



A NOTE ON POISSON LIE ALGEBROIDS

LIVIU POPESCU

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Abstract. In this paper we study some properties of a Lie algebroid and its prolongation over the vector bundle projection of the dual bundle. We generalize some results on Poisson manifolds to the level of a Lie algebroid. The notions of canonical Poisson bivector and horizontal lift are studied and their compatibility conditions are pointed out.

1. Introduction

The Lie algebroid [10] is a generalization of both concepts of Lie algebra and integrable distribution, being a vector bundle (E, π, M) with a Lie bracket on his space of sections with properties very similar to those of a tangent bundle. The Poisson manifolds are the smooth manifolds equipped with a Poisson bracket on their ring of functions. I have to remark that the cotangent bundle of a Poisson manifold has the natural structure of a Lie algebroid [13]. In the last years diverse aspects of these subjects have been studied in a lot of papers (see for instance [13], [14], [12], [1] and [7]). In the present paper we study some geometrical structures on the prolongation of a Lie algebroid to its dual bundle and investigate some aspects of the Lie algebroid geometry endowed with a Poisson structure. In this way we generalize some results on Poisson manifolds.

The paper is organized as follows. In the Section 2 we recall the Cartan calculus and the Schouten-Nijenhuis bracket at the level of a Lie algebroid and present the Poisson structures on the Lie algebroid. The Section 3 deals with the prolongation of a Lie algebroid [5], [8] to its dual bundle and continue the investigation starting in [6]. We study the properties of the canonical Poisson bivector and introduce the notion of horizontal lift. Finally, the compatibility conditions of these bivectors are given. We remark that in the particular case of the standard Lie algebroid $(E = TM, \sigma = \text{Id})$ some results of Mitric [12] are obtained.

2. Preliminaries on Lie Algebroids

Let M be a differentiable, n -dimensional manifold and (TM, π_M, M) its tangent bundle. A Lie algebroid over the manifold M is the triple $(E, [\cdot, \cdot], \sigma)$ where $\pi : E \rightarrow M$ is a vector bundle of rank m over M , whose $C^\infty(M)$ -module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot, \cdot]$ and $\sigma : E \rightarrow TM$ is a vector bundle map (called *the anchor*) which induces a Lie algebra homomorphism (also denoted σ) from $\Gamma(E)$ to $\chi(M)$, satisfying the Leibnitz rule

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2$$

for every $f \in C^\infty(M)$ and $s_1, s_2 \in \Gamma(E)$. Therefore, we have

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.$$

If $\omega \in \wedge^k(E^*)$ then the *exterior derivative* $d^E\omega \in \wedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1, \dots, s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \sigma(s_i)\omega(s_1, \dots, \hat{s}_i, \dots, s_{k+1}) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([s_i, s_j], s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{k+1}). \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it follows that $(d^E)^2 = 0$. Also, for $\xi \in \Gamma(E)$ one can define the *Lie derivative* with respect to ξ by

$$\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi$$

where i_ξ is the contraction with ξ .

If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U) \rightarrow U$ generates the local coordinates (x^i, y^α) on E . The local functions $\sigma_\alpha^i(x)$, $L_{\alpha\beta}^\gamma(x)$ on M defined by

$$\sigma(s_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L_{\alpha\beta}^\gamma s_\gamma, \quad i = 1, \dots, n, \quad \alpha, \beta, \gamma = 1, \dots, m$$

are called the *structure functions* of the Lie algebroid and satisfy the so called *structure equations* on the Lie algebroid

$$\sigma_\alpha^j \frac{\partial \sigma_\beta^i}{\partial x^j} - \sigma_\beta^j \frac{\partial \sigma_\alpha^i}{\partial x^j} = \sigma_\gamma^i L_{\alpha\beta}^\gamma, \quad \sum_{(\alpha, \beta, \gamma)} \left(\sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\delta}{\partial x^i} + L_{\alpha\eta}^\delta L_{\beta\gamma}^\eta \right) = 0. \quad (1)$$

