$

where  $Y^\dagger = -J_4 Y^T J_4$ . Matrices in  $\mathfrak{sp}(4)$  are clearly  $\theta_{\mathbb{H}}$ -matrices. Therefore, the following result is pertinent:

**Proposition 21.** *If  $Y \in M(2n, \mathbb{C})$  is a  $\theta_{\mathbb{H}}$ -matrix then its minimal and characteristic polynomials are both real polynomials.*

**Proof:** Let  $m_Y(x) = x^k + c_{k-1}x^{k-1} + \dots + c_0$ .

So from

$$Y^k + c_{k-1}Y^{k-1} + \dots + c_1Y + c_0I = 0$$

we get

$$(Y^*)^k + \bar{c}_{k-1}(Y^*)^{k-1} + \dots + \bar{c}_1Y^* + \bar{c}_0I = 0.$$

Thus  $\bar{m}_Y(x) = x^k + \bar{c}_{k-1}x^{k-1} + \dots + \bar{c}_0$  annihilates  $Y^*$ . Suppose

$$q(x) = x^l + d_{l-1}x^{l-1} + \dots + d_0$$

annihilates  $Y^*$ , with  $l < k$ . Then the same argument just used shows that  $\bar{q}$ , a polynomial of degree  $l$ , annihilates  $Y$ . Thus contradicts the minimality of  $m_Y(x)$ . Hence  $k$  is also the degree of the minimal polynomial of  $Y^*$ , and standard properties of minimal polynomials shows that the minimal polynomial of  $Y^*$  is indeed  $\bar{m}_Y(x)$ . But  $Y^\dagger$  is evidently similar to  $Y^T$ , and thus to  $Y$ . So as  $Y$  is a  $\theta_{\mathbb{H}}$ -matrix, we see that  $m_Y(x) = \bar{m}_Y(x)$ . Hence  $m_Y(x)$  is a real polynomial.

Next let  $p_Y(x) = \det(xI - A)$  be the characteristic polynomial of  $Y$ . Then the characteristic polynomial of  $Y^*$  is the complex conjugate of  $p_Y(\bar{x})$ , and hence  $p_{Y^*}(x) = p_Y(x)$ . But  $p_{Y^\dagger}(x) = p_{Y^T}(x) = p_Y(x)$ . So, as  $Y^\dagger = Y^*$ , it is evident that  $p_Y$  is also a real polynomial. ■

Matrices in  $\mathfrak{sp}(4)$  are not only  $\theta_{\mathbb{H}}$  matrices, but are also anti-Hermitian. This leads to further simplifications in their minimal polynomials

**Proposition 22.** *Let  $Y \in \mathfrak{sp}(4)$  and  $m_Y(x)$  be its minimal polynomial. Then  $m_Y(-x) = m_Y(x)$  if the degree of  $m_Y$  is even, otherwise  $m_Y(-x) = -m_Y(x)$ .*

The proof of Proposition 22 is left to the reader.

**Remark 23.** *A similar result shows that the characteristic polynomial of  $Y \in \mathfrak{sp}(4)$  is a real polynomial with only even degree terms.*

Let us now apply the foregoing results to hone our statements about  $m_Y(x)$  for  $Y \in \mathfrak{sp}(4)$ . Let

$$Y = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

Now  $Y \in \mathfrak{sp}(4)$  is equivalent to  $(A + Bj)^* = -(A + Bj)$  (here the  $*$  is Hermitian conjugation of matrices in  $M(2, \mathbb{H})$ ). This is, of course, equivalent to  $A^* = -A$  and  $B^T = B$ .

Since the characteristic polynomial of  $Y$  is of the form  $x^4 + c_2x^2 + c_0$ , we have

$$c_2 = \frac{1}{2} ([\text{Tr}(Y)]^2 - \text{Tr}(Y^2)).$$

Quite clearly  $\text{Tr}(Y) = 2\text{Re}[\text{Tr}(A)]$ . But as  $A$  is anti-Hermitian its trace is purely imaginary. So  $\text{Tr}(Y) = 0$ . Hence

$$c_2 = -\frac{1}{2}\text{Tr}(Y^2).$$

Now  $Y^2 = \theta_{\mathbb{H}}[(A + Bj)^2]$ , and

$$(A + Bj)^2 = (A^2 - B\bar{B}) + (AB + B\bar{A})j.$$

Hence

$$\text{Tr}(Y^2) = 2\text{Re}[\text{Tr}(A^2 - B\bar{B})].$$

But  $A^2 - B\bar{B} = -AA^* - BB^*$ , which is a negative semidefinite matrix, and hence a matrix with real trace. So

$$c_2 = \text{Tr}(AA^* + BB^*) = \frac{1}{2} \|Y\|_F^2.$$

So, we have an explicit formula for the characteristic polynomial of  $Y$ , viz.,

$$p_Y(x) = x^4 + \left(\frac{1}{2} \|Y\|_F^2\right)x^2 + \det(Y).$$

Since  $Y$  is diagonalizable, its minimal polynomial has distinct roots which are the eigenvalues of  $Y$ . Therefore, it suffices to examine the eigenstructure of  $Y$  with multiplicity. We find

- $Y$  has 4 distinct eigenvalues,  $ia, -ia, ib, -ib$ , iff  $\|Y\|_F^4 > 16\det(Y)$  and  $\det(Y) \neq 0$ .
- It has 3 distinct eigenvalues,  $ia, -ia, 0$  (with 0 repeated twice) iff  $\det(Y) = 0$ .
- It has 2 distinct eigenvalues,  $ia$  and  $-ia$  (each repeated twice) iff  $\|Y\|_F^4 = 16\det(Y)$  (notice that in this case  $Y$  is non-singular, since  $Y \neq 0$ , precludes  $\|Y\|_F = 0$ ). Hence its minimal polynomial in each case is given as follows
- $x^4 + \left(\frac{1}{2} \|Y\|_F^2\right)x^2 + \det(Y)$ .
- $x^3 + a^2x$ . To find  $a$ , note that the non-zero roots of the characteristic polynomial are in this case  $\frac{i}{\sqrt{2}} \|Y\|_F, -\frac{i}{\sqrt{2}} \|Y\|_F$ . So  $a^2 = \frac{1}{2} \|Y\|_F^2$ .

- $x^2 + a^2$ . In this case the roots of the characteristic polynomial are  $\frac{1}{2} \|Y\|_F$  and  $-\frac{1}{2} \|Y\|_F$ . So the minimal polynomial is  $x^2 + \frac{\|Y\|_F^2}{4}$ .

Summarizing we have:

**Theorem 24.** *Let  $Y \in \mathfrak{sp}(4)$  or  $\widehat{\mathfrak{sp}}(4)$ . Its minimal polynomial is one of the following*

- $x$ , which happens iff  $Y = 0$ .
- $x^2 + \frac{\|Y\|_F^2}{4}$ , which happens iff  $Y \neq 0$  and  $\|Y\|_F^4 = 16\det(Y)$ .
- $x^3 + \frac{1}{2}(\|Y\|_F^2)x$ , which happens iff  $Y \neq 0$ , but  $\det(Y) = 0$ .
- $x^4 + (\frac{1}{2}\|Y\|_F^2)x^2 + \det(Y)$ , which happens iff  $Y \neq 0$ ,  $\det(Y) \neq 0$ .

We emphasize that in Theorem 24 the eigenvalues of  $Y$  do not appear and therefore, one does need to know its eigenvalues in advance.

