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SYMMETRIC REDUCTION AND HAMILTON-JACOBI EQUATION OF RIGID SPACECRAFT WITH A ROTOR

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Abstract. In this paper, we consider the rigid spacecraft with an internal rotor as a regular point of reducible regular controlled Hamiltonian (RCH) system. In the cases of coincident and non-coincident centers of buoyancy and gravity, we give explicitly the equations of motion and Hamilton-Jacobi equations of reduced spacecraft-rotor system on the symplectic leaves by calculation in detail, which show the effect on controls in regular symplectic reduction and Hamilton-Jacobi theory respectively.

Contents

1. Introduction

It is well-known that the theory of controlled mechanical systems is an important subject in the recent years. It gathers together some separate areas of research such

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as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis on geometry is motivated by the aim of understanding the structure of the equations of motion of the system in a way that helps both analysis and design. There is a natural two-fold method in the study of controlled mechanical systems. First, being a special class of nonlinear control systems one can study them by using the feedback control and optimal control methods. The second, as they are also a special class of mechanical systems, one can study them combining the analysis of dynamic systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. Thus, as the theory of controlled mechanical systems presents a challenging and promising research area between the classical mechanics and modern nonlinear geometric control theory, a lot of researchers are absorbed to pour into the area and there have been a lot of interesting results. Some of them, as Bloch *et al* in [4–7], studied the symmetry and feedback control to realize a modification to the structure of a given mechanical system while, Nijmeijer and van der Schaft in [27] studied the nonlinear dynamical control systems as well as the use of feedback control to stabilize mechanical systems, and van der Schaft in [31,32] referred to the reduction and control of implicit (port) Hamiltonian systems.

In particular, we note that in Marsden *et al* [22], the authors studied regular reduction theory of controlled Hamiltonian systems with symplectic structure and symmetry as an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. In [33] Wang generalized the work in [22] by treating the singular reduction theory of regular controlled Hamiltonian systems, and Wang and Zhang in [36] generalized the work in [33] and clarify the optimal reduction theory of controlled Hamiltonian systems with Poisson structure and symmetry by using optimal momentum map and reduced Poisson tensor (or reduced symplectic form), and Ratiu and Wang [30] studied the Poisson reduction of controlled Hamiltonian system by controllability distribution. These works not only gave a variety of reduction methods for controlled Hamiltonian systems, but also showed a variety of relationships of controlled Hamiltonian equivalence of these systems.

At the same time, we note that Hamilton-Jacobi theory is an important part of classical mechanics. On one hand, it provides a characterization of the generating functions of certain time-dependent canonical transformations. On the other hand, in many cases it is possible that Hamilton-Jacobi theory provides an immediate way to integrate the equations of motion of the system, even when the problem of Hamiltonian system itself has not been or cannot be solved completely. In addition, the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and plays also an important role in the development of numerical integrators that preserve the symplectic structure and in the study of stochastic dynamical systems, see Woodhouse [37], Ge and Marsden [10], Marsden and West [23] and Lázaro-Camí and Ortega [13]. For these reasons it is a useful tool in the study of Hamiltonian system theory, and has been extensively developed during the years to become one of the most active subjects in the study of modern applied mathematics and analytical mechanics. For more details see Cariñena *et al* [8,9], Iglesias *et al* [11], León *et al* [3,14,15] and Ohsawa and Bloch [28].

The variational point of view Hamilton-Jacobi theory was originally developed by Jacobi in 1866, and states that the integral of Lagrangian of a system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the geometrical point of view is given by Abraham and Marsden in [1], and was developed in the context of time-dependent Hamiltonian system by Marsden and Ratiu in [21]. The Hamilton-Jacobi equation may be regarded as a nonlinear partial differential equation for some generating function S, and the problem becomes how to choose a time-dependent canonical transformation $\Psi : T^*Q \times \mathbb{R} \to T^*Q \times \mathbb{R}$, which transforms the dynamical vector field of the time-dependent Hamiltonian system to equilibrium, such that the generating function S of Ψ satisfies the time-dependent Hamilton-Jacobi equation. In particular, for the time-independent Hamiltonian system, one may look for a symplectic map as the canonical transformation. This work offers an important idea that one can use the dynamical vector field of the Hamiltonian system to describe the Hamilton-Jacobi equation. Moreover, assume that $\gamma : Q \to T^*Q$ is a closed one-form on the smooth configuratiomnal manifold Q, and define $X_H^\gamma = T\pi_Q \cdot X_H \cdot \gamma$, where X_H is the vector field of Hamiltonian system (T^*Q, ω, H). Then the fact that X_H^γ and X_H are γ-related, that is, $T\gamma \cdot X_H^\gamma = X_H \cdot \gamma$ is equivalent to $\mathrm{d}(H \cdot \gamma) = 0$, which is given in Cariñena *et al* [8] and Iglesias *et al* [11]. Since the Hamilton-Jacobi theory is based on the Hamiltonian picture of dynamics, Wang [34] has used the dynamical vector field of Hamiltonian system and the regular reduced Hamiltonian system to describe the Hamilton-Jacobi theory for these systems. In [35] this was extended to the Hamilton-Jacobi theory of the regular controlled Hamiltonian system, its regular reduced systems, and for clarification of the relationship between the RCH-equivalence for RCH systems and the solutions of corresponding Hamilton-Jacobi equations.

Now, it is a natural problem if there are controlled Hamiltonian systems and how to show the effect on controls in regular symplectic reduction and Hamilton-Jacobi theory of such systems. In this paper, as an application of the regular point symplectic reduction and Hamilton-Jacobi theory of RCH system with symmetry, we consider the case when the rigid spacecraft with an internal rotor is a regular point reducible RCH system given already in Marsden *et al* [22], where the rigid spacecraft with an internal rotor is modelled as a Hamiltonian system with control as presented in Bloch and Leonard [5], Bloch *et al* [7]. In the cases of coincident and non-coincident centers of buoyancy and gravity, we give explicitly the equations of motion and the Hamilton-Jacobi equation of the reduced spacecraft-rotor system on the symplectic leaves. These equations are more complex than that of the Hamiltonian systems without control and describe explicitly the effect on controls in regular symplectic reduction and Hamilton-Jacobi theory.

A brief outline of the paper is as follows. In the second section, we review some relevant definitions and basic facts about rigid spacecraft with an internal rotor, which will be used in the subsequent sections. As an application of the theoretical result concerning the symplectic reduction of RCH system given by Marsden *et al* [22], in the third section we consider the rigid spacecraft with an internal rotor as a regular point reducible RCH system on the extension of the rotation group $\mathrm{SO}(3) \times \mathrm{S}^1$ and on the that one of the Euclidean group $\mathrm{SE}(3) \times \mathrm{S}^1$, respectively, in the cases of coincident and non-coincident centers of buoyancy and gravity, and we give explicitly the equations of motion of their reduced RCH systems on the symplectic leaves. Moreover, as an application of the theoretical result about Hamilton-Jacobi theory of regular reduced RCH system obtained by Wang [35], in the fourth section, we give the Hamilton-Jacobi equations of the reduced rigid spacecraft-rotor systems on the symplectic leaves in the cases of coincident and non-coincident centers of buoyancy and gravity. This work develop the application of symplectic reduction and Hamilton-Jacobi theory of RCH systems with symmetry with an aim for much deeper understanding and recognition of the structure of the Hamiltonian systems and RCH systems.

2. The Rigid Spacecraft with a Rotor

In this paper, our goal is to give the regular point reduction and Hamilton-Jacobi theorem of rigid spacecraft with an internal rotor. In order to do this, we review some relevant definitions and basic facts about rigid spacecraft with an internal rotor. We shall follow the notation and conventions introduced in Bloch and Leonard [5], Bloch *et al* [7], Marsden [18], Marsden and Ratiu [21], and Marsden *et al* [22]. In this paper, we assume that all manifolds are real, smooth and finite dimensional and that all actions are smooth left actions. For convenience, we also assume that all controls appearing in this paper are the admissible controls.

2.1. The Spacecraft-Rotor System with Coincident Centers

We consider a rigid spacecraft (to be called the carrier body) carrying an internal rotor, and assume that the only external forces and torques acting on the spacecraft-rotor system are due to the buoyancy and the gravity. In general, it is possible that the spacecraft's center of buoyancy may not be coincident with its center of gravity. But, in this subsection we assume that the spacecraft is symmetric and has uniformly distributed mass, and that the center of buoyancy and the center of gravity are coincident. Denote by O the center of mass of the system in the body frame and let O is the origin of (orthogonal) body axes. Assume that the body coordinate axes are aligned with principal axes of the carrier body, and that the rotor is aligned along the third principal axis, see Bloch and Leonard [5] and Bloch *et al* [7]. The rotor spins under the influence of a torque u acting on the rotor. If translations are ignored and only rotations are considered, the configuration space is $Q = \mathrm{SO}(3) \times \mathrm{S}^1$, with the first factor being the attitude of the rigid spacecraft and the second factor being the angle of the rotor. The corresponding phase space is the cotangent bundle $T^*Q = T^*\mathrm{SO}(3) \times T^*\mathrm{S}^1$, where $T^*\mathrm{S}^1 \cong T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, with the canonical symplectic form.

Let $I = \mathrm{diag}(I_1, I_2, I_3)$ be the moment of inertia of the carrier body in the principal body-fixed frame, and J_3 be the moment of inertia of rotor around its rotation axis. Let $J_{3k},\ k = 1, 2$, be the moments of inertia of the rotor around the kth principal axis with $k = 1, 2$, and denote by $\bar{I}_k = I_k + J_{3k},\ k = 1, 2,\ \bar{I}_3 = I_3$. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the vector of body angular velocities computed with respect to the axes fixed in the body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let α be the relative angle of rotor and $\dot{\alpha}$ the rotor relative angular velocity about the third principal axis with respect to a carrier body fixed frame. Consider the Lagrangian of the system $L(A, \Omega, \alpha, \dot{\alpha}) : TQ \cong \mathrm{SO}(3) \times \mathfrak{so}(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is the total kinetic energy of the rigid spacecraft plus the kinetic energy of rotor, given by

$$L(A, \Omega, \alpha, \dot{\alpha}) = \frac{1}{2}(\bar{I}_1\Omega_1^2 + \bar{I}_2\Omega_2^2 + \bar{I}_3\Omega_3^2 + J_3(\Omega_3 + \dot{\alpha})^2)$$

where $A \in \mathrm{SO}(3)$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\alpha \in \mathbb{R}$, $\dot{\alpha} \in \mathbb{R}$. If we introduce the conjugate angular momentum, given by $\Pi_k = \dfrac{\partial L}{\partial \Omega_k} = \bar{I}_k\Omega_k,\ k = 1, 2$, $\Pi_3 = \dfrac{\partial L}{\partial \Omega_3} = \bar{I}_3\Omega_3 + J_3(\Omega_3 + \dot{\alpha}),\quad l = \dfrac{\partial L}{\partial \dot{\alpha}} = J_3(\Omega_3 + \dot{\alpha})$, and make use of the Legendre transformation $FL : \mathrm{SO}(3) \times \mathfrak{so}(3) \times \mathbb{R} \times \mathbb{R} \to \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}$, $(A, \Omega, \alpha, \dot{\alpha}) \to (A, \Pi, \alpha, l)$, where $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3)$, $l \in \mathbb{R}$, we end up with the Hamiltonian $H(A, \Pi, \alpha, l) : T^*Q \cong \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by the function

$$\begin{aligned} H(A,\Pi,\alpha,l) &= \Omega\cdot\Pi+\dot{\alpha}\cdot l-L(A,\Omega,\alpha,\dot{\alpha}) \\ &= \frac{1}{2}\left(\frac{\Pi_1^2}{\bar{I}_1}+\frac{\Pi_2^2}{\bar{I}_2}+\frac{(\Pi_3-l)^2}{\bar{I}_3}+\frac{l^2}{J_3}\right). \end{aligned} \tag{1}$$

In order to derive the equations of motion of spacecraft-rotor system, in the following we need to consider the symmetry and reduced symplectic structure of the configuration space $Q=\mathrm{SO}(3)\times \mathrm{S}^1$.