Locally, if $f \in C^\infty(M)$ then $d^E f = \frac{\partial f}{\partial x^i} \sigma_\alpha^i s^\alpha$, where $\{s^\alpha\}$ is the dual basis of $\{\sigma_\alpha\}$ and, if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d^E \theta = \left(\sigma_\alpha^i \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2} \theta_\gamma L_{\alpha\beta}^\gamma \right) s^\alpha \wedge s^\beta.$$

Particularly, we get

$$d^E x^i = \sigma_\alpha^i s^\alpha, \quad d^E s^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha s^\beta \wedge s^\gamma.$$

The Schouten-Nijenhuis bracket is given by [13]

$$\begin{aligned} & [X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q] \\ &= (-1)^{p+1} \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \\ & \quad \wedge X_p \wedge Y_1 \wedge \dots \wedge \hat{Y}_j \wedge \dots \wedge Y_q \end{aligned}$$

where $X_i, Y_j \in \Gamma(E)$ and a hat means the absence of a factor.

2.1. Lie Algebroids with Poisson Structure

Let us consider the bivector on E (i.e., contravariant, skew-symmetric, 2-section) $W \in \Gamma(\wedge^2 E)$ given by

$$W = \frac{1}{2} w^{\alpha\beta}(x) s_\alpha \wedge s_\beta. \quad (2)$$

Definition 1. *The bivector W is a Poisson bivector on E if and only if we have the relation $[W, W] = 0$, where $[,]$ is Schouten-Nijenhuis bracket.*

Proposition 2. *The relation $[W, W] = 0$ implies locally that*

$$\sum_{(\alpha, \varepsilon, \delta)} (w^{\alpha\beta} \sigma_\beta^i \frac{\partial w^{\varepsilon\delta}}{\partial x^i} + w^{\alpha\beta} w^{\gamma\delta} L_{\beta\gamma}^\varepsilon) = 0. \quad (3)$$

If W is a Poisson bivector then the pair (E, W) is called a *Lie algebroid with Poisson structure*. The Poisson bracket on E is given by

$$\{f_1, f_2\} = W(d^E f_1, d^E f_2), \quad f_1, f_2 \in C^\infty(E).$$

We have the bundle map $\pi^\# : E^* \rightarrow E$ defined by

$$\pi^\# \rho = i_\rho W, \quad \rho \in \Gamma(E^*).$$

Let us consider the bracket

$$[\rho, \theta]_\pi = \mathcal{L}_{\pi^\# \rho} \theta - \mathcal{L}_{\pi^\# \theta} \rho - d^E(W(\rho, \theta))$$

where \mathcal{L} is the Lie derivative and $\rho, \theta \in \Gamma(E^*)$. With respect to this bracket and the usual Lie bracket on vector fields, the map $\tilde{\sigma} : E^* \rightarrow TM$ given by

$$\tilde{\sigma} = \sigma \circ \pi^\#$$

is a Lie algebra homomorphism

$$\tilde{\sigma}[\rho, \theta]_\pi = [\tilde{\sigma}\rho, \tilde{\sigma}\theta].$$

The bracket $[\cdot, \cdot]_\pi$ satisfies also the Leibnitz rule

$$[\rho, f\theta]_\pi = f[\rho, \theta]_\pi + \tilde{\sigma}(\rho)(f)\theta$$

and it results that $(E^*, [\cdot, \cdot]_\pi, \tilde{\sigma})$ is a Lie algebroid [14].

Next, we can define the contravariant exterior differential $d^\pi : \bigwedge^k(E^*) \rightarrow \bigwedge^{k+1}(E^*)$ by

$$\begin{aligned} d^\pi \omega(s_1, \dots, s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \tilde{\sigma}(s_i) \omega(s_1, \dots, \hat{s}_i, \dots, s_{k+1}) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([s_i, s_j]_\pi, s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{k+1}). \end{aligned}$$

In fact, is obtained the cohomology of the Lie algebroid E^* with the anchor $\tilde{\sigma}$ and the bracket $[\cdot, \cdot]_\pi$ which generalize the Poisson cohomology of Lichnerowicz for Poisson manifolds [9].