**Remark 25.** *Since all quantities intervening in the above theorem are invariant under real orthogonal similarity, the theorem extends verbatim to matrices  $Y \in \widehat{\mathfrak{sp}}(4)$ . Indeed, per Remark 20, if  $Y \in \widehat{\mathfrak{sp}}(4)$ , then  $Z = U^T Y U$  is in  $\mathfrak{sp}(4)$ , where  $U$  is the explicit real orthogonal matrix in Remark 20. Thus, i) the determinants of  $Y$  and  $Z$  coincide, ii)  $\|Y\|_F = \|Z\|_F$ , and iii) the minimal polynomials of  $Y$  and  $Z$  coincide.*

**Remark 26.** *Block Structure of  $\widehat{\mathfrak{sp}}(4)$ : It will be seen that the block structure of a matrix in  $\widehat{\mathfrak{sp}}(4)$  has some benefits which matrices in  $\mathfrak{sp}(4)$  do not. Let  $X \in \widehat{\mathfrak{sp}}(4)$ . If  $X$  is written as a  $2 \times 2$  block matrix, with each block  $2 \times 2$*

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

*then i)  $A, D$  are both in  $\mathfrak{sp}(2)$ , ii)  $B = -C^*$  and iii)  $B$  is an anti- $\Theta_{\mathbb{H}}$  matrix in  $M(2, \mathbb{C})$ .*

*To see this, note that  $X = \hat{J}_4 S$  for some  $4 \times 4$  symmetric matrix  $S = \begin{pmatrix} W & Y \\ Y^T & Z \end{pmatrix}$  and  $X^* = -X$ . Since  $\hat{J}_4 = J_2 \oplus (-J_2)$ , the first of these conditions says  $A$  and  $D$  are in  $\mathfrak{sp}(2, \mathbb{C})$  and that  $B = J_2 Y$ ,  $C = -J_2 Y^T$ . Together with the second condition it follows that  $A, D \in \mathfrak{sp}(2)$  and  $B = -C^*$  and hence that  $Y^* = J_2 Y^T J_2$ . This last condition is equivalent to  $Y$  being an anti- $\Theta_{\mathbb{H}}$  matrix. Since  $B = J_2 Y$  and  $J_2$  itself is a  $\theta_{\mathbb{H}}$  matrix, it follows that  $B$  is an anti- $\Theta_{\mathbb{H}}$  matrix in  $M(2, \mathbb{C})$ . From this we can conclude the following*



1.  $\|X\|_F^2 = 2 \left( |x_{11}|^2 + |x_{12}|^2 + |x_{33}|^2 + |x_{34}|^2 \right) + 4 \left( |x_{13}|^2 + |x_{14}|^2 \right)$ .
2. *The determinant of  $X$  requires only the computation of  $2 \times 2$  determinants. To that end, first observe that an anti- $\theta_{\mathbb{H}}$  matrix is of the form  $\begin{pmatrix} \theta & \zeta \\ \bar{\zeta} & -\bar{\theta} \end{pmatrix}$ , for some  $\theta, \zeta \in \mathbb{C}$ . So it is either invertible or identically zero. Hence, representing  $X \in \widehat{\mathfrak{sp}}(4)$  as a block matrix, it follows that if  $B = 0$ , then  $\det(X) = \det(A)\det(D)$ . If  $B$  is invertible, then*

$$\det(X) = (-1)^4 \det(B) \det(-B^* - DB^{-1}A) = \det(B) \det(B^* + DB^{-1}A)$$

*which follows from the special case of the determinantal formulae of Schur mentioned in Section 2.7.*

*The last item above shows that for a determinant calculation at least  $\widehat{\mathfrak{sp}}(4)$  is more amenable than  $\mathfrak{sp}(4)$ . Indeed, if  $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{sp}(4)$ , then one will need a  $4 \times 4$  determinant calculation, when both  $A$  and  $B$  fail to be invertible, since it is now possible for  $A$  and  $B$  to be singular without being identically zero.*

### 5. $\mathfrak{su}(4)$ and $\mathfrak{so}(6, \mathbb{R})$

As is well known the spin group of  $\text{SO}(6, \mathbb{R})$  is  $\text{SU}(4)$ , and there is correspondingly an isomorphism of  $\mathfrak{so}(6, \mathbb{R})$  and  $\mathfrak{su}(4)$ . In this section we will produce a basis of one-vectors of  $\text{Cl}(0, 6)$  which is natural from the point of view of the constructions of Section 2.3 and which will enable the computation of exponentials of matrices in  $\mathfrak{so}(6, \mathbb{R})$  via a computation of exponentials of matrices in  $\mathfrak{su}(4)$ . Moreover in this construction, the matrix  $\tilde{J}_8$  naturally intervenes.

We begin with  $\text{Cl}(0, 0)$  and repeatedly apply IC1 of Section 2.3, to first produce a basis of one-vectors for  $\text{Cl}(3, 3) = \text{M}(8, \mathbb{R})$ .

Since the set of one-vectors for  $\text{Cl}(0, 0)$  is the empty set,  $\{\sigma_x, \sigma_y\}$  is what IC1 gives for a basis of one-vectors for  $\text{Cl}(1, 1)$ .

Hence a basis of one-vectors for  $\text{Cl}(2, 2)$  is then

$$\begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}, \quad \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

This produces the following basis of one-vectors for  $\text{Cl}(3, 3)$

$$\{\sigma_z \otimes \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_x \otimes I_2, \sigma_z \otimes \sigma_z \otimes i\sigma_y, \sigma_z \otimes i\sigma_y \otimes I_2, \sigma_x \otimes I_4, i\sigma_y \otimes I_4\}$$

**Table 4.** Basis of one-vectors for  $\text{Cl}(0, 6)$ .

$$\begin{aligned}
Y_1 &= (\sigma_z \otimes \sigma_z \otimes \sigma_x)(i\sigma_y \otimes \sigma_x \otimes i\sigma_y) &= \sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z) \\
Y_2 &= (I_2 \otimes I_2 \otimes \sigma_z)(i\sigma_y \otimes \sigma_x \otimes i\sigma_y) &= i\sigma_y \otimes \sigma_x \otimes \sigma_x \\
Y_3 &= (-I_2 \otimes \sigma_x \otimes \sigma_x)(i\sigma_y \otimes \sigma_x \otimes i\sigma_y) &= i\sigma_y \otimes I_2 \otimes \sigma_z \\
Y_4 &= (-\sigma_x \otimes \sigma_z \otimes \sigma_x)(i\sigma_y \otimes \sigma_x \otimes i\sigma_y) &= -\sigma_z \otimes (i\sigma_y) \otimes \sigma_z \\
Y_5 &= -I_2 \otimes i\sigma_y \otimes \sigma_x &= -I_2 \otimes i\sigma_y \otimes \sigma_x \\
Y_6 &= -i\sigma_y \otimes \sigma_z \otimes \sigma_x &= -i\sigma_y \otimes \sigma_z \otimes \sigma_x
\end{aligned}$$

Next, we use IC3 of Section 2.3, relating  $\text{Cl}(p, q)$  and  $\text{Cl}(p+1, q-1)$ , to produce, via this basis, a basis of one-vectors for  $\text{Cl}(4, 2)$

$$\begin{aligned}
\tilde{e}_1 &= \sigma_z \otimes \sigma_z \otimes \sigma_x, & \tilde{e}_4 &= (i\sigma_y \otimes I_4)(\sigma_z \otimes \sigma_z \otimes \sigma_x) \\
\tilde{e}_2 &= (\sigma_z \otimes \sigma_z \otimes i\sigma_y)(\sigma_z \otimes \sigma_z \otimes \sigma_x), & \tilde{e}_5 &= (\sigma_z \otimes \sigma_x \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_x) \\
\tilde{e}_3 &= (\sigma_z \otimes i\sigma_y \otimes I_2)(\sigma_z \otimes \sigma_z \otimes \sigma_x), & \tilde{e}_6 &= (\sigma_x \otimes I_4)(\sigma_z \otimes \sigma_z \otimes \sigma_x).
\end{aligned}$$