2.2. The Spacecraft-Rotor System with Non-Coincident Centers

Since it is possible that the spacecraft's center of buoyancy may not be coincident with its center of gravity, in this subsection then we consider the spacecraft-rotor system with non-coincident centers of buoyancy and gravity. We fix an orthogonal coordinate frame to the carrier body with origin located at the center of buoyancy and axes aligned with the principal axes of the carrier body, and the rotor is aligned along the third principal axis, see Bloch and Leonard [5], Bloch *et al* [7], and Leonard and Marsden [16]. The rotor spins under the influence of a torque u acting on the rotor. When the carrier body is oriented so that the body-fixed frame is aligned with the inertial frame, the third principal axis aligns with the direction of gravity. The vector from the center of buoyancy to the center of gravity with respect to the body-fixed frame is $h\chi$, where χ is an unit vector on the line connecting the two centers which is assumed to be aligned along the third principal axis, and h is the length of this segment. The mass of the carrier body is denoted by m, the magnitude of the gravitational acceleration by g, and let Γ be the unit vector viewed by an observer moving with the body. In this case, the configuration space is $Q=\mathrm{SO}(3)\circledS\mathbb{R}^3\times\mathrm{S}^1\cong\mathrm{SE}(3)\times\mathrm{S}^1$, where the first factor being the attitude of rigid spacecraft and the drift of spacecraft in the rotational process and the second factor being the angle of rotor. The corresponding phase space is the cotangent bundle $T^*Q=T^*\mathrm{SE}(3)\times T^*\mathrm{S}^1$, where $T^*\mathrm{S}^1\cong T^*\mathbb{R}\cong\mathbb{R}\times\mathbb{R}$, with its canonical symplectic form.

Consider the Lagrangian of the system $L(A,c,\Omega,\Gamma,\alpha,\dot{\alpha}):TQ\cong\mathrm{SE}(3)\times\mathfrak{se}(3)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, which is the total kinetic energy of the rigid spacecraft plus the kinetic energy of the rotor minus the potential energy of the system, i.e.,

$$L(A,c,\Omega,\Gamma,\alpha,\dot{\alpha})=\frac{1}{2}(\bar{I}_1\Omega_1^2+\bar{I}_2\Omega_2^2+\bar{I}_3\Omega_3^2+J_3(\Omega_3+\dot{\alpha})^2)-mgh\Gamma\cdot\chi$$

where $(A,c)\in\mathrm{SE}(3)$, $\Omega=(\Omega_1,\Omega_2,\Omega_3)\in\mathfrak{so}(3)$, $\Gamma\in\mathbb{R}^3$, $(\Omega,\Gamma)\in\mathfrak{se}(3)$, $\alpha\in\mathbb{R}$, $\dot{\alpha}\in\mathbb{R}$. If we introduce the conjugate angular momentum, given by

$\Pi_k = \dfrac{\partial L}{\partial \Omega_k} = \bar{I}_k\Omega_k,\ k=1,2, \Pi_3 = \dfrac{\partial L}{\partial \Omega_3} = \bar{I}_3\Omega_3 + J_3(\Omega_3+\dot{\alpha}), \quad l = \dfrac{\partial L}{\partial \dot{\alpha}} = J_3(\Omega_3+\dot{\alpha})$, and via the Legendre transformation $FL : \mathrm{SE}(3)\times\mathfrak{se}(3)\times\mathbb{R}\times\mathbb{R}\to\mathrm{SE}(3)\times\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}, \quad (A,c,\Omega,\Gamma,\alpha,\dot{\alpha})\to(A,c,\Pi,\Gamma,\alpha,l)$, where $\Pi=(\Pi_1,\Pi_2,\Pi_3)\in\mathfrak{so}^*(3)$, $(\Pi,\Gamma)\in\mathfrak{se}^*(3)$, $l\in\mathbb{R}$, we arrive at the Hamiltonian $H(A,c,\Pi,\Gamma,\alpha,l) : T^*Q\cong\mathrm{SE}(3)\times\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by the function

$$\begin{aligned} H(A,c,\Pi,\Gamma,\alpha,l) &= \Omega\cdot\Pi+\dot{\alpha}\cdot l - L(A,c,\Omega,\Gamma,\alpha,\dot{\alpha}) \\ &= \frac{1}{2}(\frac{\Pi_1^2}{\bar{I}_1}+\frac{\Pi_2^2}{\bar{I}_2}+\frac{(\Pi_3-l)^2}{\bar{I}_3}+\frac{l^2}{J_3})+mgh\Gamma\cdot\chi. \end{aligned} \tag{2}$$

In order to give the equation of motion of spacecraft-rotor system, in the following we need to consider the symmetry and reduced symplectic structure of the cotangent bundle of the configurational space $Q=\mathrm{SE}(3)\times\mathrm{S}^1$.

3. Symmetric Reduction of the Rigid Spacecraft with a Rotor

In the following we consider the rigid spacecraft with an internal rotor as a regular point reducible RCH system on the extension of rotation group $\mathrm{SO}(3)\times\mathrm{S}^1$ and on the extension of the Euclidean group $\mathrm{SE}(3)\times\mathrm{S}^1$, respectively, and give the equations of motion of their reduced RCH systems on the respective symplectic leaves. It is worth to note that it is different from the symmetric reduction of Hamiltonian system in Bloch and Leonard [5], Bloch *et al* [7], Marsden [18], the reductions in this paper only the controlled Hamiltonian reductions, that is, the symmetric reductions of (regular) controlled Hamiltonian systems, see Marsden *et al* [22]. We follow the notation and conventions introduced in Marsden *et al* [19, 20], Marsden and Ratiu [21], Libermann and Marle [17], Ortega and Ratiu [29].

3.1. Symmetric Reduction of Spacecraft-Rotor System with Coincident Centers

We first give the regular point reduction of spacecraft-rotor system with coincident centers of buoyancy and gravity. Assume that the Lie group $G=\mathrm{SO}(3)$ acts freely and properly on $Q=\mathrm{SO}(3)\times\mathrm{S}^1$ by left translations on the first factor $\mathrm{SO}(3)$, and via the trivial action on the second factor S^1. By using the left trivialization of $T^*\mathrm{SO}(3)=\mathrm{SO}(3)\times\mathfrak{so}^*(3)$, the action of $\mathrm{SO}(3)$ on the phase space $T^*Q=T^*\mathrm{SO}(3)\times T^*\mathrm{S}^1$ is by cotangent lift of left translations on $\mathrm{SO}(3)$ at the identity, that is, $\Phi:\mathrm{SO}(3)\times T^*\mathrm{SO}(3)\times T^*\mathrm{S}^1\cong\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathfrak{so}^*(3)\times\mathbb{R}\times\mathbb{R}\to$

$\mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}$, given by $\Phi(B, (A, \Pi, \alpha, l)) = (BA, \Pi, \alpha, l)$, for any $A, B \in \mathrm{SO}(3),\ \Pi \in \mathfrak{so}^*(3),\ \alpha, l \in \mathbb{R}$, which is also free and proper, and the orbit space $(T^*Q)/\mathrm{SO}(3)$ is a smooth manifold with $\pi : T^*Q \to (T^*Q)/\mathrm{SO}(3)$ being a smooth submersion. Since $\mathrm{SO}(3)$ acts trivially on $\mathfrak{so}^*(3)$ and $\mathbb{R}$, it follows that $(T^*Q)/\mathrm{SO}(3)$ is diffeomorphic to $\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}$.

Further we know that $\mathfrak{so}^*(3)$ is a Poisson manifold with respect to its rigid body Lie-Poisson bracket defined by

$$\{F, K\}_{\mathfrak{so}^*(3)}(\Pi) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) \tag{3}$$

where $F, K \in C^\infty(\mathfrak{so}^*(3)),\ \Pi \in \mathfrak{so}^*(3)$. For $\mu \in \mathfrak{so}^*(3)$, the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{so}^*(3)$ has an induced orbit symplectic form $\omega^-_{\mathcal{O}_\mu}$, which coincides with the restriction of the Lie-Poisson bracket on $\mathfrak{so}^*(3)$ to the coadjoint orbit $\mathcal{O}_\mu$. From the Symplectic Stratification theorem we know that the coadjoint orbits $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu}),\ \mu \in \mathfrak{so}^*(3)$, form the symplectic leaves of the Poisson manifold $(\mathfrak{so}^*(3), \{\cdot,\cdot\}_{\mathfrak{so}^*(3)})$. Let $\omega_\mathbb{R}$ be the canonical symplectic form on $T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, which is given by

$$\omega_\mathbb{R}((\theta_1, \lambda_1), (\theta_2, \lambda_2)) = \langle \lambda_2, \theta_1 \rangle - \langle \lambda_1, \theta_2 \rangle \tag{4}$$

where $(\theta_i, \lambda_i) \in \mathbb{R} \times \mathbb{R},\ i = 1, 2$, $\langle \cdot,\cdot \rangle$ is the standard inner product on $\mathbb{R} \times \mathbb{R}$. It induces a canonical Poisson bracket $\{\cdot,\cdot\}_\mathbb{R}$ on $T^*\mathbb{R}$, which is given by the formula

$$\{F, K\}_\mathbb{R}(\theta, \lambda) = \frac{\partial F}{\partial \theta}\frac{\partial K}{\partial \lambda} - \frac{\partial K}{\partial \theta}\frac{\partial F}{\partial \lambda}. \tag{5}$$

Thus, we can induce a symplectic form $\tilde{\omega}^-_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}} = \pi^*_{\mathcal{O}_\mu}\omega^-_{\mathcal{O}_\mu} + \pi^*_\mathbb{R}\omega_\mathbb{R}$ on the smooth manifold $\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}$, where the maps $\pi_{\mathcal{O}_\mu} : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} \to \mathcal{O}_\mu$ and $\pi_\mathbb{R} : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ are canonical projections, and induce a Poisson bracket $\{\cdot,\cdot\}_- = \pi^*_{\mathfrak{so}^*(3)}\{\cdot,\cdot\}_{\mathfrak{so}^*(3)} + \pi^*_\mathbb{R}\{\cdot,\cdot\}_\mathbb{R}$ on the smooth manifold $\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}$. Respectively, the maps $\pi_{\mathfrak{so}^*(3)} : \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathfrak{so}^*(3)$ and $\pi_\mathbb{R} : \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ are canonical projections, and such that $(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}, \tilde{\omega}^-_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}})$ is a symplectic leaf of the Poisson manifold $(\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}, \{\cdot,\cdot\}_-)$.

