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IMPROVED EXPLICIT BOUNDS FOR SOME FUNCTIONS OF PRIME NUMBERS

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Abstract: Using recent explicit asymptotic zero–free region and computations of zeros of the Riemann zeta function, obtained by Mossinghoff & Trudgian and Gourdon, respectively, we give an improvement for estimates of some functions related to distribution of primes, such as prime counting function, intervals containing at least one prime, Chebyshev's ψ and ϑ functions.

Keywords: Chebyshev's function, prime number, Riemann hypothesis, product involving prime number.

1. Introduction and Preliminaries

Let $\vartheta(x)$ and $\psi(x)$ be the first and second Chebyshev's functions respectively, defined by

$$\vartheta(x) = \sum_{p\leqslant x} \log p, \qquad \psi(x) = \sum_{\substack{p,m\\ p^m\leqslant x}} \log p.$$

In 1852 Chebyshev ([2], p. 379) proved in a beautiful way that for all $x > 1$

$$\vartheta(x) < \frac{6}{5}A_0x - A_0x^{\frac{1}{2}} + \frac{5}{4\log 6}\log^2 x + \frac{5}{2}\log x + 2,$$
$$\vartheta(x) > A_0x - \frac{12}{5}A_0x^{\frac{1}{2}} - \frac{5}{8\log 6}\log^2 x - \frac{15}{4}\log x - 3,$$

where $A_0 = \log(2^{1/2}3^{1/3}5^{1/5}30^{1/30}) \approx 0.92129$.

The following explicit formula which is due to Riemann gives a link between nontrivial zeros of the Riemann zeta function ζ and Chebyshev's ψ function (see [12, p. 298] and [9]):

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-\frac{1}{x^2}), \qquad (x>1,\ x\neq p^m), \tag{1}$$

2010 Mathematics Subject Classification: primary: 11A41; secondary: 11N05, 11M26, 11K65 where ρ runs over the nontrivial zeros of ζ function, $\sum_\rho \frac{x^\rho}{\rho} = \lim_{T\to\infty} \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho}$. The explicit formula (1) was proved by H. von Mangoldt in 1895.

As we note in formula (1), the numerical verification of the Riemann hypothesis (RH) and finding smaller zero–free regions for the Riemann zeta function (and L-functions in general) are of great importance in the subject of distribution of prime numbers; see Table 1 for a historical review of computations of zeros of the Riemann zeta function. Following the classical result of Hadamard and de la

Table 1. Numerical verification of zeros of the zeta function ([1], p. 39)

Year	Number of zeros	Computed by
1859 (approx)	1 (or 3)	B. Riemann
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1041	E. C. Titchmarsh
1953	1104	A. M. Turing
1956	15,000	D. H. Lehmer
1956	25,000	D. H. Lehmer
1958	35,337	N. A. Meller
1966	250,000	R. S. Lehman
1968	3,500,000	J. B. Rosser, et al.
1977	40,000,000	R. P. Brent
1979	81,000,001	R. P. Brent
1982	200,000,001	R. P. Brent, et al.
1983	300,000,001	J. van de Lune, H. J. J. te Riele
1986	1,500,000,001	J. van de Lune, et al.
2001	10,000,000,000	J. van de Lune(unpublished)
2004	900,000,000,000	S. Wedeniwski (unpublished)
2004	10^{13}	X. Gourdon & P. Demichel (unpublished)
2011	3.0610046×10^{10}	D. J. Platt [18]

Vallée-Poussin that $\zeta(s)$ does not vanish on the line $\Re s = 1$ by giving some implicit zero–free regions, Rosser and Schoenfeld gave an explicit zero–free region for the Riemann zeta function. More precisely, they determined that the first 3 502 500 zeros lie on the critical line $\Re s = 1/2$ and proved that there are no zeros of $\zeta(s)$ in the region

$$\sigma \geqslant 1 - \frac{1}{R_1 \log |t/17|}, \qquad |t| \geqslant 21, \tag{2}$$

where $R_1 = 9.645\,908\,801$ (see [23]). Using the verification of the RH up to 3 502 500–th zero and (2) Rosser and Schoenfeld gave explicit error terms in prime number theorem (PNT). Later Schoenfeld [24] proved some necessary results for the truth of the RH in terms of certain inequalities related to Chebyshev's functions. Moreover, he established the following estimate unconditionally.