Doing the requisite Kronecker multiplications this basis of one-vectors for  $\text{Cl}(4, 2)$  assumes the following form

$$\begin{aligned}
\tilde{e}_1 &= \sigma_z \otimes \sigma_z \otimes \sigma_x, & \tilde{e}_4 &= -\sigma_x \otimes \sigma_z \otimes \sigma_x \\
\tilde{e}_2 &= I_2 \otimes I_2 \otimes \sigma_z, & \tilde{e}_5 &= -I_2 \otimes i\sigma_y \otimes \sigma_x \\
\tilde{e}_3 &= -I_2 \otimes \sigma_x \otimes \sigma_x, & \tilde{e}_6 &= -i\sigma_y \otimes \sigma_z \otimes \sigma_x.
\end{aligned}$$

Finally, using IC2 of Section 2.3, relating  $\text{Cl}(p, q)$  to  $\text{Cl}(p-4, q+4)$ , we produce a basis of one-vectors for  $\text{Cl}(0, 6)$ . To that end, we first need to find  $\tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_4$ . This is given by

$$\tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_4 = i\sigma_y \otimes \sigma_x \otimes i\sigma_y.$$

The basis of one-vectors,  $\{Y_i\}$ , for  $\text{Cl}(0, 6)$  is therefore given by Table 4.

**Remark 27.** Each of the  $Y_i$  are tensor products of 3 matrices, of which two are real symmetric and one is real antisymmetric. Hence,  $Y_i^T = -Y_i$ , for all  $i$ . Since matrix transposition is an anti-involution, we find, as expected, from this that (with respect to this basis of one-vectors, Clifford conjugation on  $\text{Cl}(0, 6)$  coincides with matrix transposition.

Next a matrix form for reversion on  $\text{Cl}(0, 6)$  (with respect to the basis,  $\{Y_i; i = 1, \dots, 6\}$ , of one-vectors) will be found. We are guided in this by 3 facts:

i) the  $Y_i$  are all tensor products of 3 matrices, and the matrix  $i\sigma_y$  is one of the 3 factors in each  $Y_i$ ; ii) the matrices  $J_8$  and  $\tilde{J}_8$  are also triple tensor products with  $i\sigma_y$  again one of the factors. Specifically,  $J_8 = i\sigma_y \otimes I_4 = i\sigma_y \otimes I_2 \otimes I_2$  and  $\tilde{J}_8 = I_4 \otimes (i\sigma_y) = I_2 \otimes I_2 \otimes (i\sigma_y)$ ; and iii) Neither  $J_8$  nor  $\tilde{J}_8$  are any of the  $Y_i, i = 1, \dots, 6$ . In view of the multiplication table for the Pauli matrices, it is natural to seek reversion in the form  $M^{-1}X^T M$ , with  $M$  either  $J_8$  or  $\tilde{J}_8$ . A few calculations reveal that  $J_8^{-1}Y_i^T J_8 \neq Y_i, \forall i$ . Hence, reversion cannot be given by  $J_8^{-1}X^T J_8$ . However, we have the following proposition

**Proposition 28.** i) *The reversion anti-involution on  $\text{Cl}(0, 6)$ , with respect to the basis*

$$\begin{aligned} Y_1 &= \sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z) & Y_4 &= -\sigma_z \otimes (i\sigma_y) \otimes \sigma_z \\ Y_2 &= i\sigma_y \otimes \sigma_x \otimes \sigma_x & Y_5 &= -I_2 \otimes i\sigma_y \otimes \sigma_x \\ Y_3 &= i\sigma_y \otimes I_2 \otimes \sigma_z & Y_6 &= -i\sigma_y \otimes \sigma_z \otimes \sigma_x \end{aligned}$$

*of one-vectors is given by  $\Phi^{rev}(X) = \tilde{J}_8^T X^T \tilde{J}_8$ , for all  $X \in \text{Cl}(0, 6)$ .*

ii) *The grade involution on  $\text{Cl}(0, 6)$ , with respect to the basis  $\{Y_i; i = 1, \dots, 6\}$  of one-vectors is given by  $\Phi^{gr}(X) = \tilde{J}_8^T X \tilde{J}_8$ . Thus, the algebra of even vectors in  $\text{Cl}(0, 6)$  is the image of  $M(4, \mathbb{C})$ , under  $\theta_{\mathbb{C}}$ , in  $M(8, \mathbb{R})$ .*

**Proof:** First note that

$$\tilde{J}_8^{-1} = \tilde{J}_8^T = I_2 \otimes I_2 \otimes (-i\sigma_y).$$

Next, it suffices to check that the map  $X \rightarrow \tilde{J}_8^T X^T \tilde{J}_8$ , which is evidently an anti-involution, is the identity map on one-vectors. For this, in turn, it suffices to verify that  $\tilde{J}_8^T Y_i^T \tilde{J}_8 = Y_i$ , for all  $i = 1, \dots, 6$ . This computation is facilitated by the representations of the  $Y_i, \tilde{J}_8, \tilde{J}_8^T$  all as threefold Kronecker products. We will content ourselves with demonstrating this for  $Y_1$

$$\tilde{J}_8^T Y_1^T \tilde{J}_8 = [I_2 \otimes I_2 \otimes (-i\sigma_y)][\sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z)]^T [(I_2 \otimes I_2 \otimes (i\sigma_y))].$$

Using the fact that  $i\sigma_y$  is antisymmetric, while  $\sigma_x, \sigma_z$  are symmetric, we find that  $\tilde{J}_8^T Y_1^T \tilde{J}_8$ , is therefore

$$[I_2 \otimes I_2 \otimes (-i\sigma_y)][\sigma_x \otimes (-i\sigma_y) \otimes (-\sigma_z)][(I_2 \otimes I_2 \otimes (i\sigma_y))] = \sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z) = Y_1.$$

A similar computation reveals the result to hold for the remaining  $Y_i$ 's.

The second part of the proposition now is just a consequence of the last sentence of Remark 27. Hence, being an even vector is equivalent to  $X = \tilde{J}_8^T X \tilde{J}_8$ , i.e., to

$X^T = \tilde{J}_8^T X^T \tilde{J}_8$ , which by  $v)$  of Remark 8 says precisely that  $X = \Theta_{\mathbb{C}}(Y)$  for some  $Y \in \mathbb{M}(4, \mathbb{C})$ . ■

It now follows that  $\text{Spin}(6)$  is the collection of  $Z \in \text{Cl}(0, 6) = \mathbb{M}(8, \mathbb{R})$  satisfying

- i)  $ZZ^T = I_n$ .
- ii)  $Z$  is even, i.e.,  $Z = \Theta_{\mathbb{C}}(W)$ , for some  $W \in \mathbb{M}(4, \mathbb{C})$ .
- iii)  $ZY Z^T$  is a one-vector for all one-vectors  $Y \in \text{Cl}(0, 6)$ .

The first two conditions say that  $Z = \Theta_{\mathbb{C}}(W)$  for some  $W \in \mathbb{U}(4)$ . However, as is well known, unlike the case of  $\text{Spin}(5)$ , the last condition is no longer superfluous. Dimension considerations say that the third condition force the corresponding  $W$  to be a connected 15 dimensional subgroup of  $\mathbb{U}(4)$ . The obvious candidate is  $\text{SU}(4)$ . Within the context of the derivation above, this can be verified in one of several explicit ways. For instance, for each element  $X$  of a basis for  $\mathfrak{su}(4)$ , it suffices to check  $\theta_{\mathbb{C}}(X)Y_i - Y_i\theta_{\mathbb{C}}(X)$  is a real linear combination of the  $Y_i$ 's.