On the other hand, from the isomorphism $T^*Q \cong T^*\mathrm{SO}(3) \times T^*\mathrm{S}^1$ we know that there is a canonical symplectic form $\omega_Q = \pi^*_{\mathrm{SO}(3)}\omega_0 + \pi^*_{\mathrm{S}^1}\omega_{\mathrm{S}^1}$ on T^*Q, where ω_0 is the canonical symplectic form on $T^*\mathrm{SO}(3)$ and the maps $\pi_{\mathrm{SO}(3)} : Q = \mathrm{SO}(3) \times \mathrm{S}^1 \to \mathrm{SO}(3)$ and $\pi_{\mathrm{S}^1} : Q = \mathrm{SO}(3) \times \mathrm{S}^1 \to \mathrm{S}^1$ are canonical projections. Then the cotangent lift of the left $\mathrm{SO}(3)$-action $\Phi : \mathrm{SO}(3) \times T^*Q \to T^*Q$ is also symplectic, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q : T^*Q \to \mathfrak{so}^*(3)$ such that $\mathbf{J}_Q \cdot \pi^*_{\mathrm{SO}(3)} = \mathbf{J}_{\mathrm{SO}(3)}$, where $\mathbf{J}_{\mathrm{SO}(3)} : T^*\mathrm{SO}(3) \to \mathfrak{so}^*(3)$ is a momentum map of the left $\mathrm{SO}(3)$-action on $T^*\mathrm{SO}(3)$, and $\pi^*_{\mathrm{SO}(3)} : T^*\mathrm{SO}(3) \to T^*Q$. If $\mu \in$

$\mathfrak{so}^*(3)$ is a regular value of $\mathbf{J}_Q$, then $\mu \in \mathfrak{so}^*(3)$ is also a regular value of $\mathbf{J}_{\mathrm{SO}(3)}$ and $\mathbf{J}_Q^{-1}(\mu) \cong \mathbf{J}_{\mathrm{SO}(3)}^{-1}(\mu) \times \mathbb{R} \times \mathbb{R}$. Denote by $\mathrm{SO}(3)_\mu = \{g \in \mathrm{SO}(3);\ \mathrm{Ad}_g^* \mu = \mu\}$ the isotropy subgroup of coadjoint SO(3)-action at the point $\mu \in \mathfrak{so}^*(3)$. It follows that $\mathrm{SO}(3)_\mu$ acts also freely and properly on $\mathbf{J}_Q^{-1}(\mu)$, and that the regular point reduced space $(T^*Q)_\mu = \mathbf{J}_Q^{-1}(\mu)/\mathrm{SO}(3)_\mu \cong (T^*\mathrm{SO}(3))_\mu \times \mathbb{R} \times \mathbb{R}$ of (T^*Q, ω_Q) at μ is a symplectic manifold with symplectic form ω_μ uniquely characterized by the relation $\pi_\mu^*\omega_\mu = i_\mu^*\omega_Q = i_\mu^*\pi_{\mathrm{SO}(3)}^*\omega_0 + i_\mu^*\pi_{\mathrm{S}^1}^*\omega_{\mathrm{S}^1}$, where the map $i_\mu : \mathbf{J}_Q^{-1}(\mu) \to T^*Q$ is the inclusion and $\pi_\mu : \mathbf{J}_Q^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. Due to Abraham and Marsden [1], we know also that $((T^*\mathrm{SO}(3))_\mu, \omega_\mu)$ is symplectically diffeomorphic to $(\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}^-)$, and hence we have that $((T^*Q)_\mu, \omega_\mu)$ is symplectically diffeomorphic to $(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}, \tilde{\omega}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}^-)$, which is a symplectic leaf of the Poisson manifold $(\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}, \{\cdot,\cdot\}_-)$.

From the expression (1) for the Hamiltonian, we know that $H(A, \Pi, \alpha, l)$ is invariant under the left SO(3)-action $\Phi : \mathrm{SO}(3) \times T^*Q \to T^*Q$. In the case when $\mu \in \mathfrak{so}^*(3)$ is a regular value of $\mathbf{J}_Q$, we have the reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} (\subset \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ given by $h_\mu(\Pi, \alpha, l) = \pi_\mu(H(A, \Pi, \alpha, l)) = H(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}$. Using the rigid body Poisson bracket on $\mathfrak{so}^*(3)$ and the Poisson bracket on $T^*\mathbb{R}$, we can write the Poisson bracket on $\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}$, of $F, K : \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, in the form

$$\{F, K\}_-(\Pi, \alpha, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) + \{F, K\}_{\mathbb{R}}(\alpha, l) \tag{6}$$

see Krishnaprasad and Marsden [12]. In particular, for $F_\mu, K_\mu : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we have that $\tilde{\omega}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}^-(X_{F_\mu}, X_{K_\mu}) = \{F_\mu, K_\mu\}_-|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}$. Moreover, for reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we have the Hamiltonian vector field $X_{h_\mu}(K_\mu) = \{K_\mu, h_\mu\}_-|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}$, and hence

$$\begin{aligned}
\frac{\mathrm{d}\Pi}{\mathrm{d}t} &= X_{h_\mu}(\Pi)(\Pi, \alpha, l) = \{\Pi, h_\mu\}_-(\Pi, \alpha, l) \\
&= -\Pi \cdot (\nabla_\Pi \Pi \times \nabla_\Pi h_\mu) + \left(\frac{\partial \Pi}{\partial \alpha}\frac{\partial h_\mu}{\partial l} - \frac{\partial h_\mu}{\partial \alpha}\frac{\partial \Pi}{\partial l}\right) \\
&= -\nabla_\Pi \Pi \cdot (\nabla_\Pi h_\mu \times \Pi) = \Pi \times \Omega
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\alpha}{\mathrm{d}t} &= X_{h_\mu}(\alpha)(\Pi, \alpha, l) = \{\alpha, h_\mu\}_-(\Pi, \alpha, l) \\
&= -\Pi \cdot (\nabla_\Pi \alpha \times \nabla_\Pi h_\mu) + \left(\frac{\partial \alpha}{\partial \alpha}\frac{\partial h_\mu}{\partial l} - \frac{\partial h_\mu}{\partial \alpha}\frac{\partial \alpha}{\partial l}\right) \\
&= -\frac{(\Pi_3 - l)}{\bar{I}_3} + \frac{l}{J_3}
\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}l}{\mathrm{d}t} &= X_{h_\mu}(l)(\Pi,\alpha,l) = \{l, h_\mu\}_-(\Pi,\alpha,l) \\ &= -\Pi\cdot(\nabla_\Pi l \times \nabla_\Pi h_\mu) + (\frac{\partial l}{\partial \alpha}\frac{\partial h_\mu}{\partial l} - \frac{\partial h_\mu}{\partial \alpha}\frac{\partial l}{\partial l}) = 0\end{aligned}$$

since $\nabla_\Pi \Pi = 1,\ \nabla_\Pi \alpha = \nabla_\Pi l = 0,\ \nabla_\Pi h_\mu = \Omega$, and $\frac{\partial \Pi}{\partial \alpha} = \frac{\partial l}{\partial \alpha} = \frac{\partial h_\mu}{\partial \alpha} = 0$. If we consider the rigid spacecraft-rotor system with a control torque $u : T^*Q \to T^*Q$ acting on the rotor, and $u \in \mathbf{J}_Q^{-1}(\mu)$ is invariant under the left SO(3)-action, and its reduced control torque $u_\mu : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}$ is given by $u_\mu(\Pi,\alpha,l) = \pi_\mu(u(A,\Pi,\alpha,l)) = u(A,\Pi,\alpha,l)|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$, where $\pi_\mu : \mathbf{J}_Q^{-1}(\mu) \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}$. Thus, in the case of coincident centers of buoyancy and gravity, the equations of motion for reduced spacecraft-rotor system with the control torque u acting on the rotor are given by the system

$$\begin{aligned}\frac{\mathrm{d}\Pi}{\mathrm{d}t} &= \Pi \times \Omega \\ \frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -\frac{(\Pi_3 - l)}{\bar{I}_3} + \frac{l}{J_3} \\ \frac{\mathrm{d}l}{\mathrm{d}t} &= \upsilon(u_\mu)X_{h_\mu}.\end{aligned} \tag{7}$$

Here $\upsilon(u_\mu)X_{h_\mu} \in T(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R})$. Note that $\upsilon(u_\mu)X_{h_\mu}$ is the vertical lift of the vector field X_{h_μ} under the action of u_μ along fibers, that is

$$\upsilon(u_\mu)X_{h_\mu}(\Pi,\alpha,l) = \upsilon((Tu_\mu X_{h_\mu})(u_\mu(\Pi,\alpha,l)),(\Pi,\alpha,l)) = (Tu_\mu X_{h_\mu})^\upsilon_\sigma(\Pi,\alpha,l)$$

where σ is a geodesic in $\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}$ connecting the points $u_\mu(\Pi,\alpha,l)$ and (Π,α,l), and $(Tu_\mu X_{h_\mu})^\upsilon_\sigma(\Pi,\alpha,l)$ is just the parallel displacement of the vertical vector field $(Tu_\mu X_{h_\mu})^\upsilon(\Pi,\alpha,l)$ along the geodesic σ from $u_\mu(\Pi,\alpha,l)$ to (Π,α,l), (see Marsden *et al* [22] and Wang [35]). To sum up the above discussion, we state the following theorem.

Theorem 1. *In the case of coincident centers of the buoyancy and the gravity, the spacecraft-rotor system with the control torque u acting on the rotor, that is, the five-tuple $(T^*Q, \mathrm{SO}(3), \omega_Q, H, u)$, where $Q = \mathrm{SO}(3) \times \mathrm{S}^1$, is a regular point reducible RCH system. For a point $\mu \in \mathfrak{so}^*(3)$, the regular value of the momentum map $\mathbf{J}_Q : \mathrm{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathfrak{so}^*(3)$, the regular point reduced system is the four-tuple $(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}, \tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}, h_\mu, u_\mu)$, where $\mathcal{O}_\mu \subset \mathfrak{so}^*(3)$ is the coadjoint orbit, $\tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$ is orbit symplectic form on $\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}$, $h_\mu(\Pi,\alpha,l) = \pi_\mu(H(A,\Pi,\alpha,l)) = H(A,\Pi,\alpha,l)|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$, $u_\mu(\Pi,\alpha,l) = \pi_\mu(u(A,\Pi,\alpha,l)) = u(A,\Pi,\alpha,l)|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$, and its equations of motion are given by* (7).

Remark 2. When the rigid spacecraft does not carry any internal rotor, the configuration space is $Q = G = \mathrm{SO}(3)$, the motion of rigid spacecraft is just the rotation motion of a rigid body, the above symmetric reduction of spacecraft-rotor system is just the Marsden-Weinstein reduction of a rigid body at a regular value of momentum map, and the equation of motion (7) of reduced spacecraft-rotor system becomes the equation of motion of a reduced rigid body on a coadjoint orbit of Lie group $\mathrm{SO}(3)$. See Marsden and Ratiu [21].