Theorem 1.1 ([24], Th. 11). *Let*

$$\varepsilon_1(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} e^{-X}, \quad X = \sqrt{\log x / R_1},$$

and R_1 be defined as in (2). Then

$$|\psi(x) - x| < x\varepsilon_1(x), \qquad (x \geqslant 17), \qquad |\vartheta(x) - x| < x\varepsilon_1(x), \quad (x \geqslant 101).$$

In recent years several results appeared on the improvement of zero-free region for the Riemann zeta function and its applications, for instance see the references in Table 1. There are some zero–free regions of different type due to Littlewood,

Table 2. Zero–free regions of the Riemann zeta function

Year	R_0	Computed by
1899	34.82	de la Vallée Poussin [3]
2002	8.463	K. Ford [8]
2005	5.69693	H. Kadiri [13]
2014	5.68371	Jang, Woo-Jin; Kwon, Soun-Hi [27]
2015	5.573412	M. Mossinghoff; T. Trudgian [15]

Chudakov and Ford [8]. For example Ford gives the following explicit zero–free region

$$\beta > 1 - \frac{1}{57.54(\log|t|)^{2/3}(\log\log|t|)^{1/3}}, \qquad |t| \geqslant 3.$$

In this paper we use the zero–free region obtained by Mossinghoff and Trudgian [15] with $R_0 = 5.573412$. This zero-free region is better than the one defined in (2) for $|t| \geqslant 822$. Note that the zero-free region obtained by Ford is better than Mossinghoff and Trudgian's when $|t| \geqslant \exp(10152)$.
Recall that

$$N(T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0,\ 0 < \beta < 1,\ 0 < \gamma \leqslant T\},$$

$$F(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}.$$

The other quantity that is involved in estimating explicit bounds in Chebyshev's function is the upper bound for the difference $|N(T) - F(T)|$, which is denoted by $R(T)$. Rosser [20] proved that $|N(T) - F(T)| < R(T)$ for $T \geqslant 2$, where

$$R(T) = 0.137\log T + 0.443\log\log T + 1.588.$$

Recently Trudgian [25] improved this upper bound and showed that

$$R(T) = 0.112\log T + 0.278\log\log T + 2.510 + \frac{0.2}{T}. \tag{3}$$

Remark 1.2. During the revision of this paper we found that there is an improvement on $R(T)$ in [17], however we follow (3).

We note that the latter error term $R(T)$ is less than Rosser's for $T \geqslant \exp(18)$. In particular for

$$A = 2445999556030.342362641 > e^{28.525}$$

which is almost the height of the 10^{13}–th zero of the Riemann zeta function.

2. Improved explicit bounds for Chebyshev's Functions

In this section we shall use recent developments involving the number of zeros of the Riemann zeta function, i.e., verification of RH up to 10^{13}-th zero by Gourdon [10], explicit zero–free region for the Riemann zeta function given by Mossinghoff and Trudgian [15] and the error term in (3). In the proofs of Theorems 2.3 and 2.4 we exploit the following lemma.

Lemma 2.1 ([15], Th. 1). *The Riemann zeta function $\zeta(s)$ with $s = \sigma + it$ does not vanish in the region*

$$\sigma > 1 - \frac{1}{R_0 \log |t|}, \qquad (|t| \geqslant 2,\ R_0 = 5.573412).$$

2.1. Improved explicit bounds for large values of x

Using these two recent results we give better explicit bounds for Chebyshev's functions. The proofs of theorems in this section are essentially similar to those of Schoenfeld [24] (and extensive calculations was done by `Mathematica`), therefore we omit the proofs. To see detailed proofs and other results we refer the reader to [16].

Recall that $N(T)$, $F(T)$ and $R(T)$ are defined as in Section 1. Choose A such that $F(A) = 10^{13}$. Then

$$\begin{aligned} A &= 2\,445\,999\,556\,030.342\,362\,641, \\ \log A &= 28.525\,474\,972. \end{aligned} \tag{4}$$

Let $S_i(m, \delta)$ for $i = 1, 2, 3, 4$ be as in page 256 of [22], see also [16]. The following lemma is the basis of finding estimates for Chebyshev's function in the proofs of Rosser and Schoenfeld's results in [22], [24].