Verification of this is carried out in Theorem 29 below, since it will be needed at other points as well. It is also interesting to note that the archtypal element in the Lie algebra  $\mathfrak{u}(4)$ , but not in  $\mathfrak{su}(4)$ , viz.,  $iI_4$ , violates the linearization of the third condition for  $\text{Spin}(6)$  in a rather strong way. In other words, denoting by  $V$ , the matrix  $I_4 \otimes (i\sigma_y) = \Theta_{\mathbb{C}}(iI_4)$ , one finds that  $VY_i - Y_iV$  is not a one-vector for any  $Y_i$ . We will just demonstrate this for  $Y_1$ . Computing  $VY_1 - Y_1V$ , we find that it equals

$$\begin{aligned} (I_2 \otimes I_2 \otimes (i\sigma_y)(\sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z) - (\sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z)(I_2 \otimes I_2 \otimes (i\sigma_y) \\ = 2\sigma_x \otimes (i\sigma_y) \otimes \sigma_x. \end{aligned}$$

If we denote the end product of this computation by  $\Lambda_1$ , then  $\Lambda_1$  is, in fact, orthogonal to every one-vector, with respect to the trace inner product on defined  $\mathbb{M}(8, \mathbb{R}) = \text{Cl}(0, 6)$ . This is because a quick calculation of the matrices  $\Lambda_1^T Y_i$  reveals that each of them is a threefold Kronecker product, in which at least one factor is a multiple of one of the Pauli matrices  $\sigma_i$ ,  $i = x, y, z$ . Since the Pauli matrices are traceless, it follows that each  $\Lambda_1^T Y_i$  is traceless. Similar calculations show that  $VY_i - Y_iV$  is not a one-vector for  $i \geq 2$  also. On the other hand, the calculations below confirm that if  $V = \theta_{\mathbb{C}}(W)$ ,  $W \in \mathfrak{su}(4)$ , then  $VY_i - Y_iV$  is a one-vector,  $i = 1, \dots, 6$ .

### Computation of the Lie Algebra Isomorphism Between $\mathfrak{su}(4)$ and $\mathfrak{so}(6, \mathbb{R})$

To achieve the said computation we first need to identify the elements of  $\mathbb{M}(8, \mathbb{R})$  which arise as  $\Theta_{\mathbb{C}}(X)$ , as  $X$  runs over a basis of  $\mathfrak{su}(4)$ . The basis of  $\mathfrak{su}(4)$  we

**Table 5.**  $\Theta_{\mathbb{C}}$  embedding of  $\mathfrak{su}(4)$ .

$X \in \mathfrak{su}(4)$	$\Theta_{\mathbb{C}}(X)$	$X \in \mathfrak{su}(4)$	$\Theta_{\mathbb{C}}(X)$
$i\sigma_x \otimes I_2$	$\sigma_x \otimes I_2 \otimes (i\sigma_y)$	$I_2 \otimes (i\sigma_z)$	$I_2 \otimes \sigma_x \otimes (i\sigma_y)$
$i\sigma_y \otimes I_2$	$i\sigma_y \otimes I_2 \otimes I_2$	$i\sigma_z \otimes \sigma_z$	$\sigma_z \otimes \sigma_z \otimes (i\sigma_y)$
$i\sigma_z \otimes I_2$	$\sigma_z \otimes I_2 \otimes (i\sigma_y)$	$i\sigma_z \otimes \sigma_x$	$\sigma_z \otimes \sigma_z \otimes (i\sigma_y)$
$I_2 \otimes (i\sigma_x)$	$I_2 \otimes \sigma_x \otimes (i\sigma_y)$	$i\sigma_z \otimes \sigma_y$	$\sigma_z \otimes (i\sigma_y \otimes I_2)$
$I_2 \otimes (i\sigma_y)$	$I_2 \otimes (i\sigma_y \otimes I_2)$	$i\sigma_x \otimes \sigma_x$	$\sigma_x \otimes \sigma_x \otimes (i\sigma_y)$
$i\sigma_x \otimes \sigma_y$	$\sigma_x \otimes (i\sigma_y) \otimes I_2$	$i\sigma_y \otimes \sigma_y$	$i\sigma_y \otimes (i\sigma_y) \otimes (i\sigma_y)$
$i\sigma_x \otimes \sigma_z$	$\sigma_x \otimes \sigma_z \otimes (i\sigma_y)$	$i\sigma_y \otimes \sigma_z$	$i\sigma_y \otimes \sigma_z \otimes I_2$
$i\sigma_y \otimes \sigma_x$	$i\sigma_y \otimes \sigma_x \otimes I_2$		

**Table 6.** Lie algebra isomorphism between  $\mathfrak{su}(4)$  and  $\mathfrak{so}(6, \mathbb{R})$ .

Basis of $\mathfrak{su}(4)$	Basis of $\mathfrak{so}(6, \mathbb{R})$	Basis of $\mathfrak{su}(4)$	Basis of $\mathfrak{so}(6, \mathbb{R})$
$i\sigma_x \otimes I_2$	$2(e_1e_5^T - e_5e_1^T)$	$I_2 \otimes (i\sigma_z)$	$2(e_6e_3^T - e_3e_6^T)$
$i\sigma_y \otimes I_2$	$2(e_4e_1^T - e_1e_4^T)$	$i\sigma_z \otimes \sigma_z$	$2(e_2e_1^T - e_1e_2^T)$
$i\sigma_z \otimes I_2$	$2(e_4e_5^T - e_5e_4^T)$	$i\sigma_z \otimes \sigma_x$	$2(e_6e_1^T - e_1e_6^T)$
$I_2 \otimes (i\sigma_x)$	$2(e_3e_2^T - e_2e_3^T)$	$i\sigma_z \otimes \sigma_y$	$2(e_3e_1^T - e_1e_3^T)$
$I_2 \otimes (i\sigma_y)$	$2(e_2e_6^T - e_6e_2^T)$	$i\sigma_x \otimes \sigma_x$	$2(e_4e_6^T - e_6e_4^T)$
$i\sigma_x \otimes \sigma_y$	$2(e_4e_3^T - e_3e_4^T)$	$i\sigma_y \otimes \sigma_y$	$2(e_3e_5^T - e_5e_3^T)$
$i\sigma_x \otimes \sigma_z$	$2(e_4e_2^T - e_2e_4^T)$	$i\sigma_y \otimes \sigma_z$	$2(e_2e_5^T - e_5e_2^T)$
$i\sigma_y \otimes \sigma_x$	$2(e_5e_6^T - e_6e_5^T)$		

will work with is the basis consisting of Kronecker products of the Pauli matrices (including  $\sigma_0 = I_2$ ). We then obtain Table 5.

We can now state

**Theorem 29.** *The Lie algebra isomorphism  $\Psi_6 : \mathfrak{su}(4) \rightarrow \mathfrak{so}(6, \mathbb{R})$  is prescribed by its effect on the basis  $\{i\sigma_j \otimes I_2, I_2 \otimes (i\sigma_k), i\sigma_p \otimes \sigma_q\}$ ,  $j, k, p, q \in \{x, y, z\}$  of  $\mathfrak{su}(4)$  via Table 6.*