3. The Prolongation of a Lie Algebroid to Its Dual Bundle

Let $\tau : E^* \rightarrow M$ be the dual of $\pi : E \rightarrow M$ and $(E, [\cdot, \cdot], \sigma)$ a Lie algebroid structure over M . One can construct a Lie algebroid structure over E^* , by taking the prolongation of $(E, [\cdot, \cdot], \sigma)$ over $\tau : E^* \rightarrow M$ (see [5], [8], [11] and [6]). This structure is given by the following objects:

- The associated vector bundle is (TE^*, τ_1, E^*) where $TE^* = \cup_{u^* \in E^*} \mathcal{T}_{u^*} E^*$ with

$$\mathcal{T}_{u^*} E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*} E^* \mid \sigma(u_x) = T_{u^*} \tau(v_{u^*}), \tau(u^*) = x \in M\}$$

and the projection $\tau_1 : TE^* \rightarrow E^*$, $\tau_1(u_x, v_{u^*}) = u^*$.

- The Lie algebra structure $[\cdot, \cdot]$ on $\Gamma(\mathcal{T}E^*)$ is defined in the following way: if $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$ are such that $\rho_i(u^*) = (X_i(\tau(u^*)), U_i(u^*))$ where $X_i \in \Gamma(E), U_i \in \chi(E^*)$ and $\sigma(X_i(\tau(u^*))) = T_{u^*}\tau(U_i(u^*))$, $i = 1, 2$, then

$$[\rho_1, \rho_2](u^*) = ([X_1, X_2](\tau(u^*)), [U_1, U_2](u^*)).$$

- The anchor is the projection $\sigma^1 : \mathcal{T}E^* \rightarrow TE^*$, $\sigma^1(u, v) = v$.

Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \rightarrow E$, $\mathcal{T}\tau(u, v) = u$ then $(V\mathcal{T}E^*, \tau_1|_{V\mathcal{T}E^*}, E^*)$ with $V\mathcal{T}E^* := \ker \mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$, called the *vertical subbundle*. If (x^i, μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \rightarrow M$ then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$ where

$$\mathcal{X}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma_\alpha^i \frac{\partial}{\partial x^i} \Big|_{u^*} \right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha} \Big|_{u^*} \right). \quad (4)$$

The Lie brackets on the elements of this basis are:

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L_{\alpha\beta}^\gamma \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{P}^\alpha] = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0 \quad (5)$$

and

$$\sigma^1(\mathcal{X}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha}$$

$$d^E x^i = \sigma_\alpha^i \mathcal{X}^\alpha, \quad d^E \mu_\alpha = \mathcal{P}_\alpha, \quad d^E \mathcal{X}^\gamma = -\frac{1}{2} L_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \quad d^E \mathcal{P}_\alpha = 0$$

where $\{\mathcal{X}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$. Also, if $\rho = \rho^\alpha \mathcal{X}_\alpha + \zeta_\alpha \mathcal{P}^\alpha$ is a section of $\mathcal{T}E^*$, then

$$\sigma^1(\rho) = \sigma_\alpha^i \rho^\alpha \frac{\partial}{\partial x^i} + \zeta_\alpha \frac{\partial}{\partial \mu_\alpha}.$$

If $u^* \in E^*$ and $(u_x, v_{u^*}) \in E_x \times T_{u^*}E^*$ then

$$\theta_E(u^*)(u_x, v_{u^*}) = u^*(u_x)$$

is called the *Liouville section*. The *canonical symplectic section* ω_E is defined by

$$\omega_E = -d^E \theta_E$$

and it results that this is a nondegenerate two form and $d^E \omega_E = 0$.

In the local coordinates it follows that the Liouville section is given by

$$\theta_E = \mu_\alpha \mathcal{X}^\alpha$$

and we obtain

$$\omega_E = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma. \quad (6)$$

We remark that $VT E^*$ is Lagrangian for ω_E , i.e., $\omega_E(\rho_1, \rho_2) = 0$, for every vertical sections $\rho_1, \rho_2 \in \Gamma(VT E^*)$.