3.2. Symmetric Reduction of Spacecraft-Rotor System with Non-Coincident Centers

In the following we shall give the regular point reduction of spacecraft-rotor system with non-coincident centers of buoyancy and gravity. Because the drift in the direction of gravity breaks the symmetry and the spacecraft-rotor system is no longer $\mathrm{SO}(3)$ invariant. In this case, its physical phase space is $T^*\mathrm{SO}(3) \times T^*\mathrm{S}^1$ and the symmetry group is S^1, regarded as rotations about the third principal axis, that is, the axis of gravity. By the semidirect product reduction theorem, see Marsden *et al* [19], we know that the reduction of $T^*\mathrm{SO}(3)$ by S^1 gives a space which is symplectically diffeomorphic to the reduced space obtained by the reduction of $T^*\mathrm{SE}(3)$ by the left action of $\mathrm{SE}(3)$, that is, the coadjoint orbit $\mathcal{O}_{(\mu,a)} \subset \mathfrak{se}^*(3) \cong T^*\mathrm{SE}(3)/\mathrm{SE}(3)$. In fact, in this case, we can identify the phase space $T^*\mathrm{SO}(3)$ with the reduction of the cotangent bundle of the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3)\circledS\mathbb{R}^3$ by the Euclidean translation subgroup $\mathbb{R}^3$ and identify the symmetry group S^1 with the isotropy group $G_a = \{A \in \mathrm{SO}(3)\ ;\ Aa = a\} = \mathrm{S}^1$, which is Abelian and $(G_a)_{\mu_a} = G_a = \mathrm{S}^1$, for all $\mu_a \in \mathfrak{g}_a^*$, where a is a vector aligned with the direction of gravity and where $\mathrm{SO}(3)$ acts on $\mathbb{R}^3$ in the standard way.

Assume that the Lie group $G = \mathrm{SE}(3)$ acts freely and properly on $Q = \mathrm{SE}(3) \times \mathrm{S}^1$ by left translations on the first factor $\mathrm{SE}(3)$, and the trivial action on the second factor S^1. By using the left trivialization of $T^*\mathrm{SE}(3) = \mathrm{SE}(3) \times \mathfrak{se}^*(3)$, the action of $\mathrm{SE}(3)$ on phase space $T^*Q = T^*\mathrm{SE}(3) \times T^*\mathrm{S}^1$ is just the cotangent lift of left translations on $\mathrm{SE}(3)$ at the identity, that is, $\Phi : \mathrm{SE}(3) \times T^*\mathrm{SE}(3) \times T^*\mathrm{S}^1 \cong \mathrm{SE}(3) \times \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathrm{SE}(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}$, given by $\Phi((B,b)(A,c,\Pi,\Gamma,\alpha,l)) = (BA,c,\Pi,\Gamma,\alpha,l)$, for any $A, B \in \mathrm{SO}(3),\ \Pi \in \mathfrak{so}^*(3),\ b, c, \Gamma \in \mathbb{R}^3,\ \alpha, l \in \mathbb{R}$, which is also free and proper, and the orbit space $(T^*Q)/\mathrm{SE}(3)$ is a smooth manifold and $\pi : T^*Q \to (T^*Q)/\mathrm{SE}(3)$ is a smooth submersion. Since $\mathrm{SE}(3)$ acts trivially on $\mathfrak{se}^*(3)$ and $\mathbb{R}$, it follows that $(T^*Q)/\mathrm{SE}(3)$ is diffeomorphic to $\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}$.

We know also that $\mathfrak{se}^*(3)$ is a Poisson manifold with respect to its heavy top Lie-Poisson bracket defined by

$$\{F,K\}_{\mathfrak{se}^*(3)}(\Pi,\Gamma)=-\Pi\cdot(\nabla_\Pi F\times\nabla_\Pi K)-\Gamma\cdot(\nabla_\Pi F\times\nabla_\Gamma K-\nabla_\Pi K\times\nabla_\Gamma F) \quad (8)$$

where $F,K\in C^\infty(\mathfrak{se}^*(3))$, $(\Pi,\Gamma)\in\mathfrak{se}^*(3)$. For $(\mu,a)\in\mathfrak{se}^*(3)$, the coadjoint orbit $\mathcal{O}_{(\mu,a)}\subset\mathfrak{se}^*(3)$ has an induced orbit symplectic form $\omega^-_{\mathcal{O}_{(\mu,a)}}$, which coincides with the restriction of the Lie-Poisson bracket on $\mathfrak{se}^*(3)$ to the coadjoint orbit $\mathcal{O}_{(\mu,a)}$, and the coadjoint orbits $(\mathcal{O}_{(\mu,a)},\omega^-_{\mathcal{O}_{(\mu,a)}})$, $(\mu,a)\in\mathfrak{se}^*(3)$, form the symplectic leaves of the Poisson manifold $(\mathfrak{se}^*(3),\{\cdot,\cdot\}_{\mathfrak{se}^*(3)})$. Let $\omega_\mathbb{R}$ be the canonical symplectic form on $T^*\mathbb{R}\cong\mathbb{R}\times\mathbb{R}$ given by (4), which induces a canonical Poisson bracket $\{\cdot,\cdot\}_\mathbb{R}$ on $T^*\mathbb{R}$ given by (5). Thus, we can induce a symplectic form $\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}=\pi^*_{\mathcal{O}_{(\mu,a)}}\omega^-_{\mathcal{O}_{(\mu,a)}}+\pi^*_\mathbb{R}\omega_\mathbb{R}$ on the smooth manifold $\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$, where the maps $\pi_{\mathcal{O}_{(\mu,a)}}:\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}\to\mathcal{O}_{(\mu,a)}$ and $\pi_\mathbb{R}:\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ are the canonical projections, and the Poisson bracket on the smooth manifold $\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}$ is $\{\cdot,\cdot\}_-=\pi^*_{\mathfrak{se}^*(3)}\{\cdot,\cdot\}_{\mathfrak{se}^*(3)}+\pi^*_\mathbb{R}\{\cdot,\cdot\}_\mathbb{R}$. Here the maps $\pi_{\mathfrak{se}^*(3)}:\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}\to\mathfrak{se}^*(3)$ and $\pi_\mathbb{R}:\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ are canonical projections, and such that $(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}})$ is a symplectic leaf of the Poisson manifold $(\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R},\{\cdot,\cdot\}_-)$.

On the other hand, from $T^*Q=T^*\mathrm{SE}(3)\times T^*\mathrm{S}^1$ we know that there is a canonical symplectic form $\omega_Q=\pi^*_{\mathrm{SE}(3)}\omega_1+\pi^*_{\mathrm{S}^1}\omega_{\mathrm{S}^1}$ on T^*Q, where ω_1 is the canonical symplectic form on $T^*\mathrm{SE}(3)$ and the maps $\pi_{\mathrm{SE}(3)}:Q=\mathrm{SE}(3)\times\mathrm{S}^1\to\mathrm{SE}(3)$ and $\pi_{\mathrm{S}^1}:Q=\mathrm{SE}(3)\times\mathrm{S}^1\to\mathrm{S}^1$ are canonical projections. Then the cotangent lift of the left SE(3)-action $\Phi:\mathrm{SE}(3)\times T^*Q\to T^*Q$ is also symplectic, and admits an associated Ad^*-equivariant momentum map $\mathbf{J}_Q:T^*Q\to\mathfrak{se}^*(3)$ such that $\mathbf{J}_Q\cdot\pi^*_{\mathrm{SE}(3)}=\mathbf{J}_{\mathrm{SE}(3)}$, where $\mathbf{J}_{\mathrm{SE}(3)}:T^*\mathrm{SE}(3)\to\mathfrak{se}^*(3)$ is the momentum map of the left SE(3)-action on $T^*\mathrm{SE}(3)$, and $\pi^*_{\mathrm{SE}(3)}:T^*\mathrm{SE}(3)\to T^*Q$. If $(\mu,a)\in\mathfrak{se}^*(3)$ is a regular value of $\mathbf{J}_Q$, then $(\mu,a)\in\mathfrak{se}^*(3)$ is also a regular value of $\mathbf{J}_{\mathrm{SE}(3)}$ and $\mathbf{J}_Q^{-1}(\mu,a)\cong\mathbf{J}_{\mathrm{SE}(3)}^{-1}(\mu,a)\times\mathbb{R}\times\mathbb{R}$. Denote by $\mathrm{SE}(3)_{(\mu,a)}=\{g\in\mathrm{SE}(3)\;;\;\mathrm{Ad}^*_g(\mu,a)=(\mu,a)\}$ the isotropy subgroup of coadjoint SE(3)-action at the point $(\mu,a)\in\mathfrak{se}^*(3)$. It follows that $\mathrm{SE}(3)_{(\mu,a)}$ acts also freely and properly on $\mathbf{J}_Q^{-1}(\mu,a)$, the regular point reduced space $(T^*Q)_{(\mu,a)}=\mathbf{J}_Q^{-1}(\mu,a)/\mathrm{SE}(3)_{(\mu,a)}\cong(T^*\mathrm{SE}(3))_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$ of (T^*Q,ω_Q) at (μ,a) is a symplectic manifold with symplectic form $\omega_{(\mu,a)}$ uniquely characterized by the relation $\pi^*_{(\mu,a)}\omega_{(\mu,a)}=i^*_{(\mu,a)}\omega_Q=i^*_{(\mu,a)}\pi^*_{\mathrm{SE}(3)}\omega_1+i^*_{(\mu,a)}\pi^*_{\mathrm{S}^1}\omega_{\mathrm{S}^1}$, where the map $i_{(\mu,a)}:\mathbf{J}_Q^{-1}(\mu,a)\to T^*Q$ is the inclusion and $\pi_{(\mu,a)}:\mathbf{J}_Q^{-1}(\mu,a)\to(T^*Q)_{(\mu,a)}$ is the projection. Due to the work of Abraham and Marsden [1], we know that $((T^*\mathrm{SE}(3))_{(\mu,a)}, \omega_{(\mu,a)})$ is symplectically diffeomorphic to $(\mathcal{O}_{(\mu,a)}, \omega^-_{\mathcal{O}_{(\mu,a)}})$, and hence we have that $((T^*Q)_{(\mu,a)}, \omega_{(\mu,a)})$ is symplectically diffeomorphic to $(\mathcal{O}_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}, \tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}})$, which is a symplectic leaf of the Poisson manifold $(\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}, \{\cdot,\cdot\}_-)$.