Lemma 2.2 ([22], Lemma 8). *Let T_1 and T_2 be non-negative real numbers. Let m be a positive integer. Let $x > 1$ and $0 < \delta < (x-1)/(xm)$. Then*

$$\begin{aligned} &\frac{1}{x}\left|\psi(x) - \left\{x - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)\right\}\right| \\ &\qquad\qquad \leqslant \frac{1}{\sqrt{x}}\{S_1(m,\delta) + S_2(m,\delta)\} + S_3(m,\delta) + S_4(m,\delta) + \frac{m\delta}{2}. \end{aligned}$$

For $x \geqslant 1$ let

$$X = \sqrt{\frac{\log x}{R_0}}, \qquad R_0 = 5.573412.$$

Note that if we replace "17" with "1" in the proof of Theorem 1.1 and adapt the proof whenever it is necessary, we derive:

Theorem 2.3. *Let*

$$\varepsilon_0(x) = \sqrt{8/\pi} X^{1/2} e^{-X}.$$

Then

$$|\psi(x) - x| < x\varepsilon_0(x), \qquad (x \geqslant 3)$$

and

$$|\vartheta(x) - x| < x\varepsilon_0(x), \qquad (x \geqslant 3).$$

Note also that the error term in Theorem 1.1 has a coefficient $\sqrt{8/(17\pi)}$, which is less than $\sqrt{8/\pi}$ of Theorem 2.3 and therefore gives a better bound for lower values of x. However, since we employed a smaller zero–free region, a better bound will be obtained for $x \geqslant \exp(234)$. We mention here that the above theorem is quite similar to Corollary 1.2 of [6]. The idea of the proofs of results [6] and here are inspired by Rosser and Schoenfeld papers in the subject with slight modification and some update.

2.2. Improved explicit bounds for moderate values of x

In Theorem 2.3, the role of the verified height for RH –the number A defined in (4), and coefficients in (3) was not vigorous, but it will be more effective in estimating Chebyshev's function for moderate values of x using the next theorem. Let

$$\begin{aligned}
T_0 &= \frac{1}{\delta}\left(\frac{2R_m(\delta)}{2+m\delta}\right)^{1/m},\\
G(D) &= \sum_{0<\gamma\leqslant D} \frac{1}{(\gamma^2+1/4)^{1/2}} - \frac{1}{4\pi}\left\{\left(\log\frac{D}{2\pi} - 1\right)^2 + 1\right\}\\
&\quad + \frac{1}{D}\left\{0.112\log D + 0.278\left(\log\log D + \frac{1}{\log D}\right) + 1.265 + \frac{0.2}{D} - N(D)\right\},\\
C(D) &= 4\pi\left(0.112 + \frac{0.278}{\log D}\right),\\
\phi_m(y) &= y^{-m-1}\exp\left(-\frac{X^2}{\log y}\right),\\
q(y) &= \frac{0.112\log y + 0.278}{y\log y\log(y/2\pi)}.
\end{aligned}$$

We can give exactly the same theorem as [24, Lemma 9*], but replaced with A defined in (4).

Theorem 2.4. *Let T_0 be defined as above and satisfy $T_0 \geqslant D$, where $2 \leqslant D \leqslant A$. Let m be a positive integer and let $\delta > 0$. Then*

$$S_1(m,\delta) + S_2(m,\delta) < \Omega_1^*,$$

where

$$\Omega_1^* = \frac{2+m\delta}{4\pi}\left\{\left(\log\frac{T_0}{2\pi} + \frac{1}{m}\right)^2 + \frac{1}{m^2} + 4\pi G(D) - \frac{mC(D)}{(m+1)T_0}\right\}$$

and $G(D)$ and $C(D)$ are defined as above and

$$\Omega_2 = 2\frac{R_m(\delta)}{\delta^m}\left\{(0.159155)\int_A^\infty \phi_m(y)\log\frac{y}{2\pi}dy + (2R(Y)\phi_m(Y) - R(A)\phi_m(A))\right\},$$

where $0.159155 = 1/2\pi + q(A)$ and $Y = \max\{A, \exp\sqrt{b/(m+1)R_0}\}$. If $b > 1/2$ and $0 < \delta < (1-e^{-b})/m$, then

$$|\psi(x) - x| < \varepsilon x, \qquad (x \geqslant e^b),$$

where

$$\varepsilon = \Omega_1^* e^{-b/2} + \Omega_2 + \frac{m\delta}{2} + e^{-b}\log 2\pi.$$