Definition 3. *The Ehresmann nonlinear connection on TE^* is an almost product structure \mathcal{N} on $\tau_1 : TE^* \rightarrow E^*$ (i.e., a bundle morphism $\mathcal{N} : TE^* \rightarrow TE^*$, such that $\mathcal{N}^2 = \text{Id}$) smooth on $TE^* \setminus \{0\}$ such that $VT E^* = \ker(\text{Id} + \mathcal{N})$.*

If \mathcal{N} is a connection on TE^* then $HT E^* = \ker(\text{Id} - \mathcal{N})$ is the horizontal distribution associated to \mathcal{N} and

$$TE^* = VT E^* \oplus HT E^*.$$

Each $\rho \in \Gamma(TE^*)$ can be written as $\rho = \rho^h + \rho^v$ where ρ^h, ρ^v are sections in the horizontal and respective, vertical subbundles. A connection \mathcal{N} on TE^* induces two projectors $h, v : TE^* \rightarrow TE^*$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in \Gamma(TE^*)$. We have

$$h = \frac{1}{2}(\text{Id} + \mathcal{N}), \quad v = \frac{1}{2}(\text{Id} - \mathcal{N})$$

$$\ker h = \text{im} v = VT E^*, \quad \text{im} h = \ker v = HT E^*$$

$$h^2 = h, \quad v^2 = v, \quad hv = vh = 0, \quad h + v = \text{Id}.$$

Locally, a nonlinear connection is expressed as $\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + 2\mathcal{N}_{\alpha\beta} \mathcal{P}^\beta$ and $\mathcal{N}(\mathcal{P}^\alpha) = -\mathcal{P}^\alpha$, where $\mathcal{N}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(x, \mu)$ are the local coefficients of \mathcal{N} . The local sections \mathcal{P}^α , $\alpha = 1, \dots, m$ define a local frame of $VT E^*$, and the sections

$$\delta_\alpha^* = (\mathcal{X}_\alpha)^h = \mathcal{X}_\alpha + \mathcal{N}_{\alpha\beta} \mathcal{P}^\beta \quad (7)$$

generate a local frame of $HT E^*$. The frame $\{\delta_\alpha^*, \mathcal{P}^\alpha\}$ is a local basis of TE^* called *adapted* to the direct sum decomposition. The respective dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{P}_\alpha\}$ where

$$\delta\mathcal{P}_\alpha = \mathcal{P}_\alpha - \mathcal{N}_{\alpha\beta} \mathcal{X}^\beta. \quad (8)$$

Definition 4. *A connection \mathcal{N} is called symmetric if $HT E^*$ is Lagrangian for ω_E .*

By a straightforward computation, using (6) and (7) we get

$$\omega_E(\delta_\alpha^*, \delta_\beta^*) = \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} - \mu_\gamma L_{\alpha\beta}^\gamma \quad (9)$$

and it result that \mathcal{N} is symmetric if and only if

$$\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L_{\alpha\beta}^\gamma.$$

Proposition 5. *With respect to a symmetric nonlinear connection, the canonical symplectic structure ω_E can be written in the following form*

$$\omega_E = \mathcal{X}^\alpha \wedge \delta\mathcal{P}_\alpha + \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma.$$

Proof: Using (6) and (8) we get

$$\omega_E = \mathcal{X}^\alpha \wedge \delta\mathcal{P}_\alpha + \frac{1}{2}(\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha})\mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{1}{2}\mu_\alpha L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma$$

which ends the proof. ■

Proposition 6. *The Lie brackets of the adapted basis $\{\delta_\alpha^*, \mathcal{P}^\alpha\}$ are*

$$[\delta_\alpha^*, \delta_\beta^*] = L_{\alpha\beta}^\gamma \delta_\gamma^* + \mathcal{R}_{\alpha\beta\gamma} \mathcal{P}^\gamma, \quad [\delta_\alpha^*, \mathcal{P}^\beta] = -\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\beta} \mathcal{P}^\gamma, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0$$

where

$$\mathcal{R}_{\alpha\beta\gamma} = \delta_\alpha^*(\mathcal{N}_{\beta\gamma}) - \delta_\beta^*(\mathcal{N}_{\alpha\gamma}) - L_{\alpha\beta}^\varepsilon \mathcal{N}_{\varepsilon\gamma}. \quad (10)$$