**Proof:** For each of the  $A_k$ ,  $k = 1, \dots, 15$  in the II column of Table 5, we compute  $A_k Y_i - Y_i A_k$ , where  $\{Y_1, \dots, Y_6\}$  is the basis of one-vectors of  $\text{Cl}(0, 6)$  and express the result as a linear combination of the  $Y_l$ ,  $l = 1, \dots, 6$ . The resulting matrix is the image of  $\Psi_6(X)$ , where  $X$  is an element of the basis of  $\mathfrak{su}(4)$  listed in the I column of Table 5. This is a long calculation. We will just record the details

for  $A_2$  for illustration. We compute

$$\begin{aligned} A_2 Y_1 - Y_1 A_2 &= (i\sigma_y \otimes I_2 \otimes I_2)(\sigma_x \otimes (i\sigma_y \otimes (-\sigma_z)) \\ &\quad - (\sigma_x \otimes (i\sigma_y \otimes (-\sigma_z))(i\sigma_y \otimes I_2 \otimes I_2)) \\ &= -2\sigma_z \otimes (i\sigma_y) \otimes \sigma_z = 2Y_4 \end{aligned}$$

$$\begin{aligned} A_2 Y_2 - Y_2 A_2 &= (i\sigma_y \otimes I_2 \otimes I_2)(i\sigma_y \otimes \sigma_x \otimes \sigma_x) \\ &\quad - (i\sigma_y \otimes \sigma_x \otimes \sigma_x)(i\sigma_y \otimes I_2 \otimes I_2) = 0 \end{aligned}$$

$$\begin{aligned} A_2 Y_3 - Y_3 A_2 &= (i\sigma_y \otimes I_2 \otimes I_2)(i\sigma_y \otimes I_2 \otimes \sigma_z) \\ &\quad - (i\sigma_y \otimes I_2 \otimes \sigma_z)(i\sigma_y \otimes I_2 \otimes I_2) = 0 \end{aligned}$$

$$\begin{aligned} A_2 Y_4 - Y_4 A_2 &= i\sigma_y \otimes I_2 \otimes I_2(-\sigma_z \otimes (i\sigma_y) \otimes \sigma_z) \\ &\quad - (\sigma_z \otimes (i\sigma_y \otimes (-\sigma_z))(i\sigma_y \otimes I_2 \otimes I_2)) \\ &= 2\sigma_x \otimes (i\sigma_y) \otimes \sigma_z = -2Y_1 \end{aligned}$$

$$\begin{aligned} A_2 Y_5 - Y_5 A_2 &= i\sigma_y \otimes I_2 \otimes I_2(-I_2 \otimes (i\sigma_y) \otimes \sigma_x) \\ &\quad - (-I_2 \otimes (i\sigma_y) \otimes \sigma_x)(i\sigma_y \otimes I_2 \otimes I_2) = 0 \end{aligned}$$

$$\begin{aligned} A_2 Y_6 - Y_6 A_2 &= i\sigma_y \otimes I_2 \otimes I_2(-i\sigma_y \otimes \sigma_z \otimes \sigma_x) \\ &\quad - (-i\sigma_y \otimes \sigma_z \otimes \sigma_x)(i\sigma_y \otimes I_2 \otimes I_2) = 0. \end{aligned}$$

Hence  $\chi(i\sigma_y \otimes I_2) = 2(e_4 e_1^T - e_1 e_4^T)$ . ■

*Computing Exponentials in  $\mathfrak{so}(6, \mathbb{R})$  via those in  $\mathfrak{su}(4)$*

We finish this section with an example which illustrates the utility of passing to the  $SU(4)$  for calculating exponentials in  $\mathfrak{so}(6, \mathbb{R})$ .

**Example 30.** Consider the matrix  $X = \beta(e_4 e_6^T - e_6 e_4^T) + \delta(e_6 e_1^T - e_1 e_6^T)$ , for some  $\beta, \delta \in \mathbb{R}$ . Let us call the two summands  $X_1, X_2$ . The summands  $X_1$  and  $X_2$  do not anticommute or commute, as can be easily verified. While the individual exponentials of  $X_1$  and  $X_2$  are easily found (both have cubic minimal polynomials), their sum, without availing of the isomorphism with  $\mathfrak{su}(4)$ , presents a greater challenge. In fact,  $X$  has a quintic minimal polynomial as a brute force calculation, which we eschew, shows. On the other hand,  $\Psi_6^{-1}(X)$  has a quadratic minimal polynomial!

Computing  $W = \Psi_6^{-1}(X) \in \mathfrak{su}(4)$ , we find that it is

$$\frac{i\beta}{2}\sigma_x \otimes \sigma_x + \frac{i(\gamma - \alpha)}{2}\sigma_z \otimes \sigma_x = Z_1 + Z_2.$$

In keeping with the fact that the map  $\Psi_6$  is a Lie algebra isomorphism, we see that  $[Z_1, Z_2] \neq 0$ . However,  $Z_1 Z_2 = -Z_2 Z_1$ . Thus,  $W$ 's minimal polynomial is quadratic and one finds

$$e^W = cI_4 + \left(\frac{s}{\lambda}\right) \left( \frac{i\beta}{2} \sigma_x \otimes \sigma_x + \frac{i(\gamma - \alpha)}{2} \sigma_z \otimes \sigma_x \right)$$

where  $\lambda = \frac{1}{2} \sqrt{\beta^2 + (\gamma - \alpha)^2}$ , and  $c = \cos(\lambda)$ ,  $s = \sin(\lambda)$ . We next find  $\Lambda = \theta_{\mathbb{C}}(e^W)$ . It is given by

$$\Lambda = cI_8 + \frac{s}{\lambda} \left( \frac{\beta}{2} (\sigma_x \otimes \sigma_x \otimes i\sigma_y) + \frac{(\gamma - \alpha)}{2} (\sigma_z \otimes \otimes \sigma_x \otimes i\sigma_y) \right).$$

Hence

$$\Lambda^T = cI_8 - \frac{s}{\lambda} \left( \frac{\beta}{2} (\sigma_x \otimes \sigma_x \otimes i\sigma_y) + \frac{(\gamma - \alpha)}{2} (\sigma_z \otimes \otimes \sigma_x \otimes i\sigma_y) \right).$$

To find  $e^X$ , we compute  $\Lambda Y_i \Lambda^T$ ,  $i = 1, \dots, 6$ . Suppose  $\Lambda Y_j \Lambda^T = \sum_{i=1}^6 c_{ij} Y_i$ , then  $e^X = (c_{ij})$ . To that end, we need the following

- $\Lambda Y_1 \Lambda^T = \Lambda(\sigma_x \otimes (i\sigma_y) \otimes (-\sigma_z)) \Lambda^T$  is given by

$$\left( c^2 - \frac{s^2}{4\lambda^2} ((\gamma - \alpha)^2 - \beta^2) \right) Y_1 + \frac{2s^2\beta(\gamma - \alpha)}{4\lambda^2} Y_4 + \frac{cs(\gamma - \alpha)}{\lambda} Y_6$$

- $\Lambda Y_2 \Lambda^T = \Lambda(i\sigma_y \otimes \sigma_x \otimes \sigma_x) \Lambda^T$  is given by

$$\frac{(\beta^2 + (\gamma - \alpha)^2) c^2 + s^2\beta^2 + s^2(\gamma - \alpha)^2}{\beta^2 + (\gamma - \alpha)^2} Y_2 = Y_2$$

- $\Lambda Y_3 \Lambda^T = \Lambda(i\sigma_y \otimes I_2 \otimes \sigma_z) \Lambda^T$  is given by

$$(c^2 + \frac{s^2[\beta^2 + (\gamma - \alpha)^2]}{4\lambda^2}) Y_3 = Y_3$$

- $\Lambda Y_4 \Lambda^T = \Lambda(-\sigma_z \otimes (i\sigma_y) \otimes \sigma_x) \Lambda^T$  is given by

$$\frac{2s^2\beta(\gamma - \alpha)}{\beta^2 + (\gamma - \alpha)^2} Y_1 + \frac{(\gamma - \alpha)^2 + \beta^2(c^2 - s^2)}{\beta^2 + (\gamma - \alpha)^2} Y_4 - \frac{2cs\beta}{2\lambda} Y_6$$

- $\Lambda Y_5 \Lambda^T = \Lambda(-I_2 \otimes (i\sigma_y) \otimes \sigma_x) \Lambda^T$  is given by

$$\frac{\beta^2 + (\gamma - \alpha)^2(c^2 + s^2)}{\beta^2 + (\gamma - \alpha)^2} Y_5 = Y_5$$