Moreover, if

$$\Omega_3^* = \frac{1}{2\pi}h_3(T_2) + e_3(T_2), \qquad T_2 \geqslant A,$$

where

$$h_3(T) = \frac{2+m\delta}{2}\int_A^T \phi_0(y)\log\frac{y}{2\pi}dy + \frac{R_m(\delta)}{\delta^m}\int_T^\infty \phi_m(y)\log\frac{y}{2\pi}dy$$

and

$$e_3(T) = q(T)\left\{-\frac{2+m\delta}{2}\int_A^T \phi_0(y)\log\frac{y}{2\pi}dy + \frac{R_m(\delta)}{\delta^m}\int_T^\infty \phi_m(y)\log\frac{y}{2\pi}dy\right\}$$
$$+ R(T)\phi_0(T)\left\{2 + m\delta + 2\frac{R_m(\delta)}{(\delta T)^m}\right\},$$

then

$$|\psi(x) - x| < \varepsilon^* x, \qquad (x \geqslant e^b),$$

where

$$\varepsilon^* = \Omega_1^* e^{-b/2} + \Omega_3^* + \frac{m}{2}\delta + e^{-b}\log 2\pi.$$

Table 3 is computed according to the above Theorem with $R_0 = 5.573412$ and error term (3).

3. Improved explicit bounds for $\vartheta(x)$ and distribution of primes

Using Theorem 2.4 we get a slightly better estimate for the first Chebyshev's function (cf. [4, Th. 5.2]).

Proposition 3.1. *Let* $x_k \geqslant 8 \cdot 10^{11}$. *Then*

$$|\vartheta(x) - x| < \eta'_k \frac{x}{\log^k x}, \qquad (x \geqslant x_k),$$

where

k	0	1	2	3	4
η'_k	0.00002316833	0.000648713234	0.0181639705534	0.5085911755	1125

Proof. Let $e^b \leqslant x < e^{b+1}$. Appealing to [4, Proposition 3.1] treating $\vartheta(x) - x = \vartheta(x) - \psi(x) + \psi(x) - x$ in the following way

$$\vartheta(x) - x < -0.9999\sqrt{x} + x\varepsilon = \left(-0.9999\frac{\log^k x}{\sqrt{x}} + \varepsilon \log^k x\right) \frac{x}{\log^k x} \tag{5}$$

and

$$\begin{aligned}\vartheta(x) - x &> -1.00007\sqrt{x} - 1.78\sqrt[3]{x} - x\varepsilon \\ &= \left(-1.00007\frac{\log^k x}{\sqrt{x}} - 1.78\frac{\log^k x}{\sqrt[3]{x^2}} - \varepsilon \log^k x\right) \frac{x}{\log^k x}.\end{aligned} \tag{6}$$

To estimate η'_k, it is enough to choose $x = e^{b+1}$ in each parenthesis above. For instance, to estimate η'_1 in the interval $[8 \cdot 10^{11}, e^{28})$, we have $\varepsilon = 0.0000223228$ (see calculations below Table 3), and

$$\vartheta(x) - x < \left(-0.9999\frac{28}{\sqrt{e^{28}}} + 0.0000223228(28)\right) \frac{x}{\log x} < 0.000601759\frac{x}{\log x}, \tag{7}$$

$$\begin{aligned}\vartheta(x) - x &> \left(-1.00007\frac{28}{\sqrt{e^{28}}} - 1.78\frac{28}{\sqrt[3]{e^{2(28)}}} - 0.0000223228(28)\right) \frac{x}{\log x} \\ &> -0.000648713\frac{x}{\log x}.\end{aligned} \tag{8}$$

We choose the maximum of absolute values of (7) and (8) for this interval to get 0.000648713. Continuing this process for all intervals $[e^b, e^{b+1})$ with $b = 28, 29, \ldots$ up to $x = e^{5100}$, we get the desired results. More precisely, to complete the table above for each $k = 1, 2, 3, 4$, we did as follows:

- we fill the very beginning column with $b = 28, 29, \ldots, 5100$ quite similar to Table 3, with extra rows when it is necessary,

- corresponding to each b–row find the maximum of absolute values of the parentheses in (5) and (6) for $x = e^{b+1}$,
- compare all the values in each column correspond to η'_k, $k = 1, 2, 3, 4$ and write the maximum of that column in our table above.

For $x \geqslant \exp(5100)$ the maximum of absolute values of (5) and (6) for all $k = 1, 2, 3, 4$, when ε_0 is replaced by ε, is decreasing and strictly less than the values given in the table above. ■

Remark 3.2. The number 5100 in Table 3 is chosen as the last number, since for $x \geqslant e^{5100}$ we obtain $\varepsilon_0 < \varepsilon$, therefore we can apply then Theorem 2.3. The numbers x_k in the proposition above may be determined explicitly, and they are perhaps less than $8 \cdot 10^{11}$.