From the expression (2) of the Hamiltonian, we know that $H(A, c, \Pi, \Gamma, \alpha, l)$ is invariant under the left SE(3)-action $\Phi : \mathrm{SE}(3) \times T^*Q \to T^*Q$. In the case when $(\mu, a) \in \mathfrak{se}^*(3)$ is a regular value of $\mathbf{J}_Q$, we have the reduced Hamiltonian $h_{(\mu,a)}(\Pi, \Gamma, \alpha, l) : \mathcal{O}_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}(\subset \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ given by $h_{(\mu,a)}(\Pi, \Gamma, \alpha, l) = \pi_{(\mu,a)}(H(A, c, \Pi, \Gamma, \alpha, l)) = H(A, c, \Pi, \Gamma, \alpha, l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$. From the heavy top Poisson bracket on $\mathfrak{se}^*(3)$ and the Poisson bracket on $T^*\mathbb{R}$, we can write the Poisson bracket on $\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}$, of $F, K : \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, (see Krishnaprasad and Marsden [12]) in the form

$$\begin{aligned}\{F, K\}_-(\Pi, \Gamma, \alpha, l) = &-\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) \\ &- \Gamma \cdot (\nabla_\Pi F \times \nabla_\Gamma K - \nabla_\Pi K \times \nabla_\Gamma F) + \{F, K\}_{\mathbb{R}}(\alpha, l). \quad (9)\end{aligned}$$

In particular, for $F_{(\mu,a)}, K_{(\mu,a)} : \mathcal{O}_{(\mu,a)} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we have the symplectic form $\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}(X_{F_{(\mu,a)}}, X_{K_{(\mu,a)}}) = \{F_{(\mu,a)}, K_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$ and the reduced Hamiltonian $h_{(\mu,a)}(\Pi, \Gamma, \alpha, l) : \mathcal{O}_{(\mu,a)} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The respective Hamiltonian vector field is $X_{h_{(\mu,a)}}(K_{(\mu,a)}) = \{K_{(\mu,a)}, h_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$, and hence

$$\begin{aligned}\frac{\mathrm{d}\Pi}{\mathrm{d}t} =& X_{h_{(\mu,a)}}(\Pi)(\Pi, \Gamma, \alpha, l) = \{\Pi, h_{(\mu,a)}\}_-(\Pi, \Gamma, \alpha, l) \\ =& -\Pi \cdot (\nabla_\Pi \Pi \times \nabla_\Pi h_{(\mu,a)}) - \Gamma \cdot (\nabla_\Pi \Pi \times \nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)} \times \nabla_\Gamma \Pi) \\ &+ (\frac{\partial \Pi}{\partial \alpha}\frac{\partial h_{(\mu,a)}}{\partial l} - \frac{\partial h_{(\mu,a)}}{\partial \alpha}\frac{\partial \Pi}{\partial l}) = \Pi \times \Omega - mgh\chi \times \Gamma \\ =& \Pi \times \Omega + mgh\Gamma \times \chi\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} =& X_{h_{(\mu,a)}}(\Gamma)(\Pi, \Gamma, \alpha, l) = \{\Gamma, h_{(\mu,a)}\}_-(\Pi, \Gamma, \alpha, l) \\ =& -\Pi \cdot (\nabla_\Pi \Gamma \times \nabla_\Pi h_{(\mu,a)}) - \Gamma \cdot (\nabla_\Pi \Gamma \times \nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)} \times \nabla_\Gamma \Gamma) \\ &+ (\frac{\partial \Gamma}{\partial \alpha}\frac{\partial h_{(\mu,a)}}{\partial l} - \frac{\partial h_{(\mu,a)}}{\partial \alpha}\frac{\partial \Gamma}{\partial l}) = \nabla_\Gamma \Gamma \cdot (\Gamma \times \nabla_\Pi h_{(\mu,a)}) = \Gamma \times \Omega\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}t} =& X_{h_{(\mu,a)}}(\alpha)(\Pi, \Gamma, \alpha, l) = \{\alpha, h_{(\mu,a)}\}_-(\Pi, \Gamma, \alpha, l) \\ =& -\Pi \cdot (\nabla_\Pi \alpha \times \nabla_\Pi h_{(\mu,a)}) - \Gamma \cdot (\nabla_\Pi \alpha \times \nabla_\Gamma h_{(\mu,a)} - \nabla_\Pi h_{(\mu,a)} \times \nabla_\Gamma \alpha) \\ &+ (\frac{\partial \alpha}{\partial \alpha}\frac{\partial h_{(\mu,a)}}{\partial l} - \frac{\partial h_{(\mu,a)}}{\partial \alpha}\frac{\partial \alpha}{\partial l}) = -\frac{(\Pi_3 - l)}{\bar{I}_3} + \frac{l}{J_3}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}l}{\mathrm{d}t} =&X_{h_{(\mu,a)}}(l)(\Pi,\Gamma,\alpha,l)=\{l,h_{(\mu,a)}\}_{-}(\Pi,\Gamma,\alpha,l)\\ =&-\Pi\cdot(\nabla_{\Pi}l\times\nabla_{\Pi}h_{(\mu,a)})-\Gamma\cdot(\nabla_{\Pi}l\times\nabla_{\Gamma}h_{(\mu,a)}-\nabla_{\Pi}h_{(\mu,a)}\times\nabla_{\Gamma}l)\\ &+(\frac{\partial l}{\partial\alpha}\frac{\partial h_{(\mu,a)}}{\partial l}-\frac{\partial h_{(\mu,a)}}{\partial\alpha}\frac{\partial l}{\partial l})=0\end{aligned}$$

since $\nabla_{\Pi}\Pi=\nabla_{\Gamma}\Gamma=1,\ \nabla_{\Gamma}\Pi=\nabla_{\Pi}\Gamma=\nabla_{\Pi}\alpha=\nabla_{\Pi}l=\nabla_{\Gamma}\alpha=\nabla_{\Gamma}l=0,\ \nabla_{\Pi}h_{(\mu,a)}=\Omega$, and $\frac{\partial\Pi}{\partial\alpha}=\frac{\partial\Gamma}{\partial\alpha}=\frac{\partial l}{\partial\alpha}=\frac{\partial h_{(\mu,a)}}{\partial\alpha}=0$. If we consider the rigid spacecraft-rotor system with a control torque $u:T^*Q\rightarrow T^*Q$ acting on the rotor, such that $u\in\mathbf{J}_Q^{-1}(\mu,a)$ is invariant under the left SE(3)-action, then its reduced control torque $u_{(\mu,a)}:\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}\rightarrow\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$ is given by $u_{(\mu,a)}(\Pi,\Gamma,\alpha,l)=\pi_{(\mu,a)}(u(A,c,\Pi,\Gamma,\alpha,l))=u(A,c,\Pi,\Gamma,\alpha,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$, where $\pi_{(\mu,a)}:\mathbf{J}_Q^{-1}(\mu,a)\rightarrow\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$. Thus, in the case of non-coincident centers of buoyancy and gravity, the equations of motion for reduced spacecraft-rotor system with the control torque u acting on the rotor are given by

$$\begin{aligned}\frac{\mathrm{d}\Pi}{\mathrm{d}t}&=\Pi\times\Omega+mgh\Gamma\times\chi\\ \frac{\mathrm{d}\Gamma}{\mathrm{d}t}&=\Gamma\times\Omega\\ \frac{\mathrm{d}\alpha}{\mathrm{d}t}&=-\frac{(\Pi_3-l)}{\bar{I}_3}+\frac{l}{J_3}\\ \frac{\mathrm{d}l}{\mathrm{d}t}&=\operatorname{v}(u_{(\mu,a)})X_{h_{(\mu,a)}}\end{aligned}\tag{10}$$

where $\operatorname{v}(u_{(\mu,a)})X_{h_{(\mu,a)}}\in T(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R})$. Note that $\operatorname{v}(u_{(\mu,a)})X_{h_{(\mu,a)}}$ is the vertical lift of the vector field $X_{h_{(\mu,a)}}$ under the action of $u_{(\mu,a)}$ along fibers, that is,

$$\begin{aligned}&\operatorname{v}(u_{(\mu,a)})X_{h_{(\mu,a)}}(\Pi,\Gamma,\alpha,l)\\ &\qquad=\operatorname{v}((Tu_{(\mu,a)}X_{h_{(\mu,a)}})(u_{(\mu,a)}(\Pi,\Gamma,\alpha,l)),(\Pi,\Gamma,\alpha,l))\\ &\qquad\qquad=(Tu_{(\mu,a)}X_{h_{(\mu,a)}})^{\operatorname{v}}_{\sigma}(\Pi,\Gamma,\alpha,l)\end{aligned}$$

see Marsden *et al* [22] and Wang [35]. To sum up the above discussion, we state the following theorem.

Theorem 3. *In the case of non-coincident centers of buoyancy and gravity, the spacecraft-rotor system with the control torque* u *acting on the rotor, that is, the five-tuple* $(T^*Q,\mathrm{SE}(3),\omega_Q,H,u)$, *where* $Q=\mathrm{SE}(3)\times\mathrm{S}^1$, *is a regular point reducible RCH system. For a point* $(\mu,a)\in\mathfrak{se}^*(3)$, *which is a regular value of the momentum map* $\mathbf{J}_Q:\mathrm{SE}(3)\times\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}\rightarrow\mathfrak{se}^*(3)$, *the regular point* *reduced system is the four-tuple* $(\mathcal{O}_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}, \tilde{\omega}^{-}_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}, h_{(\mu,a)}, u_{(\mu,a)})$, *where* $\mathcal{O}_{(\mu,a)} \subset \mathfrak{se}^*(3)$ *is the coadjoint orbit,* $\tilde{\omega}^{-}_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$ *is the orbit symplectic form on* $\mathcal{O}_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}$*,* $h_{(\mu,a)}(\Pi,\Gamma,\alpha,l) = \pi_{(\mu,a)}(H(A,c,\Pi,\Gamma,\alpha,l)) = H(A,c,\Pi,\Gamma,\alpha,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$*,* $u_{(\mu,a)}(\Pi,\Gamma,\alpha,l) = \pi_{(\mu,a)}(u(A,c,\Pi,\Gamma,\alpha,l)) = u(A,c,\Pi,\Gamma,\alpha,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$*, and its equations of motion are given by* (10).

Remark 4. When the rigid spacecraft does not carry any internal rotor, the configuration space is $Q = G = \mathrm{SE}(3)$, the motion of the rigid spacecraft is just the rotation motion with drift of a rigid body, the above symmetric reduction of spacecraft-rotor system is just the Marsden-Weinstein reduction of a heavy top at a regular value of momentum map, and the equation of motion (10) of the reduced spacecraft-rotor system becomes the equation of motion of the reduced heavy top on the coadjoint orbit of Lie group $\mathrm{SE}(3)$. See Marsden and Ratiu [21].

4. Hamilton-Jacobi Equation of the Rigid Spacecraft with a Rotor

It is well-known that Hamilton-Jacobi theory provides a characterization of the generating functions of certain time-dependent canonical transformations. The solutions of such systems are extremely easy to find by reduction to the equilibrium, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [21]. In general, we know that it is not easy to find the solutions of Hamilton's equation. But, if we can get a solution of Hamilton-Jacobi equation of the Hamiltonian system, by using the relationship between the Hamilton equations and the Hamilton-Jacobi equation, it is easy to give a special solution of the Hamilton equations. Thus, it is very important to find explicitly the Hamilton-Jacobi equation of a Hamiltonian system. Recently, the present author [35] proved the following Hamilton-Jacobi theorem for regular point reducible RCH system on the generalization of a Lie group.