- $\Lambda Y_6 \Lambda^T = \Lambda((-i\sigma_y) \otimes \sigma_z \otimes \sigma_x) \Lambda^T$  is given by

$$-\frac{2cs(\gamma - \alpha)}{2\lambda} Y_1 + \frac{cs\beta}{\lambda} Y_4 + (c^2 - s^2) Y_6.$$

Hence,

$$\text{Exp}(X) = \begin{pmatrix} \frac{\beta^2 + (\gamma - \alpha)^2 (c^2 - s^2)}{\beta^2 + (\gamma - \alpha)^2} & 0 & 0 & \frac{2s^2\beta(\gamma - \alpha)}{\beta^2 + (\gamma - \alpha)^2} & 0 & -\frac{cs(\gamma - \alpha)}{\lambda} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{2s^2\beta(\gamma - \alpha)}{\beta^2 + (\gamma - \alpha)^2} & 0 & 0 & \frac{(\gamma - \alpha)^2 + \beta^2 (c^2 - s^2)}{\beta^2 + (\gamma - \alpha)^2} & 0 & \frac{cs\beta}{\lambda} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{cs(\gamma - \alpha)}{\lambda} & 0 & 0 & -\frac{cs\beta}{\lambda} & 0 & c^2 - s^2 \end{pmatrix}. \quad (1)$$

## 6. Minimal Polynomials of Matrices in $\mathfrak{su}(4)$

In this section the minimal polynomials of matrices  $X \in \mathfrak{su}(4)$ , is characterized completely. Thus, the problem of exponentiation in  $\mathfrak{su}(4)$  and hence in  $\mathfrak{so}(6, \mathbb{R})$  admit solutions which are constructive. The characterization of the minimal polynomials will involve verifiable conditions on the  $E_k(X)$ ,  $k = 2, 3, 4$ . Recall  $E_k(X)$  is the sum of all  $k \times k$  principal minors of  $X$  and these are easy to compute.

The initial observation, which follows from arguments similar to those in Proposition 21 and Proposition 22, is that the minimal polynomial,  $m_X$ , of  $X \in \mathfrak{su}(4)$ , has the following property:

- A) If the degree of  $m_X$  is even, then the coefficients of all the even powers of  $x$  in it are real, while those of the odd powers are purely imaginary.
- B) If the degree of  $m_X$  is odd, then the coefficients of all the odd powers of  $x$  in it are real, while those of the even powers are purely imaginary.

This observation can be honed into the following result

**Theorem 31.** *Let  $X$  be a non-zero matrix in  $\mathfrak{su}(4)$ . Then the structure of the minimal polynomials of  $X$  is given by*

1.  $X$  has the minimal polynomial  $x^2 + \lambda^2$ , with  $\lambda \in \mathbb{R}$  non-zero, iff  $E_3 = 0$ ,  $E_2 \neq 0$  and  $E_4 = \frac{1}{4}(E_2)^2$ .



2.  $X$  has the minimal polynomial  $x^2 + i\gamma x + \lambda^2$ , with  $\gamma, \lambda \in \mathbb{R}$  both non-zero iff  $E_2 > 0$ ,  $E_4 = -\frac{1}{12}(E_2)^2$ ,  $E_3 = 8i(\sqrt{\frac{E_2}{6}})^3$ .
3.  $X$  has the minimal polynomial  $x^3 + \theta^2 x$ , with  $\theta \in \mathbb{R}$  non-zero, iff  $E_3 = 0 = E_4$  and  $E_2 > 0$ .
4.  $X$  has minimal polynomial  $x^3 + i\gamma x^2 + \theta^2 x$ , with  $\gamma, \theta \in \mathbb{R}$  both non-zero iff  $E_2 > 0$  and  $E_3$  is either  $+2i(\sqrt{\frac{E_2}{3}})^3$  or  $-2i(\sqrt{\frac{E_2}{3}})^3$ .
5.  $X$  has minimal polynomial  $x^3 + i\gamma x^2 + \theta^2 x + i\delta$ , with  $\gamma, \theta, \delta \in \mathbb{R}$ , all non-zero iff  $E_4 \neq 0$ , and
 
$$16E_2^4 E_4 - 4E_2^3 E_3^2 - 128E_2^2 E_4^2 + 144E_2 E_3^2 E_4 - 27E_3^4 + 256E_4^3 = 0 \quad (2)$$
 and at least one of the conditions in each of items 1) and 2) above is violated.
6. The minimal polynomial of  $X$  is its characteristic polynomial iff the condition in equation (2) is violated.

Furthermore, in each of these cases the coefficients of the minimal polynomial can be determined constructively from the  $E_k$ .

**Proof:** First, since  $X$  is skew-Hermitian, so is every principal submatrix of  $X$ . Since the determinant of an even sized (resp. odd sized) skew-Hermitian matrix is real (resp. purely imaginary) it follows that  $E_2, E_4 \in \mathbb{R}$  and  $iE_3 \in \mathbb{R}$ .

Next, since  $X$  is diagonalizable its minimal polynomial has distinct roots. In view of  $E_1 = 0$  and  $X \neq 0$ , the following are the root configurations of the characteristic polynomial,  $p_X(x)$ , which lead to its minimal polynomial,  $m_X(x)$ , being of strictly lower degree than 4:

- Case 1) The two distinct roots of  $p_X$  are  $ia$  and  $-ia$ , each with multiplicity 2, and  $a \in \mathbb{R}$  non-zero. In this case  $m_X = x^2 + a^2$ .
- Case 2) The two distinct roots of  $p_X$  are  $ia$  and  $ib$  with *i*)  $a, b$  non-zero real, and *ii*) the former repeated thrice and the latter once. In this case, necessarily  $b = -3a$ . In this case  $m_X = x^2 + 2iax + 3a^2$ .
- Case 3) The three distinct roots of  $p_X$  are 0 (repeated twice) and  $ia$  and  $-ia$  of multiplicity one each (with  $a \in \mathbb{R}$  non-zero). In this case  $m_X = x^3 + a^2 x$ .
- Case 4) The three distinct roots of  $p_X$  are  $ia, ib$  and 0, with first repeated twice and the latter two of multiplicity one each. Once again  $a, b \in \mathbb{R}$  are non-zero. In this case, necessarily  $b = -2a$  and  $m_X = x^3 + ia x^2 + 2a^2 x$ .

Case 5) The three distinct roots of  $p_X$  are  $ia$ ,  $ib$  and  $ic$ , with  $i$ )  $a, b, c \in \mathbb{R}$  and  $abc \neq 0$ , and  $ii$ ) the multiplicity of  $ia$  is two, while that of the other roots is one each. In this case necessarily,  $b + c = -2a$ . Furthermore,

$$m_X = (x-ia)(x-ib)(x-ic) = x^3 - i(a+b+c)x^2 + (-a^2 - ab - ac)x + iabc.$$

Since,  $b + c = -2a$ , this simplifies, for the moment, to

$$m_x = x^3 + iax^2 + a^2x + iabc.$$

Case 6) All roots of  $p_X(x)$  are distinct. In this case the minimal polynomial is  $p_X$ .

To now characterize these root configurations, without having to find the roots, we note that  $E_k = S_k, \forall k$ , where  $S_k$  is, of course, the  $k$ th elementary symmetric polynomial of the roots of the characteristic polynomial. So we have

Case 1) In this case  $E_2 = -a^2 + a^2 + a^2 + a^2 + a^2 - a^2 = 2a^2$ . Similarly  $E_3 = 0$  and  $E_4 = a^4$ . So for  $X$  to have the minimal polynomial  $x^2 + \lambda^2$ , it is necessary that  $E_3 = 0$ ,  $E_2 > 0$  and  $E_4 = \frac{1}{4}(E_2)^2$ . Furthermore,  $\lambda = \sqrt{\frac{E_2}{2}}$ . The converse is also true. If these conditions on the  $E_k$  hold,

$$p_X = x^4 + E_2x^2 + E_4 = x^4 + E_2x^2 + \frac{E_2^2}{4}.$$

Quite clearly this is a quadratic in  $x^2$ , leading to the eigenvalues being of the form  $ia$  and  $-ia$ , each repeated twice, with  $a$  the positive square root of  $\frac{E_2}{2}$ , which, of course leads to  $m_X = x^2 + \frac{E_2}{2}$ .