Applying the previous proposition, we obtain estimates for $\pi(x)$.

Proposition 3.3. *Let $x \geqslant 8 \cdot 10^{11}$. Then*

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.0794}{\log x}\right),$$
$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.265}{\log^2 x}\right).$$

Proof. By Abel's summation formula

$$\begin{aligned}\pi(x) &= \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(y)}{y \log^2 y} dy \\ &< \frac{x}{\log x}\left(1 + \frac{\eta'_k}{\log^k x}\right) + \int_2^x \frac{1}{\log^2 y}\left(1 + \frac{\eta'_k}{\log^k y}\right) dy, \qquad (k = 1, 2). \qquad (9)\end{aligned}$$

We are looking for inequality of this type:

$$\pi(x) < A_2(x), \qquad (x \geqslant 8 \cdot 10^{11}),$$

where

$$A_2(x) = \frac{x}{\log x}\left(1 + \frac{c}{\log x}\right),$$

and c is a constant, which will be determined in the following. Let $A_1(x)$ be the right-hand side of (9). Therefore it is enough to have $A_1(x) < A_2(x)$ for $x \geqslant 8 \cdot 10^{11}$. To have this inequality it is enough to have $A_1(x_0) \leqslant A_2(x_0)$ with $x_0 = 8 \cdot 10^{11}$ and $A'_1(x) < A'_2(x)$ for $x \geqslant x_0$. Indeed,

$$A'_1(x) = \frac{1}{\log x} + \frac{2\eta'_k x}{\log^{k+1} x} - \frac{\eta'_k(-1 + x + kx)}{\log^{k+2} x}$$

and

$$A'_2(x) = \frac{1}{\log x} + \frac{-1 + c}{\log^2 x} - \frac{2c}{\log^3 x}.$$

We apply the case η_1' in Proposition 3.1, and get for $x \geqslant 8 \cdot 10^{11}$

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.0794}{\log x}\right)$$

or if we let

$$A_2(x) = \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{c'}{\log x}\right),$$

by a similar method we arrive at

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.265}{\log^2 x}\right). \qquad \blacksquare$$

Note that if the values of the function $\mathrm{li}(x)$ can be calculated in some way, we could use the following formula

$$\mathrm{li}(x) - \mathrm{li}(2) = \int_2^x \frac{1}{\log y} dy = \left[\frac{y}{\log y} + \frac{y}{\log^2 y} + \frac{2!y}{\log^3 y} + \frac{3!y}{\log^4 y} + \cdots + \frac{j!y}{\log^{j+1} y}\right]_2^x + (j+1)! \int_2^x \frac{1}{\log^{j+2} y} dy, \qquad (j = 0, 1, \ldots)$$

to calculate the integral in (9) instead of the method of differential calculus which we applied above.

Other interesting problems in the realm of distribution of primes are determining intervals that contain at least one prime number and the difference between consecutive prime numbers see [19], [14] and [5] for more details about effective intervals containing primes and primes in arithmetic progressions. In this paper we shall concern with the explicit bounds. In the next proposition we shall give an upper bound for the length of intervals containing at least one prime.

Proposition 3.4. *For all $x \geqslant 492\,227$, there exists at least one prime p such that*

$$x < p \leqslant x\left(1 + \frac{3}{100\log^2 x}\right).$$

Note that $3/100$ can be refined to 0.00290013. We informed later about the following result of Trudgian and therefore we omitted our proof which was quite similar to his. Note also that in our proof we did a direct computation for $492\,227 \leqslant x < 5\,402\,962$. Trudgian [26] gives a shorter interval but for higher values of x. He proves that for all $x \geqslant 2\,898\,242$, there is a prime in the interval

$$\left[x, x\left(1 + \frac{1}{111\log^2 x}\right)\right]$$

One can derive the following formula for sum of reciprocal over primes using Abel's summation formula by (see [21, p. 74])

$$\sum_{p \leqslant x} \frac{1}{p} - \log\log x - B = \frac{\vartheta(x) - x}{x\log x} - \int_x^\infty \{\vartheta(t) - t\}\left(\frac{1}{t^2\log^2 t} + \frac{1}{t^2\log t}\right) dt, \quad (10)$$

where B is Meissel-Mertens constant (see [11, p. 23]) defined by

$$B = \gamma + \sum_p \left\{ \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} \approx 0.261497. \tag{11}$$

Let

$$|\vartheta(x) - x| < \eta_k' \frac{x}{\log^k x}, \qquad (x \geqslant x_k).$$

Then using (10) one can prove the following proposition.