Theorem 5. *Let us have a regular point reducible* RCH *system* $(T^*Q, G, \omega_Q, H, F, W)$ *on* $Q = G \times V$*, where* G *is a Lie group,* V *is a* k*-dimensional vector space,* $\gamma : Q \to T^*Q$ *is an one-form on* Q*,* $\gamma^* : T^*T^*Q \to T^*Q$ *is symplectic with an induced symplectic form* $\pi_Q^*\omega_Q$ *on* T^*T^*Q*,* $\pi_Q^* : T^*Q \to T^*T^*Q$*,* $\pi_Q : T^*Q \to Q$*. Let us assume that* γ *is closed with respect to* $T\pi_Q : TT^*Q \to TQ$*, and* $\tilde{X}^\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma$*, where* $\tilde{X} = X_{(T^*Q,G,\omega_Q,H,F,u)}$ *is the dynamical vector field of the regular point reducible* RCH *system* $(T^*Q, G, \omega_Q, H, F, W)$ *with a control law* u*. Let us assume also that* $\mu \in \mathfrak{g}^*$ *is the regular reducible point of the* RCH *system, and that the image* $\mathrm{im}(\gamma) \subset \mathbf{J}_Q^{-1}(\mu)$*, is* G_μ*-invariant,* $\bar{\gamma} = \pi_\mu(\gamma) : Q \to$

$\mathcal{O}_\mu \times V \times V^*$, *where* G_μ *is the isotropy subgroup of coadjoint action at* μ, *and* $\pi_\mu : \mathbf{J}_Q^{-1}(\mu) \to \mathcal{O}_\mu \times V \times V^*$. *Then the following two assertions are equivalent*

i) $\tilde{X}^\gamma$ *and* $\tilde{X}_\mu$ *are* $\bar{\gamma}$*-related, where* $\tilde{X}_\mu = X_{(\mathcal{O}_\mu \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_\mu \times V \times V^*}, h_\mu, f_\mu, u_\mu)}$ *is the dynamical vector field of regular point reduced* RCH *system* $(\mathcal{O}_\mu \times V \times V^*, \tilde{\omega}^-_{\mathcal{O}_\mu \times V \times V^*}, h_\mu, f_\mu, u_\mu)$

ii) $X_{h_\mu \cdot \bar{\gamma}} + \upsilon(f_\mu \cdot \bar{\gamma}) + \upsilon(u_\mu \cdot \bar{\gamma}) = 0$, *or* $X_{h_\mu \cdot \bar{\gamma}} + \upsilon(f_\mu \cdot \bar{\gamma}) + \upsilon(u_\mu \cdot \bar{\gamma}) = \tilde{X}^\gamma$.

Here γ is a solution of the Hamilton-Jacobi equation $X_{H \cdot \gamma} + \upsilon(F \cdot \gamma) + \upsilon(u \cdot \gamma) = 0$, if and only if $\bar{\gamma}$ is a solution of the Hamilton-Jacobi equation $X_{h_\mu \cdot \bar{\gamma}} + \upsilon(f_\mu \cdot \bar{\gamma}) + \upsilon(u_\mu \cdot \bar{\gamma}) = 0$. For convenience, $\upsilon(f_\mu \cdot \bar{\gamma}) X_{h_\mu \cdot \bar{\gamma}}$ and $\upsilon(u_\mu \cdot \bar{\gamma}) X_{h_\mu \cdot \bar{\gamma}}$ are written simply as $\upsilon(f_\mu \cdot \bar{\gamma})$ and $\upsilon(u_\mu \cdot \bar{\gamma})$.

As an application of the above result, we consider the rigid spacecraft with an internal rotor as a regular point reducible RCH system on the generalization of rotation group $\mathrm{SO}(3) \times \mathrm{S}^1$ and on the generalization of Euclidean group $\mathrm{SE}(3) \times \mathrm{S}^1$, respectively, and give the Hamilton-Jacobi equations of their reduced RCH systems on the symplectic leaves which show the effect on controls in Hamilton-Jacobi theory.

4.1. H-J Equation of Spacecraft-Rotor System with Coincident Centers

In the following we first derive the Hamilton-Jacobi equation for regular point reduced spacecraft-rotor system with coincident centers of buoyancy and gravity. From the expression for the Hamiltonian (1), we know that $H(A, \Pi, \alpha, l)$ is invariant under the left SO(3)-action. When $\mu \in \mathfrak{so}^*(3)$ is a regular value of $\mathbf{J}_Q$, the reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is given by $h_\mu(\Pi, \alpha, l) = \pi_\mu(H(A, \Pi, \alpha, l)) = H(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}$, and the reduced Hamiltonian vector field $X_{h_\mu}(K_\mu) = \{K_\mu, h_\mu\}_-|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}}$.

Assume that $\gamma : \mathrm{SO}(3) \times \mathrm{S}^1 \to T^*(\mathrm{SO}(3) \times \mathrm{S}^1)$ is an one-form on $\mathrm{SO}(3) \times \mathrm{S}^1$, the $\gamma^* : T^*T^*(\mathrm{SO}(3) \times \mathrm{S}^1) \to T^*(\mathrm{SO}(3) \times \mathrm{S}^1)$ is symplectic, γ is closed with respect to $T\pi_{\mathrm{SO}(3) \times \mathrm{S}^1} : TT^*(\mathrm{SO}(3) \times \mathrm{S}^1) \to T(\mathrm{SO}(3) \times \mathrm{S}^1)$, $\mathrm{im}(\gamma) \subset \mathbf{J}_Q^{-1}(\mu)$, is $\mathrm{SO}(3)_\mu$-invariant, and $\bar{\gamma} = \pi_\mu(\gamma) : \mathrm{SO}(3) \times \mathrm{S}^1 \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}$. Denote by $\bar{\gamma}(A, \alpha) = (\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4, \bar{\gamma}_5)(A, \alpha) \in \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R} (\subset \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R})$, and $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)(A, \alpha) \in \mathcal{O}_\mu$, then $h_\mu \cdot \bar{\gamma} : \mathrm{SO}(3) \times \mathrm{S}^1 \to \mathbb{R}$ is given by

$$h_\mu \cdot \bar{\gamma}(A, \alpha) = H \cdot \bar{\gamma}(A, \alpha)|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}} = \frac{1}{2}\left(\frac{\bar{\gamma}_1^2}{\bar{I}_1} + \frac{\bar{\gamma}_2^2}{\bar{I}_2} + \frac{(\bar{\gamma}_3 - \bar{\gamma}_5)^2}{\bar{I}_3} + \frac{\bar{\gamma}_5^2}{J_3}\right) \quad (11)$$

and the Hamiltonian vector field is

$$
\begin{aligned}
&X_{h_\mu\cdot\bar{\gamma}}(\Pi) = \{\Pi, h_\mu\cdot\bar{\gamma}(A,\alpha)\}_-|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}\\
&= -\Pi\cdot(\nabla_\Pi\Pi\times\nabla_\Pi(h_\mu\cdot\bar{\gamma})) + \{\Pi, h_\mu\cdot\bar{\gamma}(A,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}\\
&= -\nabla_\Pi\Pi\cdot(\nabla_\Pi(h_\mu\cdot\bar{\gamma})\times\Pi) + \left(\frac{\partial\Pi}{\partial\alpha}\frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial l} - \frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial\Pi}{\partial l}\right)\\
&= (\Pi_1,\Pi_2,\Pi_3)\times\left(\frac{\bar{\gamma}_1}{\bar{I}_1},\ \frac{\bar{\gamma}_2}{\bar{I}_2},\ \frac{(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3}\right)\\
&= \left(\frac{\bar{I}_2\Pi_2(\bar{\gamma}_3-\bar{\gamma}_5)-\bar{I}_3\Pi_3\bar{\gamma}_2}{\bar{I}_2\bar{I}_3}, \frac{\bar{I}_3\Pi_3\bar{\gamma}_1-\bar{I}_1\Pi_1(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3\bar{I}_1},\ \frac{\bar{I}_1\Pi_1\bar{\gamma}_2-\bar{I}_2\Pi_2\bar{\gamma}_1}{\bar{I}_1\bar{I}_2}\right)
\end{aligned}
$$

since $\nabla_\Pi\Pi=1$, and $\nabla_{\Pi_j}(h_\mu\cdot\bar{\gamma})=\bar{\gamma}_j/\bar{I}_j,\ j=1,2,\ \nabla_{\Pi_3}(h_\mu\cdot\bar{\gamma})=(\bar{\gamma}_3-\bar{\gamma}_5)/\bar{I}_3$, and $\frac{\partial\Pi}{\partial\alpha}=\frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial\alpha}=0$. Then we have also

$$
\begin{aligned}
X_{h_\mu\cdot\bar{\gamma}}(\alpha) &= \{\alpha, h_\mu\cdot\bar{\gamma}(A,\alpha)\}_-|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}\\
&= -\Pi\cdot(\nabla_\Pi\alpha\times\nabla_\Pi(h_\mu\cdot\bar{\gamma})) + \{\alpha, h_\mu\cdot\bar{\gamma}(A,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}\\
&= -\nabla_\Pi\alpha\cdot(\nabla_\Pi(h_\mu\cdot\bar{\gamma})\times\Pi) + (\frac{\partial\alpha}{\partial\alpha}\frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial l} - \frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial\alpha}{\partial l})\\
&= -\frac{(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3}+\frac{\bar{\gamma}_5}{J_3}
\end{aligned}
$$

$$
\begin{aligned}
X_{h_\mu\cdot\bar{\gamma}}(l) &= \{l, h_\mu\cdot\bar{\gamma}(A,\alpha)\}_-|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}\\
&= -\Pi\cdot(\nabla_\Pi l\times\nabla_\Pi(h_\mu\cdot\bar{\gamma})) + \{l, h_\mu\cdot\bar{\gamma}(A,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}\\
&= -\nabla_\Pi l\cdot(\nabla_\Pi(h_\mu\cdot\bar{\gamma})\times\Pi) + (\frac{\partial l}{\partial\alpha}\frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial l} - \frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial l}{\partial l}) = 0
\end{aligned}
$$

since $\nabla_\Pi\alpha=0,\ \nabla_\Pi l=0$, and $\frac{\partial l}{\partial\alpha}=\frac{\partial\alpha}{\partial l}=\frac{\partial(h_\mu\cdot\bar{\gamma})}{\partial\alpha}=0$. If we consider the rigid spacecraft-rotor system with a control torque $u: T^*Q\to T^*Q$ acting on the rotor, and $u\in\mathbf{J}_Q^{-1}(\mu)$ is invariant under the left SO(3)-action, then its reduced control torque $u_\mu:\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}\to\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}$ is given by $u_\mu(\Pi,\alpha,l)=\pi_\mu(u(A,\Pi,\alpha,l))=u(A,\Pi,\alpha,l)|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$, where $\pi_\mu:\mathbf{J}_Q^{-1}(\mu)\to\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}$. The dynamical vector field of the regular point reduced spacecraft-rotor system $(\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}},h_\mu,u_\mu)$ is given by

$$X_{(\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}},h_\mu,u_\mu)} = X_{h_\mu}+\upsilon(u_\mu)$$

where $\upsilon(u_\mu)=\upsilon(u_\mu)X_{h_\mu}\in T(\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R})$. Assume that

$$\upsilon(u_\mu\cdot\bar{\gamma})X_{h_\mu}(A,\alpha)=(U_1,U_2,U_3,U_4,U_5)(A,\alpha)\in T(\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}).$$

Thus, in the case of coincident centers of buoyancy and gravity, the Hamilton-Jacobi equations for the reduced spacecraft-rotor system with control torque u acting on the rotor are

$$\begin{aligned}
\bar{I}_2\Pi_2(\bar{\gamma}_3-\bar{\gamma}_5)-\bar{I}_3\Pi_3\bar{\gamma}_2+\bar{I}_2\bar{I}_3U_1=0\\
\bar{I}_3\Pi_3\bar{\gamma}_1-\bar{I}_1\Pi_1(\bar{\gamma}_3-\bar{\gamma}_5)+\bar{I}_3\bar{I}_1U_2=0\\
\bar{I}_1\Pi_1\bar{\gamma}_2-\bar{I}_2\Pi_2\bar{\gamma}_1+\bar{I}_1\bar{I}_2U_3=0\\
-J_3(\bar{\gamma}_3-\bar{\gamma}_5)+\bar{I}_3\bar{\gamma}_5+\bar{I}_3J_3U_4=0\\
U_5=0.
\end{aligned} \tag{12}$$

To sum up the above discussion, we state the following theorem.