Case 2) In this case  $E_2 = 6a^2$ , while  $E_3 = 8ia^3$  and finally,  $E_4 = -3a^4$ . From this it follows that a necessary condition for  $X$  to have the minimal polynomial

$$m_X(x) = x^2 + i\gamma x + c^2$$

is that  $E_2 > 0$ ,  $E_3 = 8i\left(\frac{E_2}{6}\right)^{3/2}$  and  $E_4 = -\frac{E_2^2}{12}$ .

The converse also holds. Indeed, in this case,  $p_X$  has a triple root. Hence  $p'_X$  has a double root and this double root is one of the roots of  $p''_X$ . Now

$$p''_X = 12x^2 + 2E_2.$$

Its roots are  $i\sqrt{\frac{E_2}{6}}$  and  $-i\sqrt{\frac{E_2}{6}}$ . Only one of these can be a root of  $p_X$ , since neither is  $-3$  times the other and  $p_X$  has only one multiple root. We calculate

$$p_X\left(i\sqrt{\frac{E_2}{6}}\right) = \frac{E_2^2}{36} - \frac{E_2^2}{6} + \frac{8E_2^2}{36} - \frac{E_2^2}{12} = 0.$$

Here we have made use of the necessary conditions  $E_2 > 0$ ,  $E_3 = 8i\sqrt{\left(\frac{E_2}{6}\right)^3}$  and  $E_4 = -\frac{1}{12}(E_2)^2$ .

Thus, sufficiency has also been verified. Finally, note that the coefficients of the minimal polynomial satisfy  $\gamma = 2a$ ,  $c^2 = 3a^2$ . Both can be obtained **without** finding  $a$ . Clearly,  $c^2 = \frac{E_2}{2}$  and to find  $\gamma$  we look at the sign of the purely imaginary number  $E_3$ . Its sign coincides with the sign of  $\gamma$ , and the actual value of  $\gamma$  is then found from, say, just  $E_2$ .

Case 3) In this case, we find  $E_2 = a^2$  and that  $E_3 = 0 = E_4$ . So the stated conditions are obviously necessary. They are also sufficient, since under these conditions the characteristic polynomial is

$$p_X(x) = x^4 + E_2x^2 = x^2(x^2 + E_2).$$

Since  $E_2 > 0$ , its roots are obviously 0 (repeated twice) and  $i\sqrt{E_2}$  and  $-i\sqrt{E_2}$ .

Finally, the minimal polynomial, in this case, is  $m_X = x^3 + c^2x$ , and  $c^2$  is evidently uniquely determined as  $c^2 = E_2$ .

Case 4) In this case  $E_2 = 3a^2$ ,  $E_3 = 2ia^3$ ,  $E_4 = 0$ . So necessarily  $E_2 > 0$  and  $E_3$  is plus or minus  $2i\left(\frac{E_2}{3}\right)^{\frac{3}{2}}$  and  $E_4 = 0$ .

To verify the converse note that, if the stated conditions on  $E_2, E_3, E_4$  hold then

$$p_X(x) = x^4 + E_2x^2 - 2i\left(\frac{E_2}{3}\right)^{\frac{3}{2}}x = x\left(x^3 + E_2x - 2i\left(\frac{E_2}{3}\right)^{\frac{3}{2}}\right)$$

So 0 is a single root and the remaining roots of  $p_X$  are the roots of

$$q(x) = x^3 + E_2x - 2i\left(\frac{E_2}{3}\right)^{\frac{3}{2}}.$$

To show that  $q(x)$ , and thus  $p_X$ , has a double root we compute

$$q'(x) = 3x^2 + E_2.$$

Its roots are  $x = i\sqrt{\frac{E_2}{3}}$  and  $x = -i\sqrt{\frac{E_2}{3}}$ . We check if one of these roots is a root of  $p_X$ . We find, if  $E_3 = 2i\left(\frac{E_2}{3}\right)^{\frac{3}{2}}$ , then

$$p\left(i\sqrt{\frac{E_2}{3}}\right) = 0.$$

If  $E_3 = -2i\left(\frac{E_2}{3}\right)^{\frac{3}{2}}$ , then

$$p\left(-i\sqrt{\frac{E_2}{3}}\right) = 0.$$

So indeed the stated conditions are sufficient as well.

Finally, to determine the coefficients of  $m_X(x) = x^3 + i\gamma x^2 + \theta^2 x$ , we note that since  $m_X$  is also  $x^3 + ia x^2 + 2a^2 x$ , we must have  $\theta^2 = 2a^2 = \frac{2}{3}E_2$  and that  $\gamma$  is plus or minus  $i\sqrt{\frac{E_2}{3}}$ , depending on the sign of the non-zero purely imaginary number  $E_3$ .

Case 5)  $X$  has a minimal polynomial, which is of lower degree than 4, iff  $p_X$  has a repeated root. Now  $p_X$  has a repeated root iff it and its derivative have a common root. The latter condition obtains iff the resultant of  $p_X$  and  $p'_X$  vanish. This condition is precisely the validity of equation (2). The remaining conditions ensure that this repeated root configuration is not one of the preceding cases, and thus has to correspond to the root configuration  $\{ia, ia, ib, ic\}$ , with  $abc \neq 0$ .

To determine the coefficients of  $m_X$ , we first note that, since  $c = -(b + 2a)$  one has

$$m_X = x^3 + (ia)x^2 + (2a^2 - bc)x + iabc.$$

Let us write this as

$$m_X = x^3 + c_1 x^2 + c_2 x + c_3.$$

Now,  $E_2 = -a^2 - 2a(b+c) - bc = 3a^2 - bc$ . Thus,  $c_2 = E_2 - a^2$ . Similarly,  $c_3 = i\frac{E_4}{a}$ . Hence,

$$m_X = x^3 + (ia)x^2 + (E_2 - a)x + i\frac{E_4}{a}.$$

So to fully find  $m_X$  we need  $a$ . To that end, we proceed as follows. Note first that

$$E_3 = i(2a^3 - 2abc).$$

Since  $E_2 = 3a^2 - bc$ , we find

$$E_3 = i(-4a^3 + (2a)E_2).$$

Equivalently,  $iE_3 = 4a^3 - (2a)E_2$ . Hence,  $a$  is a root of the cubic

$$c(x) = 4X^3 - (2E_2)x - iE_3 = 0. \quad (3)$$

Since  $E_2$  and  $iE_3$  are real, this cubic has at least one real root. If this cubic has only one real root then, that real root gives  $a$  and we are done. If it has three real roots, say  $\alpha, \beta, \gamma$ , then by construction precisely one of  $\{i\alpha, i\beta, i\gamma\}$  is a double root of  $p_X$ . So we evaluate  $p_X$  and  $p'_X$  at these points and see at which of these both vanish. That gives  $a$  and hence  $m_X$ . ■

**Remark 32.**  $e^X$  can be found for any  $X \in \mathfrak{su}(4)$  satisfying the first three cases of Theorem 31 by using the formulae presented in Theorem 14. For cases 4) and 5) of Theorem 31 one can use Lagrange interpolation, i.e.,  $e^X$  is that polynomial in  $X$  which takes on the value  $e^{ir}$  at a root  $ir, r \in \mathbb{R}$  of the corresponding minimal polynomial. Note that the proof of Theorem 31 supplies, as a by-product, recipes to find the roots of the minimal polynomial in cases 4) and 5). For case 6), if  $E_3 = 0$ , then one can invoke case IV) of Theorem 14. Similarly, in Case 6) if  $E_4 = \det(X) = 0$ , then one can easily find the roots of the characteristic polynomial. They are given by  $0, i\alpha, i\beta, -i(\alpha + \beta)$ , with  $\alpha\beta \neq 0$  and  $\alpha \neq \beta$  and  $\alpha \neq -\beta$ . These can be found by solving a cubic. Finally, in Case 6), if neither  $E_3$  nor  $E_4$  is zero, then one has to solve a quartic to find the eigenvalues, which, albeit, complicated, can be found in closed form. One can then use Lagrange interpolation to find  $e^X$ .