Proposition 3.5. *Let η_k' and x_k be defined as in Proposition 3.1. Then*

$$\left| \sum_{p \leqslant x} \frac{1}{p} - \log\log x - B \right| < \frac{\eta_k'}{k \log^k x} + \left(1 + \frac{1}{k+1}\right) \frac{\eta_k'}{\log^{k+1} x}, \qquad (x \geqslant x_k). \tag{12}$$

A similar inequality can be found in the proofs of [4, Th. 6.10] and [7, Th. 5.6].

3.1. Explicit estimates for $\prod_{p \leqslant x}(1 + 1/p)$

Before starting the argument to prove the next proposition and its corollary, we give the following lemma. Recall that

$$1 + t < e^t < \frac{1}{1-t}, \qquad (t < 1). \tag{13}$$

Lemma 3.6. *Let c_1, c_2 and k be fixed positive numbers. Then there exists a positive number x_0 depending on c_1, c_2 and k such that for all $x \geqslant x_0$ we have*

$$\exp\left(\frac{c_1}{\log^k x} + \frac{c_2}{x \log x} \right) < \frac{1}{1 - (c_1/\log^k x)}.$$

First, we determine some values for which we encounter later. Let

$$S(x) = \sum_{p > x} \left\{ \log\left(1 + \frac{1}{p}\right) - \frac{1}{p} \right\} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{p > x} \frac{1}{p^n}.$$

Hence,

$$\sum_{p > x} \left(\frac{1}{2p^2} - \frac{1}{3p^3} \right) < -S(x) < \sum_{p > x} \frac{1}{2p^2}.$$

Using Abel's summation formula and estimates for $\vartheta(x)$ one obtains

$$\sum_{p > x} \frac{1}{2p^2} < \frac{1}{x \log x}$$

and

$$\sum_{p>x}\left(\frac{1}{2p^2}-\frac{1}{3p^3}\right)>\frac{1}{2x\log x}-\frac{5}{x\log^2 x}.$$

From

$$\prod_p\left(1+\frac{1}{p}\right)=\prod_p\left(1-\frac{1}{p^2}\right)/\prod_p\left(1-\frac{1}{p}\right)$$

and definition of B in (11) we have

$$\sum_p\left\{\log\left(1+\frac{1}{p}\right)-\frac{1}{p}\right\}=\log\frac{6}{\pi^2}+\gamma-B.$$

Therefore,

$$\sum_{p\leqslant x}\frac{1}{p}-B=\sum_{p\leqslant x}\log\left(1+\frac{1}{p}\right)+\sum_{p>x}\left\{\log\left(1+\frac{1}{p}\right)-\frac{1}{p}\right\}-\log\frac{6}{\pi^2}-\gamma.$$

Now by Proposition 3.1

$$\left|\sum_{p\leqslant x}\log\left(1+\frac{1}{p}\right)+S(x)-\log\frac{6}{\pi^2}-\gamma-\log\log x\right|<C_k(x), \tag{14}$$

where $C_k(x)$ is the right-hand side of (12). Expanding terms inside absolute value (14), we get

$$\sum_{p\leqslant x}\log\left(1+\frac{1}{p}\right)<\log\frac{6}{\pi^2}+\gamma+\log\log x+C_k(x)-S(x), \tag{15}$$

$$\sum_{p\leqslant x}\log\left(1+\frac{1}{p}\right)>\log\frac{6}{\pi^2}+\gamma+\log\log x-C_k(x)-S(x). \tag{16}$$

Note that $-S(x)<1/x\log x$ and it is less than the order of $C_k(x)$. We take exponential of both sides of (15) and (16) and use Lemma 3.6 to derive the following proposition.

Proposition 3.7. *Let x_k be as in Proposition 3.1. Then for all $x\geqslant x_k$ we have*

$$\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)<\frac{6e^\gamma}{\pi^2}\frac{1}{1-C_k(x)}\log x,$$

$$\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)>\frac{6e^\gamma}{\pi^2}\{1-C_k(x)\}\log x.$$

Second proof of Proposition 3.7. Apart from the argument explained before Proposition 3.7 we can treat the proof of this proposition in a different way. In this argument we do not use the estimates for $\prod_p(1-1/p^2)$, $\prod_p(1-1/p)$ or $\prod_{p\leqslant x}(1-1/p)$.