Theorem 6. *In the case of coincident centers of buoyancy and gravity, for a point* $\mu\in\mathfrak{so}^*(3)$*, which is a regular value of the momentum map* $\mathbf{J}_Q:\mathrm{SO}(3)\times\mathfrak{so}^*(3)\times\mathbb{R}\times\mathbb{R}\to\mathfrak{so}^*(3)$*, the reduced system of spacecraft-rotor system with the control torque* u *acting on the rotor* $(T^*Q,\mathrm{SO}(3),\omega_Q,H,u)$*, where* $Q=\mathrm{SO}(3)\times\mathrm{S}^1$*, is the four-tuple* $(\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}},h_\mu,u_\mu)$*, in which* $\mathcal{O}_\mu\subset\mathfrak{so}^*(3)$ *is the coadjoint orbit,* $\tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$ *is orbit symplectic form on* $\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}$*,* $h_\mu(\Pi,\alpha,l)=H(A,\Pi,\alpha,l)|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$*, and* $u_\mu(\Pi,\alpha,l)=u(A,\Pi,\alpha,l)|_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}}$*. Assume that* $\gamma:\mathrm{SO}(3)\times\mathrm{S}^1\to T^*(\mathrm{SO}(3)\times\mathrm{S}^1)$ *is one-form on* $\mathrm{SO}(3)\times\mathrm{S}^1$*,* $\gamma^*:T^*T^*(\mathrm{SO}(3)\times\mathrm{S}^1)\to T^*(\mathrm{SO}(3)\times\mathrm{S}^1)$ *is symplectic,* γ *is closed with respect to* $T\pi_Q:TT^*Q\to TQ$*, and* $\mathrm{im}(\gamma)\subset\mathbf{J}_Q^{-1}(\mu)$*, is* $\mathrm{SO}(3)_\mu$*-invariant, where* $\mathrm{SO}(3)_\mu$ *is the isotropy subgroup of coadjoint* $\mathrm{SO}(3)$*-action at the point* $\mu\in\mathfrak{so}^*(3)$*, and* $\bar{\gamma}=\pi_\mu(\gamma):\mathrm{SO}(3)\times\mathrm{S}^1\to\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}$*. Then* $\bar{\gamma}$ *is a solution to either of the Hamilton-Jacobi equation of reduced spacecraft-rotor system given by* (12)*, or to the equation* $X_{h_\mu\cdot\bar{\gamma}}+v(u_\mu\cdot\bar{\gamma})=\tilde{X}^\gamma$*, if and only if* $\tilde{X}^\gamma$ *and* $\tilde{X}_\mu$ *are* $\bar{\gamma}$*-related, where* $\tilde{X}^\gamma=T\pi_Q\cdot\tilde{X}\cdot\gamma$*,* $\tilde{X}=X_{(T^*Q,\mathrm{SO}(3),\omega_Q,H,u)}$*, and* $\tilde{X}_\mu=X_{(\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_\mu\times\mathbb{R}\times\mathbb{R}},h_\mu,u_\mu)}$*.*

Remark 7. When the rigid spacecraft does not carry any internal rotor, the configuration space is $Q=G=\mathrm{SO}(3)$, the above Hamilton-Jacobi equation (12) of the reduced spacecraft-rotor system is just the Lie-Poisson Hamilton-Jacobi equation of the Marsden-Weinstein reduced Hamiltonian system $(\mathcal{O}_\mu,\omega^-_{\mathcal{O}_\mu},h_{\mathcal{O}_\mu})$ on the coadjoint orbit of Lie group $\mathrm{SO}(3)$ since the symplectic structure on the coadjoint orbit $\mathcal{O}_\mu$ is induced by the (-)-Lie-Poisson brackets on $\mathfrak{so}^*(3)$. See Marsden and Ratiu [21], Ge and Marsden [10], and Wang [34].

4.2. H-J Equation of Spacecraft-Rotor System with Non-Coincident Centers

In the following we shall give the Hamilton-Jacobi equation for regular point reduced spacecraft-rotor system with non-coincident centers of buoyancy and gravity. From the expression (2) of the Hamiltonian, we know that $H(A,c,\Pi,\Gamma,\alpha,l)$ is invariant under the left SE(3)-action. In the case when $(\mu,a)\in\mathfrak{se}^*(3)$ is a regular value of $\mathbf{J}_Q$, we have the reduced Hamiltonian $h_{(\mu,a)}(\Pi,\Gamma,\alpha,l):\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}(\subset\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R})\to\mathbb{R}$, which is given by $h_{(\mu,a)}(\Pi,\Gamma,\alpha,l)=\pi_{(\mu,a)}(H(A,c,\Pi,\Gamma,\alpha,l))=H(A,c,\Pi,\Gamma,\alpha,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$, and the reduced Hamiltonian vector field $X_{h_{(\mu,a)}}(K_{(\mu,a)})=\{K_{(\mu,a)},h_{(\mu,a)}\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$.

Assume that $\gamma:\mathrm{SE}(3)\times\mathrm{S}^1\to T^*(\mathrm{SE}(3)\times\mathrm{S}^1)$ is an one-form on $\mathrm{SE}(3)\times\mathrm{S}^1$, $\gamma^*:T^*T^*(\mathrm{SE}(3)\times\mathrm{S}^1)\to T^*(\mathrm{SE}(3)\times\mathrm{S}^1)$ is symplectic, γ is closed with respect to $T\pi_{\mathrm{SE}(3)\times\mathrm{S}^1}:TT^*(\mathrm{SE}(3)\times\mathrm{S}^1)\to T(\mathrm{SE}(3)\times\mathrm{S}^1)$, $\mathrm{im}(\gamma)\subset\mathbf{J}^{-1}((\mu,a))$, which is $\mathrm{SE}(3)_{(\mu,a)}$-invariant, and $\bar{\gamma}=\pi_{(\mu,a)}(\gamma):\mathrm{SE}(3)\times\mathrm{S}^1\to\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$. If we denote by $\bar{\gamma}(A,c,\alpha)=(\bar{\gamma}_1,\bar{\gamma}_2,\bar{\gamma}_3,\Gamma_1,\Gamma_2,\Gamma_3,\bar{\gamma}_4,\bar{\gamma}_5)(A,c,\alpha)\in\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}(\subset\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R})$, then $h_{(\mu,a)}\cdot\bar{\gamma}:\mathrm{SE}(3)\times\mathrm{S}^1\to\mathbb{R}$ is given by the formula

$$\begin{aligned} h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha) &= H\cdot\bar{\gamma}(A,c,\alpha)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}} \\ &= \frac{1}{2}\left(\frac{\bar{\gamma}_1^2}{\bar{I}_1}+\frac{\bar{\gamma}_2^2}{\bar{I}_2}+\frac{(\bar{\gamma}_3-\bar{\gamma}_5)^2}{\bar{I}_3}+\frac{\bar{\gamma}_5^2}{J_3}\right)+mgh\Gamma\cdot\chi \end{aligned} \tag{13}$$

and the Hamiltonian vector field is

$$\begin{aligned} X_{h_{(\mu,a)}\cdot\bar{\gamma}}(\Pi) &= \{\Pi,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}} \\ &= -\Pi\cdot(\nabla_\Pi\Pi\times\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma}))-\Gamma\cdot(\nabla_\Pi\Pi\times\nabla_\Gamma(h_{(\mu,a)}\cdot\bar{\gamma}) \\ &\quad -\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma})\times\nabla_\Gamma\Pi)+\{\Pi,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}} \\ &= -\nabla_\Pi\Pi\cdot(\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma})\times\Pi)-\nabla_\Pi\Pi\cdot(\nabla_\Gamma(h_{(\mu,a)}\cdot\bar{\gamma})\times\Gamma) \\ &\quad +(\frac{\partial\Pi}{\partial\alpha}\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial l}-\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial\Pi}{\partial l}) \\ &= (\Pi_1,\Pi_2,\Pi_3)\times(\frac{\bar{\gamma}_1}{\bar{I}_1},\frac{\bar{\gamma}_2}{\bar{I}_2},\frac{(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3})+mgh(\Gamma_1,\Gamma_2,\Gamma_3)\times(\chi_1,\chi_2,\chi_3) \\ &= (\frac{\bar{I}_2\Pi_2(\bar{\gamma}_3-\bar{\gamma}_5)-\bar{I}_3\Pi_3\bar{\gamma}_2}{\bar{I}_2\bar{I}_3}+mgh(\Gamma_2\chi_3-\Gamma_3\chi_2), \\ &\quad \frac{\bar{I}_3\Pi_3\bar{\gamma}_1-\bar{I}_1\Pi_1(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3\bar{I}_1}+mgh(\Gamma_3\chi_1-\Gamma_1\chi_3), \\ &\quad \frac{\bar{I}_1\Pi_1\bar{\gamma}_2-\bar{I}_2\Pi_2\bar{\gamma}_1}{\bar{I}_1\bar{I}_2}+mgh(\Gamma_1\chi_2-\Gamma_2\chi_1)) \end{aligned}$$

since $\nabla_\Pi \Pi=1,\ \nabla_\Gamma \Pi=0,\ \frac{\partial \Pi}{\partial \alpha}=\frac{\partial \Pi}{\partial l}=0,\ \Gamma=(\Gamma_1,\Gamma_2,\Gamma_3),\ \chi=(\chi_1,\chi_2,\chi_3)$, and $\nabla_{\Pi_j}(h_{(\mu,a)}\cdot\bar{\gamma})=\bar{\gamma}_j/\bar{I}_j,\ j=1,2,\ \nabla_{\Pi_3}(h_{(\mu,a)}\cdot\bar{\gamma})=(\bar{\gamma}_3-\bar{\gamma}_5)/\bar{I}_3$.
In a similar way we have

$$\begin{aligned}
&X_{h_{(\mu,a)}\cdot\bar{\gamma}}(\Gamma)=\{\Gamma,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}\\
&=-\Pi\cdot(\nabla_\Pi\Gamma\times\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma}))-\Gamma\cdot(\nabla_\Pi\Gamma\times\nabla_\Gamma(h_{(\mu,a)}\cdot\bar{\gamma})\\
&\quad-\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma})\times\nabla_\Gamma\Gamma)+\{\Gamma,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}\\
&=\nabla_\Gamma\Gamma\cdot(\Gamma\times\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma}))+\left(\frac{\partial\Gamma}{\partial\alpha}\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial l}-\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial\Gamma}{\partial l}\right)\\
&=(\Gamma_1,\Gamma_2,\Gamma_3)\times\left(\frac{\bar{\gamma}_1}{\bar{I}_1},\frac{\bar{\gamma}_2}{\bar{I}_2},\frac{(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3}\right)\\
&=\left(\frac{\bar{I}_2\Gamma_2(\bar{\gamma}_3-\bar{\gamma}_5)-\bar{I}_3\Gamma_3\bar{\gamma}_2}{\bar{I}_2\bar{I}_3},\ \ \frac{\bar{I}_3\Gamma_3\bar{\gamma}_1-\bar{I}_1\Gamma_1(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3\bar{I}_1},\ \ \frac{\bar{I}_1\Gamma_1\bar{\gamma}_2-\bar{I}_2\Gamma_2\bar{\gamma}_1}{\bar{I}_1\bar{I}_2}\right)
\end{aligned}$$