Proof: Using (7) we obtain

$$[\delta_\alpha^*, \delta_\beta^*] = \left(\sigma_\alpha^i \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial x^i} - \sigma_\beta^i \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial x^i} + \mathcal{N}_{\alpha\delta} \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\delta} - \mathcal{N}_{\beta\delta} \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\delta} \right) \mathcal{P}^\gamma + L_{\alpha\beta}^\varepsilon \mathcal{X}_\varepsilon$$

and putting $\mathcal{X}_\varepsilon = \delta_\varepsilon^* - \mathcal{N}_{\varepsilon\gamma} \mathcal{P}^\gamma$ we get $[\delta_\alpha^*, \delta_\beta^*] = L_{\alpha\beta}^\gamma \delta_\gamma^* + \mathcal{R}_{\alpha\beta\gamma} \mathcal{P}^\gamma$. ■

The curvature of a connection \mathcal{N} on $\mathcal{T}E^*$ is given by $\Omega = -\mathbb{N}_h$ where h is horizontal projector and \mathbb{N}_h is the Nijenhuis tensor of h , given by

$$\mathbb{N}_h(\theta, \rho) = [h\theta, h\rho] - h[h\theta, \rho] - h[\theta, h\rho] + h^2[\theta, \rho].$$

Remark 7. *In the local coordinates we get*

$$\Omega = -\frac{1}{2} \mathcal{R}_{\alpha\beta\gamma} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma$$

where $\mathcal{R}_{\alpha\beta\gamma}$ is given by (10) and is called the *curvature tensor* of \mathcal{N} .

Proof: Since $h^2 = h$ we obtain

$$\Omega(h\rho_1, h\rho_2) = -v[h\rho_1, h\rho_2], \quad \Omega(h\rho_1, v\rho_2) = \Omega(v\rho_1, v\rho_2) = 0$$

and in local coordinates we get

$$\Omega(\delta_\alpha^*, \delta_\beta^*) = -v[\delta_\alpha^*, \delta_\beta^*] = -\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma$$

which concludes the proof. ■

Remark 8. *The curvature satisfies the Bianchi identity*

$$\mathcal{R}_{\alpha\beta\gamma} + \mathcal{R}_{\beta\gamma\alpha} + \mathcal{R}_{\gamma\alpha\beta} = 0.$$

Proof: By direct computation, using relation (10) and structure equations given by (1). ■

The curvature is an obstruction to the integrability of $HT E^*$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E^*$. We have

Remark 9. *$HT E^*$ is integrable if and only if the curvature vanishes.*

The integrability conditions for the almost product structure \mathcal{N} is given by the vanishing of the associated Nijenhuis tensor $N_{\mathcal{N}}$. By a straightforward computation we obtain

$$N_{\mathcal{N}}(\mathcal{P}^\alpha, \mathcal{P}^\beta) = 0, \quad N_{\mathcal{N}}(\delta_\alpha^*, \mathcal{P}^\beta) = 0, \quad N_{\mathcal{N}}(\delta_\alpha^*, \delta_\beta^*) = 4\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma.$$

Thus

$$N_{\mathcal{N}} = -2\mathcal{R}_{\alpha\beta\gamma}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma$$

and it results that the distribution $HT E^*$ is integrable if and only if the almost product structure \mathcal{N} is integrable.