Recall that for $t>0$ we have

$$\frac{1}{t+1/2}<\log\left(1+\frac{1}{t}\right)<\frac{1}{2}\left(\frac{1}{t}+\frac{1}{t+1}\right).$$

Let

$$\begin{aligned}\sum_{p\leqslant x}\left\{\log\left(1+\frac{1}{p}\right)-\frac{1}{2}\left(\frac{1}{p}+\frac{1}{p+1}\right)\right\}&=-a_x,\\ \sum_{p\leqslant x}\left\{\log\left(1+\frac{1}{p}\right)-\frac{1}{p+1/2}\right\}&=b_x.\end{aligned}$$

It is clear that

$$a_x+b_x=\frac{1}{2}\sum_{p\leqslant x}\left(\frac{1}{p}-\frac{2}{p+1/2}+\frac{1}{p+1}\right).$$

Therefore,

$$\begin{aligned}\log\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)&=\sum_{p\leqslant x}\log\left(1+\frac{1}{p}\right)=\frac{1}{2}\sum_{p\leqslant x}\left(\frac{1}{p}+\frac{1}{p+1}\right)-a_x\\ &=\sum_{p\leqslant x}\frac{1}{p}-\frac{1}{2}\sum_{p\leqslant x}\frac{1}{p(p+1)}-a_x\\ &<\log\log x+B+C_k(x)-\frac{1}{2}\sum_{p\leqslant x}\frac{1}{p(p+1)}-a_x,\\ \log\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)&=\sum_{p\leqslant x}\log\left(1+\frac{1}{p}\right)=\sum_{p\leqslant x}\frac{1}{p+1/2}+b_x\\ &=\sum_{p\leqslant x}\frac{1}{p}-\frac{1}{2}\sum_{p\leqslant x}\frac{1}{p(p+1/2)}+b_x\\ &>\log\log x+B-C_k(x)-\frac{1}{2}\sum_{p\leqslant x}\frac{1}{p(p+1/2)}+b_x.\end{aligned}$$

Taking exponential of both sides in each inequality and using (13) we get the bounds in the proposition.

Extending this argument we arrive at

$$\log \prod_{x<p\leqslant y}\left(1+\frac{1}{p}\right) < \log\log y - \log\log x + C_k(x) + C_k(y) - \frac{1}{2}\sum_{x<p\leqslant y}\frac{1}{p(p+1)} - (a_y - a_x)$$

and

$$\log \prod_{x<p\leqslant y}\left(1+\frac{1}{p}\right) > \log\log y - \log\log x - C_k(x) - C_k(y) - \frac{1}{2}\sum_{x<p\leqslant y}\frac{1}{p(p+1/2)} + (b_y - b_x),$$

which are slightly better than the bounds in Corollary 3.8. However, for simplicity, we used the estimates just before Proposition 3.7.

We conclude this paper by the following corollary with a similar proof for the above proposition. Note that x_0 in the second inequality of the following corollary is not necessarily equal to x_k in Proposition 3.1 and might be larger than x_k, however we did not compute x_0 here.

Corollary 3.8. *We have*

$$\prod_{x<p\leqslant y}\left(1+\frac{1}{p}\right) < \frac{\log y}{\log x}\left\{\frac{1}{1-C_k(x)-C_k(y)}\right\}, \qquad (x \geqslant x_k)$$

and

$$\prod_{x<p\leqslant y}\left(1+\frac{1}{p}\right) > \frac{\log y}{\log x}\left\{1-C_k(x)-C_k(y)\right\}, \qquad (x \geqslant x_0 \geqslant x_k),$$

where x_k *depends on* η'_k.

Remark 3.9. In Theorem 2.4 we take $D = 2500$. For $b = \log(8\cdot 10^{11}) \approx 27.4079$, we have $m = 1$, $\delta = 8.99\cdot 10^{-8}$ and $\varepsilon = 2.8477\cdot 10^{-5}$.

For $b = 28$, we have $m = 1$, $\delta = 7.21\cdot 10^{-6}$ and $\varepsilon = 2.23228\cdot 10^{-5}$.