since $\nabla_\Gamma\Gamma=1,\ \nabla_\Pi\Gamma=0,\ \frac{\partial\Gamma}{\partial\alpha}=\frac{\partial\Gamma}{\partial l}=0.$
Finally

$$\begin{aligned}
&X_{h_{(\mu,a)}\cdot\bar{\gamma}}(\alpha)=\{\alpha,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}\\
&=-\Pi\cdot(\nabla_\Pi\alpha\times\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma}))-\Gamma\cdot(\nabla_\Pi\alpha\times\nabla_\Gamma(h_{(\mu,a)}\cdot\bar{\gamma})\\
&\quad-\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma})\times\nabla_\Gamma\alpha)+\{\alpha,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}\\
&=(\frac{\partial\alpha}{\partial\alpha}\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial l}-\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial\alpha}{\partial l})=-\frac{(\bar{\gamma}_3-\bar{\gamma}_5)}{\bar{I}_3}+\frac{\bar{\gamma}_5}{J_3}
\end{aligned}$$

and

$$\begin{aligned}
&X_{h_{(\mu,a)}\cdot\bar{\gamma}}(l)=\{l,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_-|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}\\
&=-\Pi\cdot(\nabla_\Pi l\times\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma}))-\Gamma\cdot(\nabla_\Pi l\times\nabla_\Gamma(h_{(\mu,a)}\cdot\bar{\gamma})\\
&\quad-\nabla_\Pi(h_{(\mu,a)}\cdot\bar{\gamma})\times\nabla_\Gamma l)+\{l,\ h_{(\mu,a)}\cdot\bar{\gamma}(A,c,\alpha)\}_{\mathbb{R}}|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}\\
&=(\frac{\partial l}{\partial\alpha}\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial l}-\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial\alpha}\frac{\partial l}{\partial l})=0
\end{aligned}$$

since $\nabla_\Pi\alpha=\nabla_\Gamma\alpha=0,\ \nabla_\Pi l=\nabla_\Gamma l=0,\ \frac{\partial\alpha}{\partial l}=\frac{\partial l}{\partial\alpha}=0$, and $\frac{\partial(h_{(\mu,a)}\cdot\bar{\gamma})}{\partial\alpha}=0$.
If we consider the spacecraft-rotor system with a control torque $u:T^*Q\to T^*Q$ acting on the rotor, and $u\in\mathbf{J}_Q^{-1}((\mu,a))$ is invariant under the left SE(3)-action, then its reduced control torque $u_{(\mu,a)}:\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}\to\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$ is given by $u_{(\mu,a)}(\Pi,\Gamma,\alpha,l)=\pi_{(\mu,a)}(u(A,c,\Pi,\Gamma,\alpha,l))=u(A,c,\Pi,\Gamma,\alpha,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$, where $\pi_{(\mu,a)}:\mathbf{J}_Q^{-1}((\mu,a))\to\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$. The dynamical vector field of regular

point reduced spacecraft-rotor system $(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}},h_{(\mu,a)},u_{(\mu,a)})$ is given by

$$X_{(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}},h_{(\mu,a)},u_{(\mu,a)})}=X_{h_{(\mu,a)}}+\upsilon(u_{(\mu,a)})$$

where $\upsilon(u_{(\mu,a)})=\upsilon(u_{(\mu,a)})X_{h_{(\mu,a)}}\in T(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R})$. Assume that

$$\upsilon(u_{(\mu,a)}\cdot\bar{\gamma})X_{h_{(\mu,a)}}(A,c,\alpha)=(U_1,U_2,U_3,U_4,U_5,U_6,U_7,U_8)(A,c,\alpha).$$

Thus, in the case of non-coincident centers of buoyancy and gravity, the Hamilton-Jacobi equations for reduced spacecraft-rotor system with the control torque u acting on the rotor are given by the following system of equations

$$\begin{aligned}
\bar{I}_2\Pi_2(\bar{\gamma}_3-\bar{\gamma}_5)-\bar{I}_3\Pi_3\bar{\gamma}_2+mgh\bar{I}_2\bar{I}_3(\Gamma_2\chi_3-\Gamma_3\chi_2)+\bar{I}_2\bar{I}_3U_1&=0\\
\bar{I}_3\Pi_3\bar{\gamma}_1-\bar{I}_1\Pi_1(\bar{\gamma}_3-\bar{\gamma}_5)+mgh\bar{I}_3\bar{I}_1(\Gamma_3\chi_1-\Gamma_1\chi_3)+\bar{I}_3\bar{I}_1U_2&=0\\
\bar{I}_1\Pi_1\bar{\gamma}_2-\bar{I}_2\Pi_2\bar{\gamma}_1+mgh\bar{I}_1\bar{I}_2(\Gamma_1\chi_2-\Gamma_2\chi_1)+\bar{I}_1\bar{I}_2U_3&=0\\
\bar{I}_2\Gamma_2(\bar{\gamma}_3-\bar{\gamma}_5)-\bar{I}_3\Gamma_3\bar{\gamma}_2+\bar{I}_2\bar{I}_3U_4&=0\\
\bar{I}_3\Gamma_3\bar{\gamma}_1-\bar{I}_1\Gamma_1(\bar{\gamma}_3-\bar{\gamma}_5)+\bar{I}_3\bar{I}_1U_5&=0\\
\bar{I}_1\Gamma_1\bar{\gamma}_2-\bar{I}_2\Gamma_2\bar{\gamma}_1+\bar{I}_1\bar{I}_2U_6&=0\\
-J_3(\bar{\gamma}_3-\bar{\gamma}_5)+\bar{I}_3\bar{\gamma}_5+\bar{I}_3J_3U_7&=0\\
U_8&=0.
\end{aligned}\tag{14}$$

To sum up the above discussion, we formulate the following proposition theorem.

Theorem 8. *In the case of non-coincident centers of buoyancy and gravity and when the point* $(\mu,a)\in\mathfrak{se}^*(3)$*, is as regular value of the momentum map* $\mathbf{J}_Q:\mathrm{SE}(3)\times\mathfrak{se}^*(3)\times\mathbb{R}\times\mathbb{R}\to\mathfrak{se}^*(3)$*, one obtains a regular point reduced system of spacecraft-rotor system with the control torque* u *acting on the rotor* $(T^*Q,\mathrm{SE}(3),\omega_Q,H,u)$*, where* $Q=\mathrm{SE}(3)\times\mathrm{S}^1$*, presented by the four-tuple* $(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R},\ \tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}},h_{(\mu,a)},u_{(\mu,a)})$*, in which* $\mathcal{O}_{(\mu,a)}\subset\mathfrak{se}^*(3)$ *is the coadjoint orbit,* $\tilde{\omega}^-_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$ *is orbit symplectic form on* $\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$*,* $h_{(\mu,a)}(\Pi,\Gamma,\theta,l)=H(A,v,\Pi,\Gamma,\theta,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$*, and* $u_{(\mu,a)}(\Pi,\Gamma,\theta,l)=u(A,v,\Pi,\Gamma,\theta,l)|_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}}$*. Assume that* $\gamma:\mathrm{SE}(3)\times\mathrm{S}^1\to T^*(\mathrm{SE}(3)\times\mathrm{S}^1)$ *is an one-form on* $\mathrm{SE}(3)\times\mathrm{S}^1$*, and* $\gamma^*:T^*T^*(\mathrm{SE}(3)\times\mathrm{S}^1)\to T^*(\mathrm{SE}(3)\times\mathrm{S}^1)$ *is symplectic, and* γ *is closed with respect to* $T\pi_Q:TT^*Q\to TQ$*, and* $\mathrm{im}(\gamma)\subset\mathbf{J}^{-1}(\mu,a)$*, and it is* $\mathrm{SE}(3)_{(\mu,a)}$*-invariant, where* $\mathrm{SE}(3)_{(\mu,a)}$ *is the isotropy subgroup of coadjoint* $\mathrm{SE}(3)$*-action at the point* $(\mu,a)\in\mathfrak{se}^*(3)$*, and* $\bar{\gamma}=\pi_{(\mu,a)}(\gamma):\mathrm{SE}(3)\times\mathrm{S}^1\to\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}$*.*

Then $\bar{\gamma}$ is a solution to either the Hamilton-Jacobi equation of reduced spacecraft-rotor system given by (14)*, or to the equation* $X_{h_{\mathcal{O}_{(\mu,a)}}\cdot\bar{\gamma}} + \upsilon(u_{(\mu,a)}\cdot\bar{\gamma}) = \tilde{X}^{\gamma}$*, if and only if* $\tilde{X}^{\gamma}$ *and* $\tilde{X}_{(\mu,a)}$ *are* $\bar{\gamma}$*-related, where* $\tilde{X}^{\gamma} = T\pi_Q \cdot \tilde{X}\cdot\gamma$*,* $\tilde{X} = X_{(T^*Q,SE(3),\omega_Q,H,u)}$*, and* $\tilde{X}_{(\mu,a)} = X_{(\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R},\tilde{\omega}^{-}_{\mathcal{O}_{(\mu,a)}\times\mathbb{R}\times\mathbb{R}},h_{(\mu,a)},u_{(\mu,a)})}$*.*

Remark 9. When the rigid spacecraft does not carry any internal rotor, the configuration space is $Q = G = \mathrm{SE}(3)$, the above Hamilton-Jacobi equation (14) of reduced spacecraft-rotor system is just the Lie-Poisson Hamilton-Jacobi equation of the Marsden-Weinstein reduced Hamiltonian system $(\mathcal{O}_{(\mu,a)}, \omega^{-}_{\mathcal{O}_{(\mu,a)}}, h_{\mathcal{O}_{(\mu,a)}})$ on the coadjoint orbit of Lie group $\mathrm{SE}(3)$, and the symplectic structure on the coadjoint orbit $\mathcal{O}_{(\mu,a)}$ is induced by the (-)-Lie-Poisson brackets on $\mathfrak{se}^*(3)$.

5. Conclusions

In this paper, as an application of the symplectic reduction and Hamilton-Jacobi theory of regular controlled Hamiltonian systems with symmetry, in both cases of coincident and non-coincident centers of the buoyancy and gravity, we have presented explicitly the equations of the motion and the Hamilton-Jacobi equations of the reduced spacecraft-rotor systems on the symplectic leaves which show the effect of controls in regular symplectic reduction and Hamilton-Jacobi theory. We have to note also that in [24–26], the authors study the dynamics of a rigid spacecraft under the influence of gravity torques and solve the dynamical equations in a first-order form with a special coefficient matrix.

In the future, we hope to study the stabilization of rigid spacecraft with an internal rotor and to describe the action of controls of the system. On the other hand, if we define a controlled Hamiltonian system on the cotangent bundle T^*Q by using the Poisson structure, see Wang and Zhang in [36] and Ratiu and Wang in [30], and the symplectic reduction for regular controlled Hamiltonian system cannot be used, what and how we could do? This is a problem worthy to be considered in detail. In addition, we have to mention also that there have been a lot of beautiful results concerning the reduction theory of Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. Thus, it is an important topic to study the application of reduction theory of controlled Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. These will be our goals in the future research.

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