**Remark 33.** Spin (5) Reconsidered: Section 3 started with a basis of one-vectors for  $\text{Cl}(3, 0)$  (namely the Pauli basis) and applied the natural constructions in Section 2.3 to arrive at a basis of one-vectors for  $\text{Cl}(0, 5)$ . The ability to produce a basis of one-vectors for  $\text{Cl}(0, 6)$ , starting from  $\text{Cl}(0, 0)$ , which lead to  $\tilde{J}_8$  playing a role in reversion, naturally raises the question whether following that set of iterative constructions could lead to something similar for  $\text{Cl}(0, 5)$ . We show below that this is the case and more importantly that a slight variation of this construction reveals a role in reversion for yet another matrix in the  $\mathbb{H} \otimes \mathbb{H}$  basis for  $M(4, \mathbb{R})$ , viz., the matrix  $M_{j \otimes 1}$ ! In the process, a natural interpretation of the matrix  $X(z_0, z_1, z_2)$  of Remark 3 is also found.

- a) Start with  $\text{Cl}(0, 1)$  and apply the construction IC1 of Section 2.3 twice to arrive at a basis of one-vectors for  $\text{Cl}(2, 3)$ . Next we use IC3 of Section 2.3 to arrive at a basis of one-vectors for  $\text{Cl}(4, 1)$ , and then finally use IC2 of Section 2.3 to arrive at a basis of one-vectors for  $\text{Cl}(0, 5)$ . We begin with  $\{i\}$  as the obvious basis of one-vectors for  $\text{Cl}(0, 1)$ . The basis of one-vectors for  $\text{Cl}(2, 3)$  produced in this manner

$$\begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}, \quad \begin{pmatrix} i\sigma_z & 0 \\ 0 & -i\sigma_z \end{pmatrix}, \quad \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Applying IC3 and then IC2 (after removing inessential negative signs) yields the following basis for  $\text{Cl}(0, 5)$

$$g_1 = \{\sigma_x \otimes I_2, g_2 = i\sigma_y \otimes \sigma_y, g_3 = i\sigma_y \otimes \sigma_z, g_4 = i\sigma_z \otimes I_2, g_5 = i\sigma_y \otimes \sigma_x\}.$$

With respect to this basis reversion on  $\text{Cl}(0, 5) = \text{M}(4, \mathbb{C})$  is described by

$$\Phi^{rev}(X) = \tilde{J}_4^{-1} X^T \tilde{J}_4.$$

- b) A typical one-vector, with respect to the basis  $\{g_k\}$  in part a) is given by the following matrix

$$\begin{pmatrix} id & 0 & c + ia & e - bi \\ 0 & id & e + ib & -c + ia \\ -c + ia & -e + ib & -id & 0 \\ -e - ib & c + ia & 0 & -id \end{pmatrix}$$

(with  $a, b, c, d, e \in \mathbb{R}$ ). But this matrix is precisely  $X(z_0, z_i, z_2)$  described in Remark 3, with  $z_0 = c + ia$ ,  $z_1 = e + ib$ ,  $z_2 = id$ . This gives a different motivation for this matrix in [20]. It should be pointed out that the basis  $\{g_i; i = 1, \dots, 5\}$  given in part a) is not present in [20], since for identification of  $\text{Spin}(5)$ , [20] works in  $\text{Cl}(0, 4)$ .

- c) We now discuss a slight variation on the construction in part a). Everything remains verbatim up to the basis of one-vectors for  $\text{Cl}(2, 3)$ . However, for the production of a basis of one-vectors for  $\text{Cl}(4, 1)$  using IC3 we have interchanged the roles of  $\sigma_z \otimes \sigma_x$  and  $\sigma_x \otimes I_2$  - the two one-vectors in  $\text{Cl}(2, 3)$  which square to  $+1$ , cf., Remark 6. This then yields yet another basis of one-vectors for  $\text{Cl}(0, 5)$  given by

$$\hat{f}_1 = i\sigma_z \otimes \sigma_x, \hat{f}_2 = I_2 \otimes i\sigma_z, \hat{f}_3 = -I_2 \otimes i\sigma_y, \hat{f}_4 = -i\sigma_x \otimes \sigma_x, \hat{f}_5 = i\sigma_y \otimes \sigma_x$$

Now the form of reversion on  $\text{Cl}(0, 5)$  with respect to this basis of one-vectors is

$$\Phi_{rev}(X) = \check{J}_4^{-1} X^T \check{J}_4$$

where  $\check{J}_4 = M_{\mathbb{J} \otimes 1}$ .

## 7. Conclusions

In this note we have derived explicit matrix realizations of the reversion automorphism for  $\text{Cl}(0, 5)$  and  $\text{Cl}(0, 6)$ , with respect to bases of one-vectors which are

natural from the point of view of the standard iterative procedures, described in Section 2.3. This also leads to a first principles approach to the spin groups in these dimensions, in the sense that they are obtained by working entirely in  $\text{Cl}(0, 5)$  and  $\text{Cl}(0, 6)$  respectively. These constructions are then used to find closed form expressions for the exponentials of real antisymmetric matrices of size  $5 \times 5$  and  $6 \times 6$ . This is facilitated by the derivation of explicit expressions for the minimal polynomials of matrices in the Lie algebras of the corresponding spin groups. These expressions do not require any spectral knowledge of the matrices in question. An important by-product of this note is that it provides further evidence for the importance of the isomorphism between  $\mathbb{H} \otimes \mathbb{H}$  and  $M(4, \mathbb{R})$ . It would be interesting to examine whether the explicit relations between the spin group for  $n = 5, 6$  and the special orthogonal group for  $n = 5, 6$ , established in this work, can be used to shed light constructively on further relations between these groups, analogous to those discussed in [10].

There are some questions whose study this work naturally suggests. We mention two here:

- It would be useful to obtain expressions for minimal polynomials of matrices in  $\mathfrak{su}(4)$  directly from their  $\mathbb{H} \otimes \mathbb{H}$  representations, analogous to the formulae in [23]. Specifically, if one writes an  $X \in \mathfrak{su}(4)$  as  $Y + iZ$  with  $Y, Z$  real matrices, then  $Y^T = -Y$  and  $Z^T = Z$ . This is significant because any such work will also yield formulae for minimal polynomials of the real matrix  $Y + Z$ . Since such a matrix is the most general traceless real  $4 \times 4$  matrix, the benefits are obvious. In Section 6, while no knowledge of eigenvalues or eigenvectors was needed, the diagonalizability of matrices in  $\mathfrak{su}(4)$  was heavily used. On the other hand, the methods in [23] never used any such information. Since there are many important non-diagonalizable matrices in  $M(4, \mathbb{R})$ , this would be of high utility.
- It is important to be able to invert the covering maps  $\Phi_5$  and  $\Phi_6$ . One application of this would be the ability to deduce factorizations of matrices in  $\text{SO}(n, \mathbb{R})$ , for  $n = 5, 6$ , from those for matrices in their spin groups. The inversion of these maps requires solving a system polynomial equations in several variables which are essentially quadratic.

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