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Table 3. $|\psi(x) - x| < x\varepsilon, \quad (x \geqslant e^b)$, for Theorem 2.4

b	m	δ	ε		b	m	δ	ε
18.42	1	4.77(-4)	1.14768(-3)		900	22	2.07(-12)	2.38363(-11)
18.43	1	4.75(-4)	1.14315(-3)		950	21	2.15(-12)	2.34913(-11)
18.44	1	4.73(-4)	1.13863(-3)		1000	21	2.11(-12)	2.31553(-11)
18.45	1	4.71(-4)	1.13413(-3)		1050	21	2.08(-12)	2.28241(-11)
18.5	1	4.61(-4)	1.11188(-3)		1100	20	2.14(-12)	2.24818(-11)
18.7	1	4.22(-4)	1.02704(-3)		1150	20	2.11(-12)	2.21454(-11)
19.0	1	3.70(-4)	9.11453(-4)		1200	20	2.08(-12)	2.18145(-11)
19.5	1	2.96(-4)	7.46327(-4)		1250	19	2.15(-12)	2.14786(-11)
20	1	2.37(-4)	6.10463(-4)		1300	19	2.11(-12)	2.11424(-11)
21	1	1.52(-4)	4.07193(-4)		1350	19	2.08(-12)	2.08104(-11)
22	1	9.68(-5)	2.70582(-4)		1400	19	2.05(-12)	2.04844(-11)
23	1	6.17(-5)	1.79185(-4)		1450	18	2.12(-12)	2.01427(-11)
24	1	3.93(-5)	1.18300(-4)		1500	18	2.09(-12)	1.98118(-11)
25	1	2.51(-5)	7.79136(-5)		1550	18	2.05(-12)	1.94848(-11)
26	1	1.61(-5)	5.12454(-5)		1600	17	2.13(-12)	1.91489(-11)
27	1	1.06(-5)	3.37337(-5)		1650	17	2.09(-12)	1.88163(-11)
28	1	7.21(-6)	2.23228(-5)		1700	17	2.05(-12)	1.84905(-11)
29	1	5.25(-6)	1.49678(-5)		1750	17	2.02(-12)	1.81689(-11)
30	2	1.26(-6)	9.41361(-6)		1800	16	2.10(-12)	1.78269(-11)
35	2	1.22(-7)	1.05465(-6)		1850	16	2.06(-12)	1.74996(-11)
40	3	7.81(-9)	1.16281(-7)		1900	16	2.02(-12)	1.71784(-11)
45	4	5.59(-10)	1.23361(-8)		1950	15	2.11(-12)	1.68432(-11)
50	7	3.44(-11)	1.30116(-9)		2000	15	2.06(-12)	1.65151(-11)
75	26	2.20(-12)	2.96551(-11)		2100	15	1.98(-12)	1.58784(-11)
100	26	2.18(-12)	2.94369(-11)		2200	14	2.03(-12)	1.52134(-11)
150	26	2.15(-12)	2.90426(-11)		2300	13	2.08(-12)	1.45634(-11)
200	26	2.12(-12)	2.86726(-11)		2400	13	1.99(-12)	1.39241(-11)
250	25	2.18(-12)	2.83103(-11)		2500	12	2.04(-12)	1.32746(-11)
300	25	2.15(-12)	2.79499(-11)		2600	12	1.95(-12)	1.26487(-11)
350	25	2.12(-12)	2.75989(-11)		2700	11	2.00(-12)	1.19999(-11)
400	25	2.10(-12)	2.72546(-11)		3000	10	1.84(-12)	1.01468(-11)
450	24	2.15(-12)	2.69005(-11)		3200	9	1.79(-12)	8.92633(-12)
500	24	2.12(-12)	2.65560(-11)		3500	7	1.78(-12)	7.10311(-12)
550	24	2.10(-12)	2.62153(-11)		3700	6	1.70(-12)	5.95398(-12)
600	23	2.16(-12)	2.58692(-11)		4000	5	1.46(-12)	4.37393(-12)
650	23	2.13(-12)	2.55256(-11)		4500	3	1.06(-12)	2.11979(-12)
700	23	2.10(-12)	2.51874(-11)		4700	2	9.60(-13)	1.44058(-12)
750	22	2.16(-12)	2.48462(-11)		5000	2	5.22(-13)	7.83769(-13)
800	22	2.14(-12)	2.45044(-11)		5050	2	4.72(-13)	7.08228(-13)
850	22	2.10(-12)	2.41467(-11)		5100	2	4.27(-13)	6.39987(-13)

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