3.1. Canonical Poisson Structure

On the Lie algebroid $(\mathcal{T}E^*, [,], \sigma^1)$ we have the canonical symplectic section ω_E given by (6) which induces a vector bundle isomorphism

$$\flat_{\omega_E} : E^* \rightarrow E, \quad i_\zeta \omega_E \in E^* \rightarrow \zeta \in E.$$

Definition 10. *The canonical Poisson bivector is given by*

$$\Lambda = \natural_{\omega_E} \omega_E.$$

It follows that

$$\Lambda(dF, dG) = -\omega_E(\natural(dF), \natural(dG)), \quad F, G \in C^\infty(E^*)$$

and in local coordinates we get

$$\Lambda = \mathcal{P}^\alpha \wedge \mathcal{X}_\alpha + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{P}^\beta \wedge \mathcal{P}^\gamma.$$

Remark 11. *The Schouten-Nijenhuis bracket $[\Lambda, \Lambda]$ leads, locally, to the expression*

$$\frac{1}{3} \sum_{(\alpha, \beta, \gamma)} (\sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\varepsilon}{\partial x^i} + L_{\alpha\delta}^\varepsilon L_{\beta\gamma}^\delta) \mu_\varepsilon \mathcal{P}^\beta \wedge \mathcal{P}^\alpha \wedge \mathcal{P}^\gamma$$

and $[\Lambda, \Lambda] = 0$ follows from the structure equations on the Lie algebroid (1).

Definition 12. *Let us consider a Poisson bivector on E given by (2), then the horizontal lift of W to TE^* is the bivector defined by*

$$W^H = \frac{1}{2} w^{\alpha\beta}(x) \delta_\alpha^* \wedge \delta_\beta^*.$$

Proposition 13. *The horizontal lift W^H is a Poisson bivector if and only if W is a Poisson bivector on E and*

$$w^{\alpha\beta} w^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} = 0.$$

Proof: The Poisson condition $[W, W] = 0$ leads to the relation

$$\sum_{(\alpha, \varepsilon, \delta)} (w^{\alpha\beta} w^{\gamma\delta} L_{\beta\gamma}^\varepsilon + w^{\alpha\beta} \sigma_\beta^i \frac{\partial w^{\varepsilon\delta}}{\partial x^i}) = 0$$

and $[W^H, W^H] = 0$ yields

$$\sum_{(\varepsilon, \delta, \alpha)} \left(w^{\alpha\beta} w^{\gamma\delta} L_{\beta\gamma}^\varepsilon + w^{\alpha\beta} \sigma_\beta^i \frac{\partial w^{\varepsilon\delta}}{\partial x^i} \right) \delta_\varepsilon^* \wedge \delta_\alpha^* \wedge \delta_\delta^* + w^{\alpha\beta} w^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} \mathcal{P}^\varepsilon \wedge \delta_\alpha^* \wedge \delta_\gamma^* = 0$$

which ends the proof. ■

Recall that two Poisson structures are said to be *compatible* if the bivectors w_1 and w_2 satisfy the condition

$$[w_1, w_2] = 0.$$

Proposition 14. *If W^H is a Poisson bivector and \mathcal{N} is a symmetric nonlinear connection, then W^H is compatible with the canonical Poisson structure Λ if and only if the following relations fulfilled*

$$\sigma_\gamma^i \frac{\partial \omega^{\alpha\beta}}{\partial x^i} + \omega^{\varepsilon\alpha} \left(\frac{\partial N_{\varepsilon\gamma}}{\partial \mu_\beta} - L_{\varepsilon\gamma}^\beta \right) - \omega^{\varepsilon\beta} \left(\frac{\partial N_{\varepsilon\gamma}}{\partial \mu_\alpha} - L_{\varepsilon\gamma}^\alpha \right) = 0 \quad (11)$$

$$\omega^{\varepsilon\alpha} \mathcal{R}_{\alpha\gamma\delta} = 0. \quad (12)$$

Proof: If \mathcal{N} is symmetric then $\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L_{\alpha\beta}^\gamma$ and with respect with the basis $\{\delta_\alpha^*, \mathcal{P}^\alpha\}$ it results

$$\Lambda = \mathcal{P}^\alpha \wedge \delta_\alpha^*.$$

By a straightforward computation we obtain that the relation $[W^H, \Lambda] = 0$ is equivalent with relations (11) and (12). ■

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Liviu Popescu

Department of Applied Mathematics in Economy

University of Craiova

13, A. I. Cuza st., 200585 Craiova

ROMANIA

E-mail address: liviupopescu@central.ucv.ro